**Introduction to Point Set Topology, (Part I)**
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**Lecture - 55**
**The Hierarchy**

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Welcome to module 55 of Point Set Topology Part I. So, today we shall study the separation axioms, Frechetness Hausdorffness regularity etc comparing one with the other. So, that is why the word Hierarchy here which one is stronger than which; so, that is the main question here ok. For example: you have already seen that Hausdorffness implies Frechetness. On the other hand, we cannot compare directly regularity or normality ok. Similarly, we can't compare Hausdorffness and regularity ok.

However, if you mix up these two ok, then something quite surprising thing comes out. There is a complete hierarchy you can make. So, once there is such a hierarchy, we will have to use numbers to indicate them. I mean classically they have been done like that; so, we have to follow it, there is no other choice ok.

I want to warn you that there are some authors who do the other way round which is totally unexplainable, I do not know how they have got into that mess; even very good book like Simon's book has a different definitions altogether.

So, my connotation is different from that. So, you have to be a bit careful about that. So, I would prefer the terminology I am following, which is more logical than the other one. So, I just want to warn you that is all. So, we shall call a Frechet space a $T_1$ space and a Hausdorff space a $T_2$ space ok.

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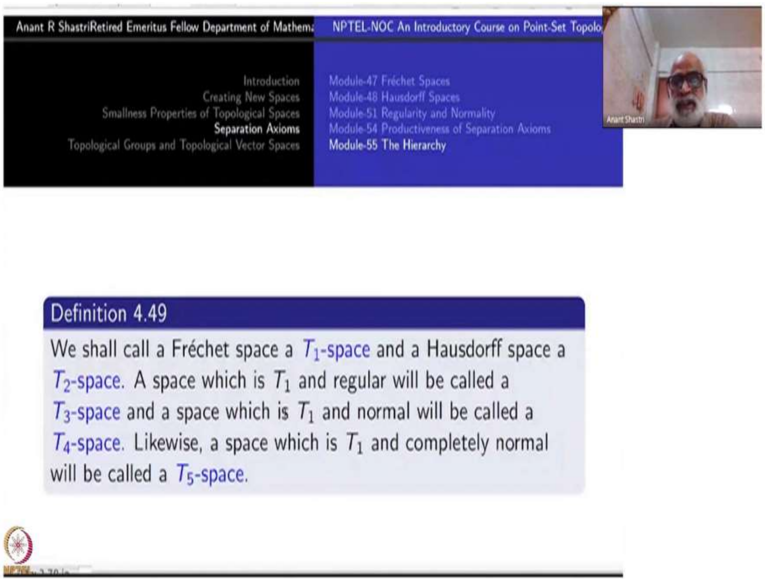


A space which is $T_1$ and regular will be called $T_3$ space and a space which is $T_1$ and normal will be called $T_4$ space. So, this is what I meant by mixing up, $T_1$ and regular will have the name, namely $T_3$. And, $T_1$ and normal will have the name $T_4$, $T_1$ and completely normal will be called $T_5$.

Now, why these numbers these numbers have been chosen with some results already in mind namely a $T_n$, where $n > m$ will always implies $T_m$. So, $T_5$ implies $T_4$ implies $T_3$ implies $T_2$ implies $T_1$. So, that is the hierarchy. That looks like a beautiful way of putting it and easy to remember thing also ok. So, that is the theorem, the first theorem here ok.

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So, I will come back to these two things a little later.

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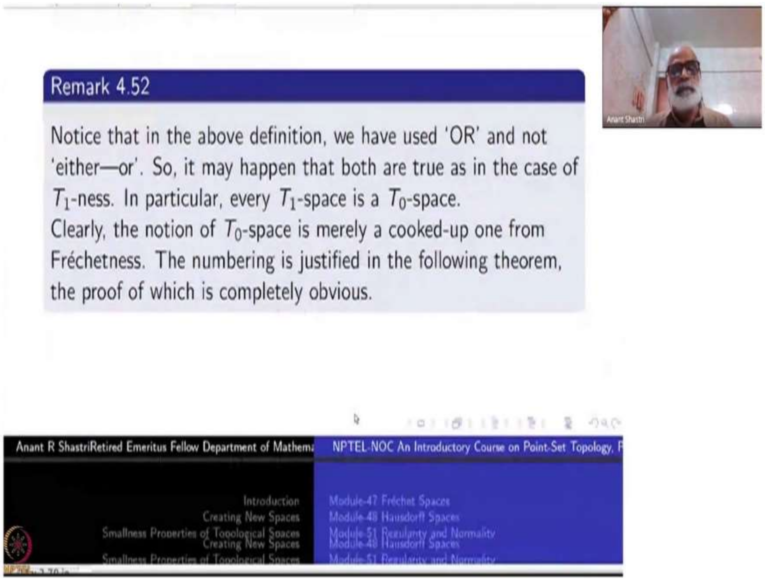


So, I will come back to that one. So, first let me go through this theorem.

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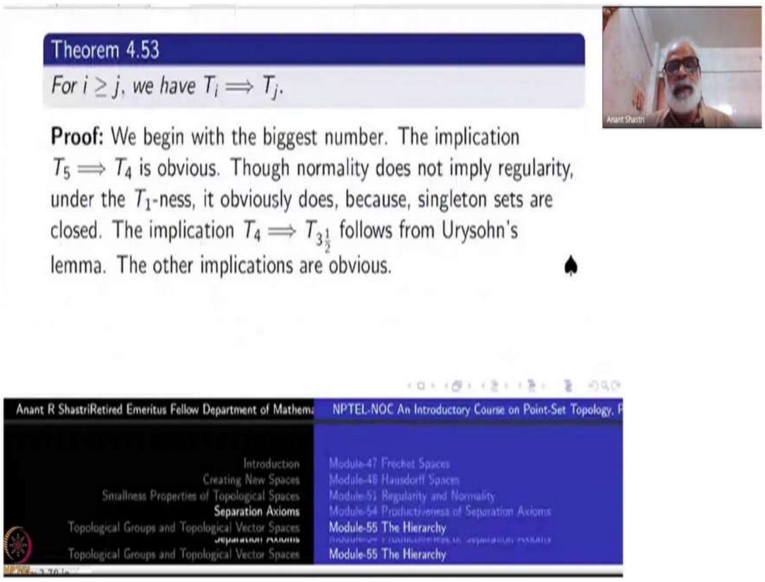


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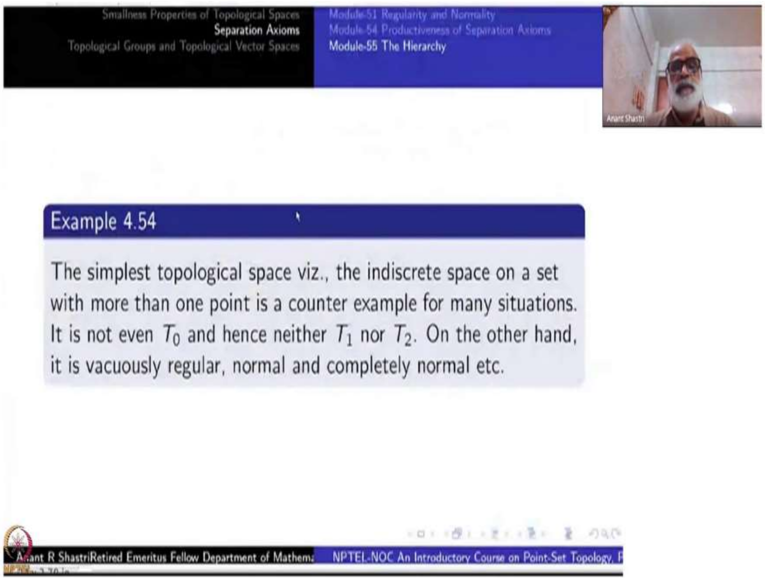


So, first let me go through this theorem. $i > j$, we have $T_i$ implies $T_j$ ok. So, $T_5$ implies $T_4$ is obvious, $T_5$ being completely normal plus $T_1, T_4$ is normal plus $T_1$ ok. So, completely normal implies already normal so, that is obvious.

Next, though normality itself does not imply regularity, if you put $T_1$ on both sides, there is an implication. Why? because as soon as $T_1$ is there ok, singleton sets become closed.

Therefore, if you have a point and a closed set disjoint from that, its as if we are having two disjoint closed subsets. Therefore, once you put normality also, there are open subsets around them which are disjoint. So, $T_1$ness assures that singleton points are closed, that is why this works alright.

Now, $T_4$ implies another one, I will come to that one later on. First let us complete this.

So, now, $T_3$ similarly implies $T_2$ why? Because, $T_3$ is to a closed set and a single point, but single point is closed and when you take two distinct points, they are both closed subsets. So, we can apply regularity to get two open subsets around that one; so, that imply Hausdorffness.

So, $T_1$ness helps to derive Hausdorffness from regularity. Only under $T_1$, otherwise it is not true.

And, already we have seen that $T_2$ implies $T_1$, ok.

Now, I go back to these other numbers here. so, So, we are introducing little more few more numbers here, let me come back here.

A $T_1$ space which is completely regular, remember there was a regularity and complete regularity also we have introduced ok. So, $T_1$ space which is completely regular will be called as $T_{3\frac{1}{2}}$.

Unfortunately, there is no integer between 4 and 3. So, we have to use three and half Ok? The whole idea here is that $T_4$ implies $T_{3\frac{1}{2}}$ and $T_{3\frac{1}{2}}$ automatically implies $T_3$. Because, complete regularity implies regularity, add $T_1$ on both sides you get $T_{3\frac{1}{2}}$ implies $T_3$. But, it true is that $T_4$ implies $T_{3\frac{1}{2}}$ because of Urysohn's characterization. Remember that $T_3$ under this complete regularity was actually an adopted version of Urysohn's characterization right.

So, that is the whole idea. So, this $T_{3\frac{1}{2}}$ has another name: it is called Tychonoff space ok. So, there was no integer to accommodate it. So, people cooked up this $T_{3\frac{1}{2}}$ name for it that is all.

But, there is another thing one can do, a weaker version of $T_1$ ok. So, let us define that one, we have not done that one yet. So, there is no regularity, normality anything, it is weaker than $T_1$ space. What is it? A topological space is called $T_0$ space, (so, this time we are jumping not $T_{1/2}$ and so on) $T_0$ space, if for every pair $x, y$ of points in $X$ ok, when I say a pair I meant $x$ and $y$ are distinct, there exist an open set $U$ containing $x$ and not containing $y$ or an open set containing $y$ and not containing $x$. I repeat: given two distinct points, you know you may have an open set around the first one not containing the second one or it may happen that there is an open set containing the second one, but not the first one.

It just means that both of them can also occur, I am not saying `either... or...' ok. I am not saying only this or only that, No. The point is both of them can also occur. I do not have to tell that, but I want to make that one clear ok. In the definition above, we have not used the words `either.... or...'. So, it may happen that both are true as in the case of $T_1$ness, in the case of $T_1$ness if you have two distinct points, there is a neighbourhood about one which does not contain the other.

Now, I do not say which one ok; therefore, it is applicable to both the points there right? So, that is why a $T_1$ness automatically implies $T_0$, but $T_0$ may not imply $T_1$ ok. So, this $T_0$ space its just looks like a cooked-up notion. That is my opinion, cooked-up notion from Frechetness. There is only one instance wherein with some extra hypothesis there, $T_0$ necessarily imply $T_1$ness, we will see that one ok. There is only one instance of that one.

So, anyway, the numbering is completely justified because of our theorem now, $T_5$ implies $T_4$ implies $T_{3\frac{1}{2}}$ implies $T_3$ implies $T_2$ implies $T_1$ implies $T_0$ ok. So, this is a complete hierarchy alright, whenever $i > j$ gives $T_i$ implies $T_j$, ok.

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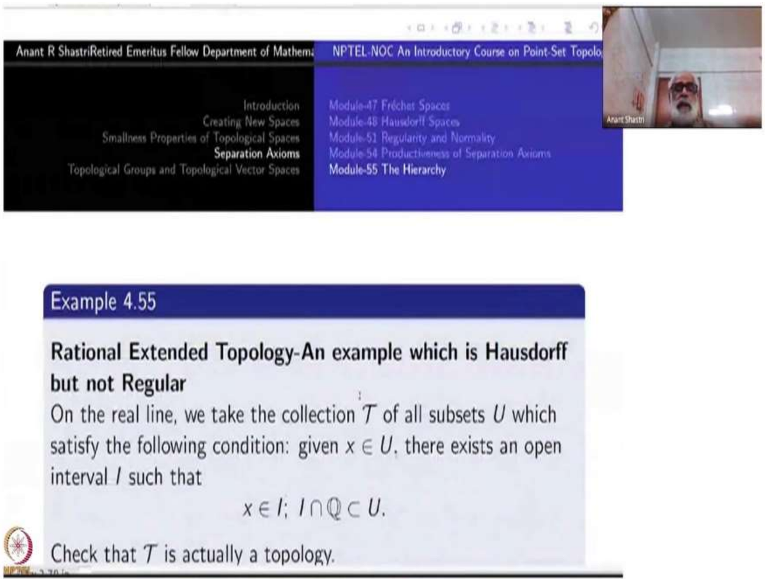


So, here is another example now which is a Hausdorff space, but not a regular space ok. See Hausdorffness does not imply regularity, regularity does not imply Hausdorffness either, but regularity plus $T_1$ implies Hausdorffness ok; Hausdorffness same thing as $T_2$. So, here is an example which is Hausdorffness but not regular.

Again on the real line, we take the collection tau of all subsets $U$ which satisfy the following condition.

Given $x$ belonging to $U$, there exists an open interval $I$ such that $x$ is inside $I$, but instead of saying that $I$ is contained inside $U$ which will be the usual topology, what we say $I \cap \mathbb{Q}$ is contained inside $U$, a much weaker condition ok. If the whole of $I$ is contained inside $U$ well and good, that will be usual topology, but this is much weaker condition ok.

Nevertheless, this condition defines a topology on $\mathbb{R}$, with this topology $\mathbb{R}$ will be called rationally extended topology ok; that is the name. Obviously, it is finer than the usual topology, because usual topology also satisfy this condition right. The whole of $I$ will be contained inside $U$.

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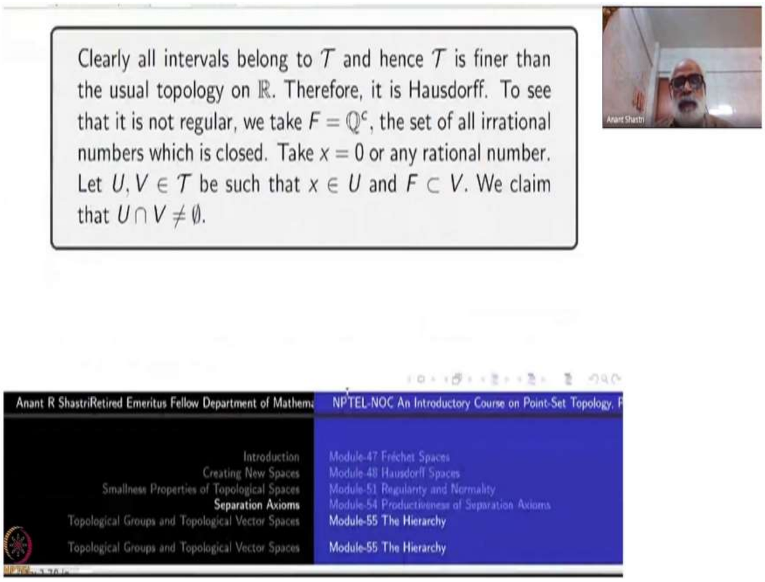


Once it is finer than the usual topology, it is Hausdorff, anything finer than Hausdorff space is Hausdorff. So, half part is over. So, what we want to prove is that it is not regular ok. For seeing that it is not regular, we take $F$ equal to $\mathbb{Q}^c$ set of all irrational numbers, the set of rational numbers is an open subset here right. Because, take a point in the rational numbers, take any interval all the rational points in the interval are contained inside $\mathbb{Q}$ that is all. So, $\mathbb{Q}$ itself is open therefore, $\mathbb{Q}^c$ is a closed set ok.

Now, take x equal to 0 or any rational number for that matter, let us take $x$ equal to 0 that is outside $F$ right? So, we must find, what we must find? $U$ and $V$ such that $x$ belongs to $U$ and $F$ is contained inside $V$, and $U \cap V$ is empty, that is regularity. But, now we have to show that no matter what $U$ and $V$ are, the moment they are open and contain $x$ and $F$, their intersection is non-empty. That is what I have to show, so as conclude that the space is not regular. We could have chosen any other point, any other closed set, but this is our choice; so, $F$ is $\mathbb{Q}^c$. So, we will try to do this one. If it fails it does not mean that it is regular because, we have made a choice which may be wrong ok.

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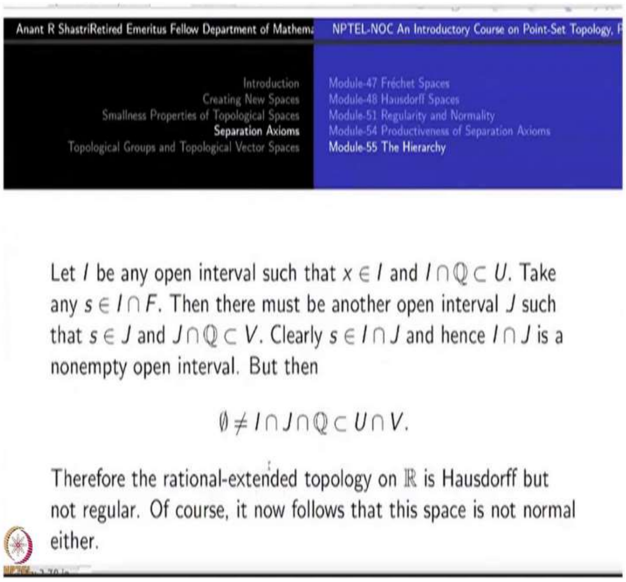


So, assume that $U$ is an open subset containing $x = 0$ and $V$ is an open set containing all the irrational number that is what we have started. So, since $U$ is an open subset, there will be an open interval $I$ such that this $x$ is inside $I$, remember $x$ is just $0$, or any rational number; $x$ belongs to $I$ and $I \cap \mathbb{Q}$ is contained inside $U$ ok? So, $I$ is an open interval therefore, you can take any $s \in I \cap F$, which has to non empty.

What is $F$? $F$ was set of all irrational numbers. So, it has lot of irrational numbers ok; obviously, this $s$ will be different from $x$ no problem. Then there must be another open interval $J$ such that $s$ is inside $J$, $J \cap \mathbb{Q}$ is inside $V$, because $V$ is an open subset containing $F$, by our assumption ok. Look at these two intervals $I$ and $J$ ok, they have a common point $s$, $I$ and $J$ are open intervals. This is a common point essentially they are intersecting.

So, intersection of two open intervals if it is non-empty, it is another interval only right? Therefore, $I \cap J$ is a non-empty open interval, but then this non-empty open interval intersection with $\mathbb{Q}$ is also non-empty. Now, if you look at $J \cap \mathbb{Q}$ that is inside $V$, but if you look at intersect $I \cap \mathbb{Q}$ that is inside $U$. Therefore, this intersection is both inside $U$ and $V$ so, it is inside $U \cap V$.

Now, that is the thing that we wanted to prove, that such open subsets cannot be disjoint. So, that proves that the rational extended topology on $\mathbb{R}$ is Hausdorff, but not regular ok. Can it be normal?

Student: No.

Teacher: Why?

Student: Because, we have seen that $T_3$ implies $T_4$.

Teacher: The other way around $T_4$!

Student: Yeah, $T_4$ implies $T_3$.

Normal plus $T_1$ implies regular plus $T_1$. So, $T_4$ implies $T_3$, but we have seen that it is not regular, but it is Hausdorff; so, it is $T_1$ ok. So, it follows that this cannot be normal. So, that is a corollary, since you have proved that it is not regular ok.

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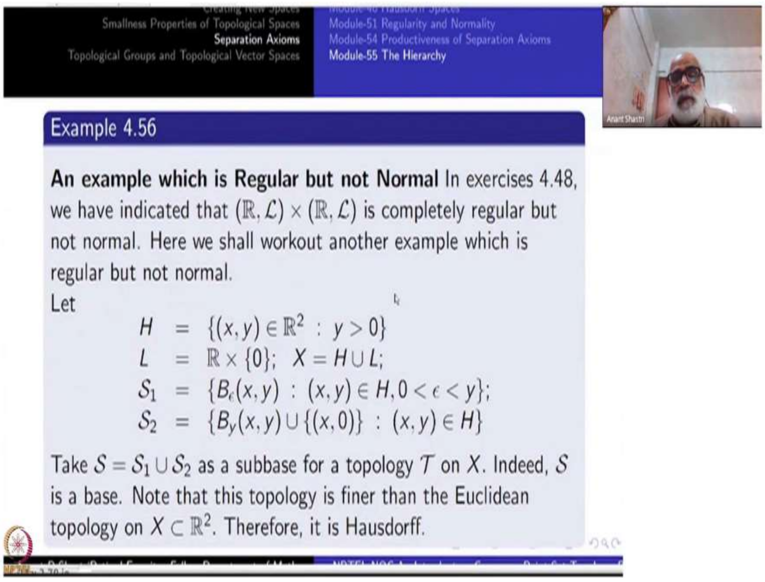

Now, we will take another example which will give you regularity does not imply normality. This space is regular, but not normal ok. We have already seen such an example, but we would like to do this one for reason that this is again another modification of the real topology, Euclidean topology. So, in exercise 4.48, we have indicated that the semi interval topology product with itself is completely regular, but not normal.

See what we have proved is regular and not normal, but it is actually completely regular. That is what we have indicated in the exercise, there it was an exercise ok. But now, we will prove this one the other example, which is regular, but not normal ok. So, what I do? I take the upper half plane $H$ or $(x, y)$ belonging to $\mathbb{R}^2$, with $y$ positive ok. The second coordinate is positive, the upper half plane, open upper half plane, I am denoting $L$, the real line $\mathbb{R} \times \{0\}, y$ equal to 0 ok.

I am including that also with $H$ and that is my $X$. So, this is the closed upper half plane, but I do not want to call it so, because, (this part I can call it as upper half plane), but here around $L$, I am going to change the topology. So, I am using a different notation $X$ here. So, here is the topology coming now, two families are declared, together making a subbase.

$\mathcal{S}_1$ is equal to set of all open balls ok, around points inside $H$ namely $y$ coordinate is positive $x$ coordinate anything; $B_\epsilon(x, y)$. It must be contained inside $H$ therefore, the radius must be less than $y$ that is all, $0 < \epsilon < y$. So, I am taking all the open balls completely contained inside the upper half plane ok. These are standard open balls right. The second one is slightly different, that is where the crux of the matter lies.

They are open balls with $(x, y)$ center, the radius is equal to $y$ not some $\epsilon$. So, $y$-coordinate becomes the radius of that ok. So, it is touching the $x$-axis right in one point. What is that point? $(x, 0)$. So, now you include that also that point is not there in the open ball here, it is tangential. So, include that point also that is the elements in this set such that $(x, y)$'s are inside $H$, ok.

So, start with a point in the upper half plane, take the maximum open ball contained inside that that is the meaning of this $B_y(x, y)$ ok. You cannot take bigger than that, then it will go below the $x$-axis, that is not allowed right. So, if you take maximum open ball this is what it is, then put that point $(x, 0)$ also in that. So, this is going to be one of the sets inside this $\mathcal{S}_2$, take the collection of all of them; so, that is your $\mathcal{S}_2$. Now, you put the union of these two, call that as $\mathcal{S}$ a sub base for a topology on $X$. Any collection of subsets can be declared as subbase that we know ok.

So, this is sub base for topology on $X$. Whatever that topology is it has the property that, by the way I have made a wrong remark here, namely this is actually a base. Let us not bother about this, this is subbase is enough for us. Note that this topology is finer than the Euclidean topology, because you see on the on the upper half part this is actually Euclidian topology ok. Everything open in the $\mathbb{R}^2$ is there and vice versa.

And On the $x-$axis you intersect these balls with the $x-$axis, what is it? It is just the single point $(x, 0)$ therefore, each singleton point on the $x-$axis becomes an open set. Therefore, the induced topology on the $x-$axis is discrete, in any case its finer than the usual topology alright. Therefore, this entire topology is finer than the usual topology ok, in particular it is Hausdorff. So, Hausdorffness is already there. Alright.

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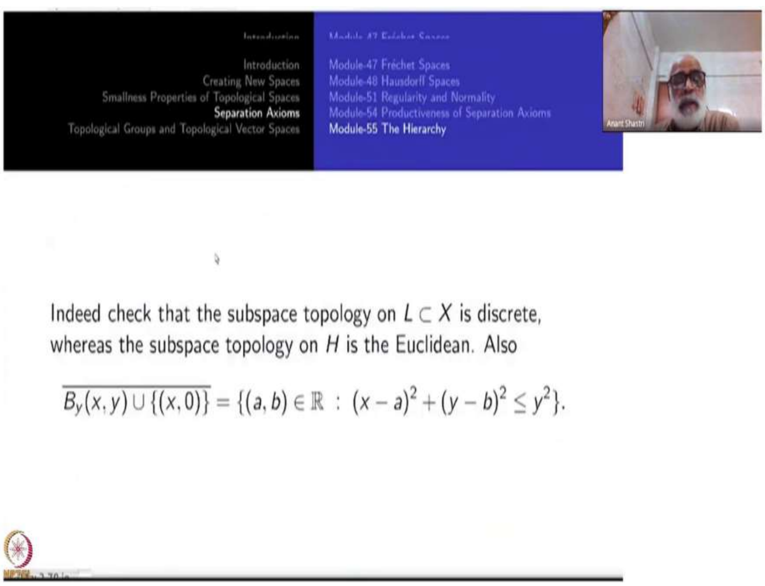


Now, we have to show that this is not what this is not normal. So, I observe this namely if you take $B_y(x, y)$ union $\{(x, 0)\}$ ok, that is an open subset of this topology. It is in the $\mathcal{S}_2$

part, its closure is all those $(a, b)$ belong to $\mathbb{R}^2$ such that $(x-a)^2+(y-b)^2 \leq y^2$; the full closed ball will come, when $a, b$ ranges over or $\mathbb{R}$ ok.

So, the closure will be just the closed ball that is all. $(x, 0)$ is already there, but the closure will contain all the rest of the circle also that is all.

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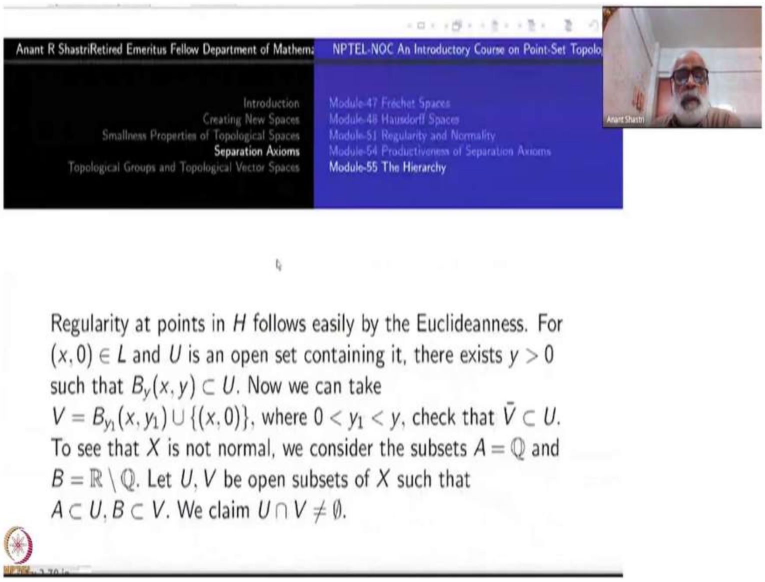


Regularity at points of $H$ follows easily by the Euclideaness of the upper half plane. Take a point in the upper half plane, take an open subset, you do not have to worry about any weird open subset, but you can take one inside that you can take a usuual open ball right and verify the regularity, its already Euclidean space; so, it is regular. So, there there is no problem there. The problem arises when you take $(x, 0) \in L$, namely on the $x$-axis ok.

For $(x, 0) \in L$ and $U$ is an open set containing it ok, there exists a $y > 0$ such that $B_y(x, y)$ is inside $U$. this is these are because members of $\mathcal{S}_2$ where the point $x$ is fixed from a local base at $(x, 0)$. Now, we can take $V$ equal to $B_y(x, y_1) \cup \{(x, 0)\}$, where the $y$ coordinate $y_1$ is less than $y$ ok.

And, then check that $\bar{V}$ is contained inside $U$ ok. So, even for points on $L$ one has verified regularity. To see that $X$ is not normal is our next task here ok. So, we again take $A$ equal to the entire $\mathbb{Q} \times \{0\}$ and $B$ equal to $(\mathbb{R} \setminus \mathbb{Q}) \times \{0\}$ ok. Similar to the earlier example ok, we will show that there is no open subsets containing $A$ and $B$ which are disjoint ok. So, that is what we want to show. This is similar to $\mathbb{R} \times L$ but, some somewhat easier maybe you can see.

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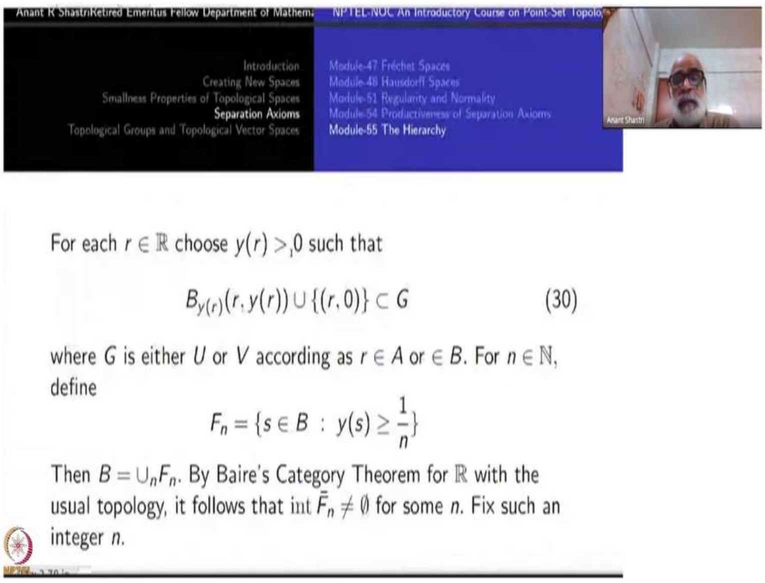


For each $r \in \mathbb{R}$, choose $y_r$ and take that $B_{y_r}(r, y_r)$. Since $y_r$ is positive, $(r, y_r)$ will be in the upper half plane right. Now, I am taking the full ball maximum ball of whatever possible radius, that will be of radius $y_r$. Put the point $(r, 0)$ also, this is an open subset now. So, such an open subset will be inside $G$, where $G$ is either $U$ or $V$ according as $r$ is inside $U$ or inside $V$; that means, rational or irrational ok, for both the choice is done the same way.

So, you can check you can use open balls, there are always such open balls. Each contained either inside $U$ or inside $V$, according $r$ inside $A$ or $B$. Now, for $n \in \mathbb{N}$, let us define $F_n$ to be all $s$ inside $B$ such that $y_s > 1/n$. See for each each $r$, I have a $y_r$ that I have chosen. So, I look at all those $s$ such that the corresponding $y_s > 1/n$.

So, that is my definition of $F_n$. Sicne for every $r, y_r$ is positive it follows that the entire of $B$ will be union of $F_n$'s. After all once it is positive it will be bigger than some $1/n$. So, $B$ is union of $F_n$'s. Now, by Baire's Category Theorem on $\mathbb{R}$ with the usual topology, see remember what is $B$? $B$ is set of all irrational numbers. It follows that interior of $\overline{F_n}$ cannot be empty for all of them right?

We have shown that the entire of irrational numbers cannot be written as countable union of nowhere dense sets. Because, then you can another set of countable numbers of singletons of rationals, to get the whole of $\mathbb{R}$ as a countable union of nowhere dense sets. So, that is why one of the $F_n$'s must have the property that interior of $\overline{F_n}$ is nonempty. You fix such an integer now ok.

So far we have not used anything other than the fact that $U$ and $V$ are open ok. But, now one of this has interior of $\overline{F_n}$ is nonempty ok. Choose an open interval $I$ contained inside $\overline{F_n}$, interior in the usual topology. There are subsets of $\mathbb{R}$ now.

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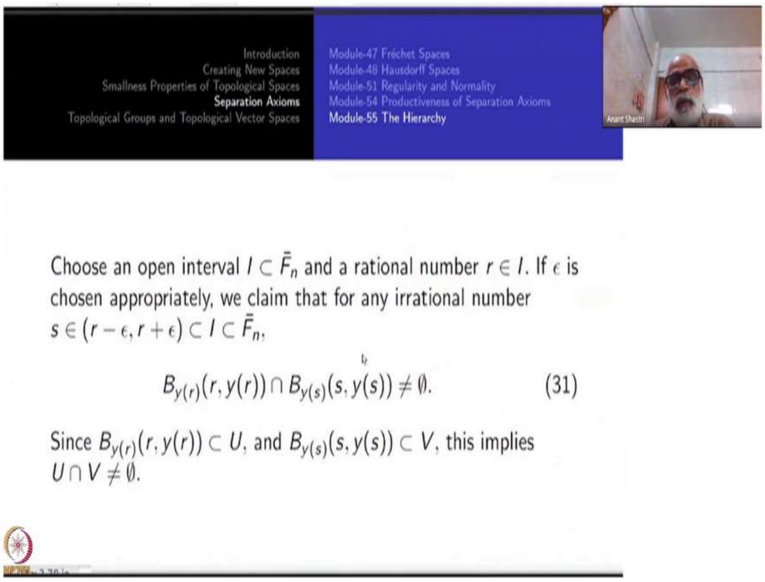


So, there will be an open interval $I$ contained in $F_n$, and a rational number, say $r$ belonging to $I$. If $\epsilon$ is chosen appropriately, we claim that that something goes wrong.

For any rational number $r \in I$, we can always choose $\epsilon$ positive such that $(r-\epsilon, r+\epsilon)$ is contained in $I$. And then there are plenty of irrational numbers inside that interval.

But I am not saying that for all $\epsilon$ this is true, appropriately chosen $\epsilon$. What we claim is that $B_{y_r}(r, y_r)$ intersection with the corresponding ball for $s$, viz, $B_{y_s}(s, y_s)$ will be non-empty. That is the contradiction because these these two balls are supposed to be contained inside two disjoint open sets ok? This one says $r$ is rational number and $s$ is irrational number ok. So, that will imply that one is in $U$ another other is in $V$ and hence $U \cap V$ is nonempty ok? So, this is our claim: how to choose $\epsilon$ is the point, so that this will happen.

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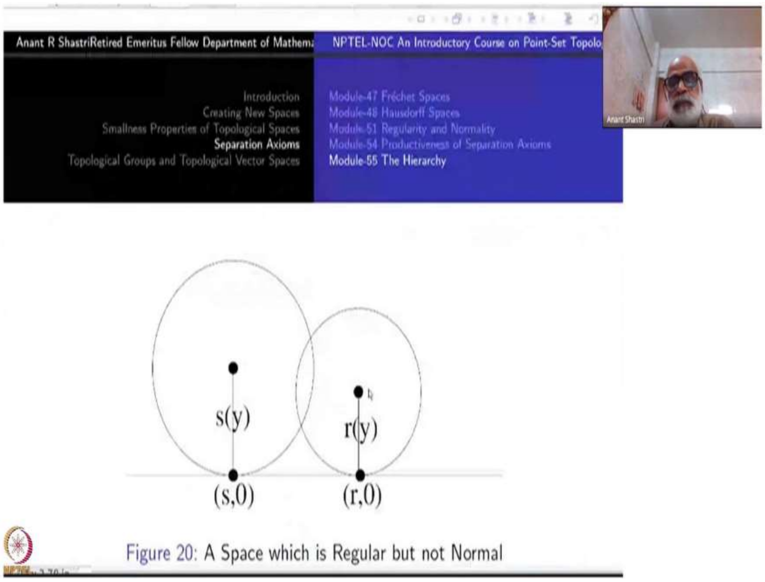


So, here is a picture what is happening here whatever $r$ and $s$, whatever they are there is some $y_r$ ok. So, this whole open ball along with this point is inside $U$ or inside $V$, that is how we have got it. Now, I want that $s$ and $r$ are chosen such that they are intersecting here ok. $r$ has been already chosen, so $s$ should be chosen close to $r$ in such a way such that the corresponding ball will intersect.

Note that $s(y)$'s are already chosen, I have no control over that, but I can choose $s$ itself closer and closer to $r$. So, how close I should choose is indicated in this picture ok. So, now, I have just worked out 12th standard mathematics here.

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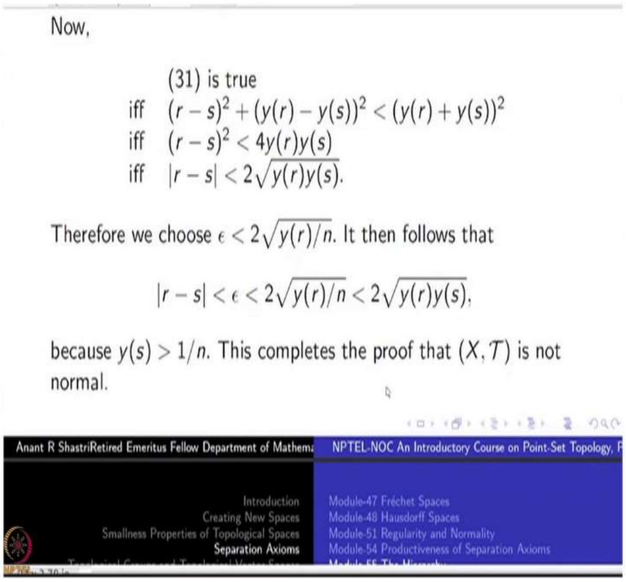


So, 31 is true; that means, the intersection is non empty if and only if $(r-s)^2+(y_r-y_s)^2$, which is the square of distance between the two centres, is less than $(y_r+y_s)^2$, the square of the sum of the radii. So, this is what I am playing. Go back to this picture: the distance between the two centres must be less than the sum of the two radii.

So, that is the distance between these two, this distance total distance must be less than the length of this one plus the radius of this plus radius of that which is $y_r+y_s$. So, there is notational difference here, $y_r$ this is the $y_r$ and $y_s$ that is all ok. So so, that is the first condition $y_r+y_s$, they are the radius sum total, if I take the square root of this, this will be square root of that.

So, as I have taken the squares on both sides ok. This is same thing as now simplify $(r-s)^2$ is less than $4y_ry_s$, you take this one to this side $y_r^2y_s^2$ will cancel out ok.

Student: Ok.

$2y_ry_s$ and $2y_ry_s$ will add up, this will be $4y_ry_s$. It is same thing as now taking square root $|r-s|$ should be less than $2\sqrt{y_ry_s}$ ok? Therefore, choose now $\epsilon$ to be less than $2\sqrt{y_r/n}$ ok.

Suppose, you choose this $y_r/n$, it follows that now what is this $n$ remember, this $n$ was fixed such that the interval is contained inside the interior of $\overline{F_n}$.

So, that n appears here, it follows that $r-s$ is less than $\epsilon$ if you if you if you have this one, this $\epsilon$ is less than $2\sqrt{y_r/n}$ that will be less than $2\sqrt{y_r y_s}$. I want this one, I want the last thing. If I have satisfied this $r-s$ less than this one, then the intersections will be non-empty, the 31 will be true. So, now, I choose this $\epsilon$ to be less than this one, then $r-s$ is less than $\epsilon$ will satisfy this property ok.

So, this is because $y_s$ is bigger than $1/n$. So, that $y_s$ part disappears here, you see the some condition should not be depending on s; so, that I am choosing $s$. So, this is purely in terms of $r$ now ok. So, that will be automatically less than this one because $y_s$ part is less than $1/n$. So, this completes the proof that $(X, \mathcal{T})$ is not normal.

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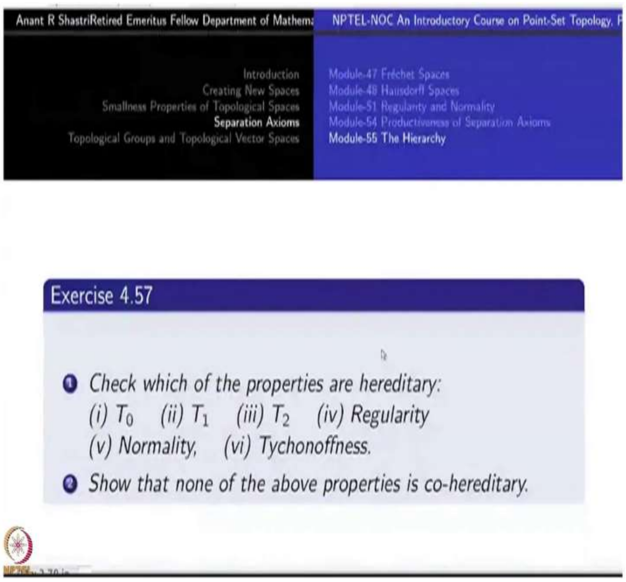


So, here is some exercises, maybe you can have your own exercise also, but I would not like to encourage you to go on studying just counter examples. Nevertheless, if you are interested in, there is a very good book written on this one, long long back ok. I have given the reference right in the beginning. So, I will indicate it to you later. So, you can read that book ok.

So, thank you. We will meet next hour now ok.