Introduction to Point Set Topology, (Part I)
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Lecture - 54
Productiveness of Separation Axioms

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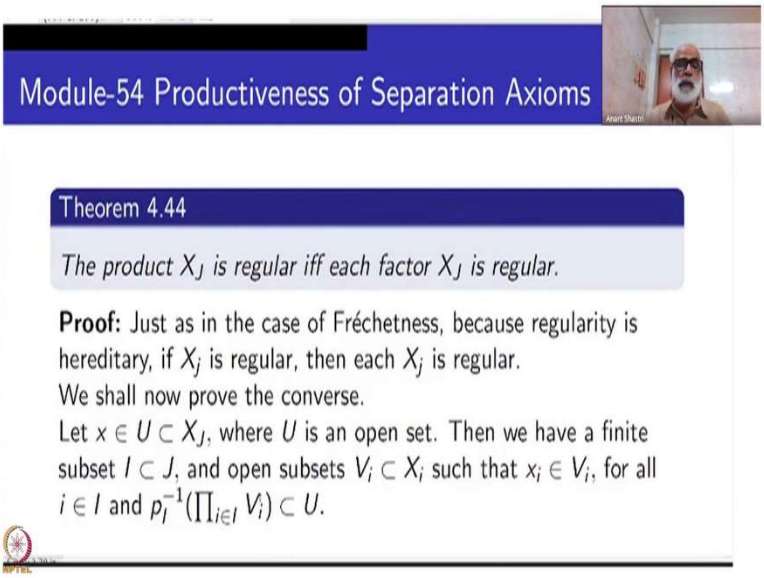

Welcome to module 54, we shall continue the study of Separation Axioms this time, concentrating on checking whether they are productive. The very first theorem here is that regularity is productive ok? So, this is stated as follows: product X_J is regular if and only if each factor X_j is regular.

Just as in the case of Frechetness because regularity is hereditary, if X_J is regular then each X_j is regular, why? Because you can always think of each coordinate space, each factor space as a subspace of the product space right? $X \times \{y\}$ is contained $X \times Y$, via $x \mapsto (x, y)$. So, that is the subspace you know X can be identified with that subspace. That is what we have been using already.

So, if the product space is regular, the subspace will be regular; that means, each factor is regular that is easy.

We shall now prove the converse; pick up x belonging to U where U is open in X_J ok. So, we must produce another subset, open subset subset whose closure is contained inside U and this open subset contains x.

So, between $\{x\}$ and U one must squeeze another open set ok. So, in the product topology something is open means its a neighbourhood of x means, there is a basic open nbd. And basic open set looks like what? We have a finite subset I of J and open nbds V_i of x_i in X_i such that the coordinate is inside $p_i(U)$ for all $i \in I$. This I is a finite set and look at the intersection of $p_i^{-1}(V_i)$, i ranges from 1 to n.

That obviously contains x ok and that will be contained inside U. So, actually regularity can be checked by just using basic open sets. If I show that inside this, I can get another onbd say W such that $\bar{W}$ is contained inside this one, that is same thing as doing it for U ok. So, you can you assume that U itself is a basic open set that is all.

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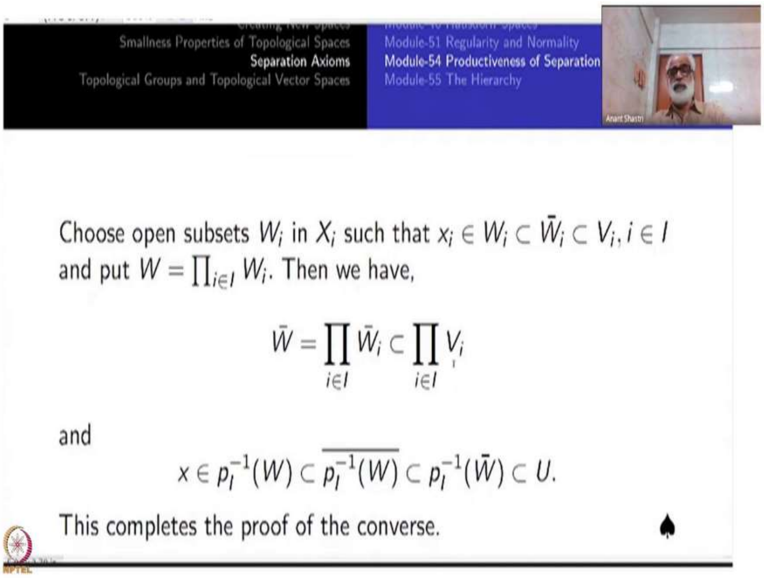

Now, for each X_i being regular for each x_i contained inside V_i, V_i is open, you can find an open subset W_i in X_i such that x_i is inside W_i inside $\bar{W}_i$ inside V_i. Between x_i and V_i, we can squeeze another open set V_i. So, this we can do for all i inside I, and then I take the product of this finitely many W_i's that I am calling as W. Then one thing which we have already checked all the time is the closure of this product is the product of the closures, each closure here is contained in corresponding V_i. So, this product will be contained inside product of V_i's ok.

Now, if we take p_I inverse of this one, x will be inside $p_I^{-1}(W)$ because each i^{th} coordinate x_I is inside W by the very choice x_i's are inside W. So, x is inside $p_I^{-1}(W)$, automatically it will be contained this is closure what I need to show here it is contained inside U, right.

Now, this is a closed sub set containing this one and this is also a closed subset p_I is a continuous map. So, p_I inverse of a closed subset is closed and it contains $p_I^{-1}(W)$. Therefore, this closure is contained inside this one, actually these are equal here, but I am just taking the easy way, no doubt this is contained inside and this has been chosen such that this is inside U because it is this $p_I^{-1}(\bar{W})$. See $\bar{W}$ is in the inside this one. So, $p_I^{-1}(\bar{W})$ is p_I inverse of these and these things are contained inside U, that is the choice of V_i's. That completes the proof.

So, not much hard work here, regularity is productive. So, in that sense it is nearer to Hausdorffness right? Indeed there is two schools of thought in topology, people from one use Hausdorffness whenever they have difficulty, the other school uses regularity whenever they have difficulty and they achieve similar results, in fact, identical results ok. So, in that sense, regularity is quite near Hausdorffness. But the two concepts work in slightly different way, that is what I wanted to say ok.

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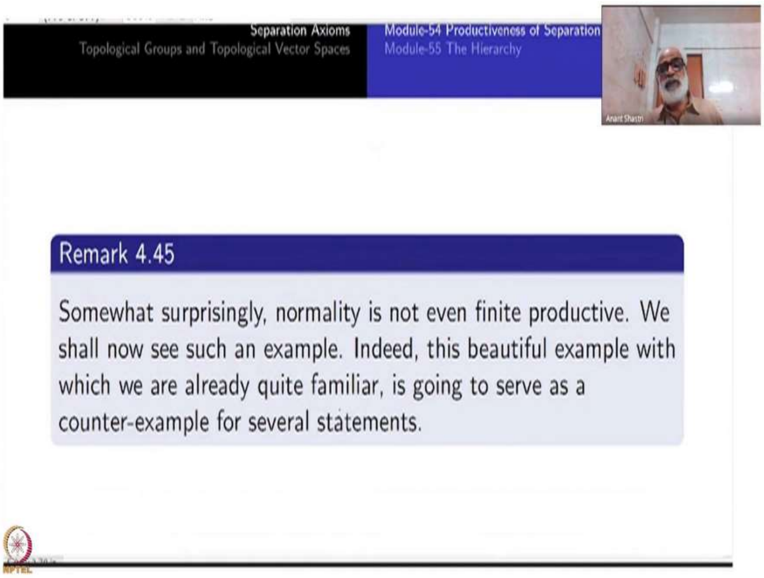

But, somewhat surprisingly, normality which we tend to think is nearer to regularity, it has weird properties. Normality is not even finite productive. The reason may be because it is not even hereditary ok. Even one way it fails. Even if $X \times Y$ is normal it does not imply that X is normal and Y is normal, that is the funny thing here. So, we shall see now such an example. The example that we are going to see quite a beautiful example, which you are already familiar with and we have studied quite a few properties of that; that is the semi open interval topology.

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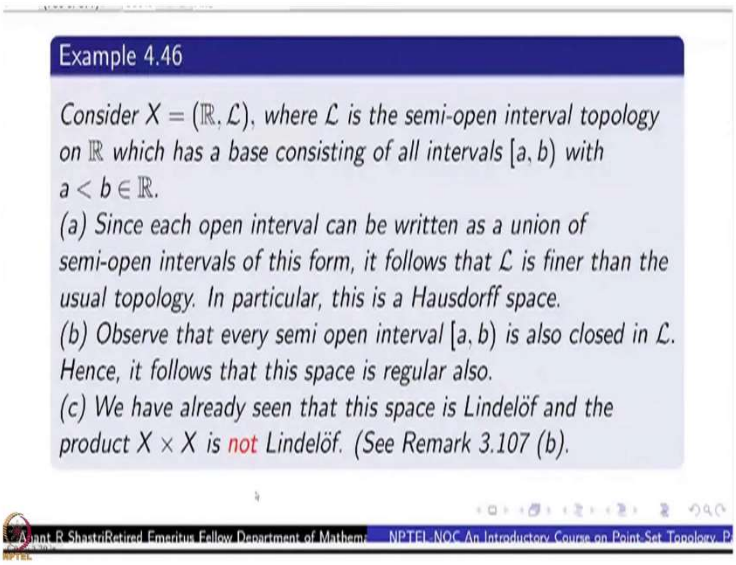

X is the real line with the semi interval topology, namely with the base consists of half-closed intervals $[a, b)$. Very specific! That is why it is denoted by $\mathcal{L}$, semi intervals; you could have denoted by $\mathbb{R}$ also, but that would mean you are taking $(a, b], a$ open and b closed. These two are homeomorphic to each other. So, if you study one of them, you know the other. By just taking x goes to $-x$ you will get a homeomorphism, ok?

So, concentrate on this semi-interval topology, each open interval can be written as a union of semi open intervals of this form right? Suppose I want to get (a, b). Then all that I have to do is $[a + 1/n, b)$ union overall n will be equal to (a, b), right.

So, all open intervals can be written as this one, it follows that this topology is finer than the usual topology because all the open subsets in the usual topology will be also open in this topology. However, these half closed intervals are not open in the usual topology.

So, this is finer, strictly finer than the usual topology. Anything finer than a Hausdorff space will be also Hausdorff. Therefore, this space is Hausdorff ok? Now, observe that every semi open interval is also a closed, closed in $\mathcal{L}$ ok? So, when you have a base consisting of closed sets, the topology therefore will have some very strong properties. And people have studied such things.

So, all those properites will be available for this $(\mathbb{R}, \mathcal{L})$, ok? Closed intervals are closed in inside usual topology, but here even half closed intervals are both open as well as closed in this topology. Why this is closed? Because what is its complement? $[b, \infty)$, is by definition is an open set $(-\infty, a)$ is open, that is also an open subset because you have just seen that every open subset in $\mathbb{R}$ is also open here ok.

So, now if x belonging to such a basic open set $[a, b)$, you can W to be $[a, b)$ itself so that x is in W is in $\bar{W}$ is in $[a, b)$. See, regularity is satisfied trivially, here. You do not have to choose another W at all. We have already seen that this space is Lindelof and the product is not Lindelof, do you remember that? This was a remark. Let me just show you that.

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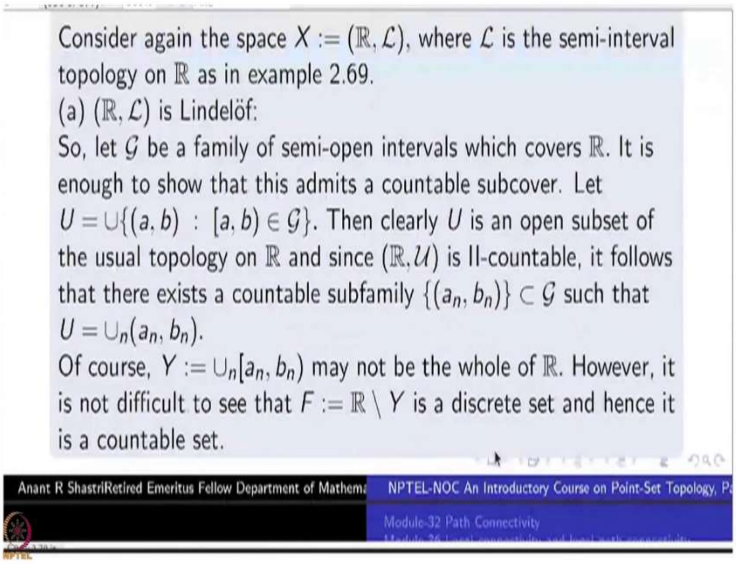

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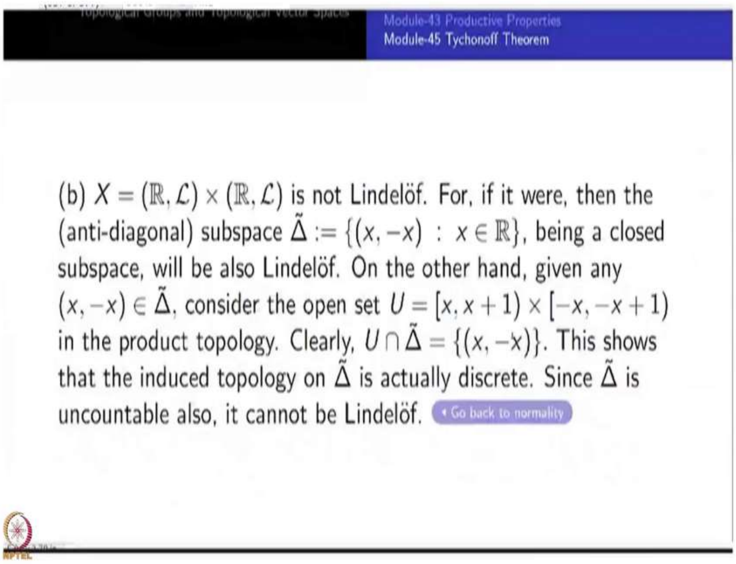

We have already done this one. Then by showing that the diagonal is discrete in the product, remember that this was the example right?

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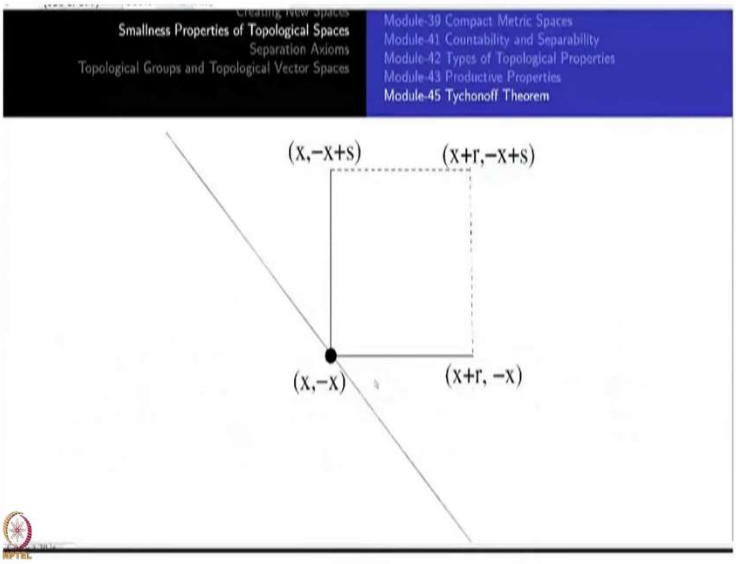

So, here we had shown that the product is not Lindelof. Therefore, you know you can introduce a lot of things. Of course, this itself is Lindelof we have seen. So, this was not all that trivial ok? You have to use the second countability of the usual topology and so on ok. So, but I do not have to repeat that ok? we have seen this one that the space is Lindelof, but the product is not Lindelof alright.

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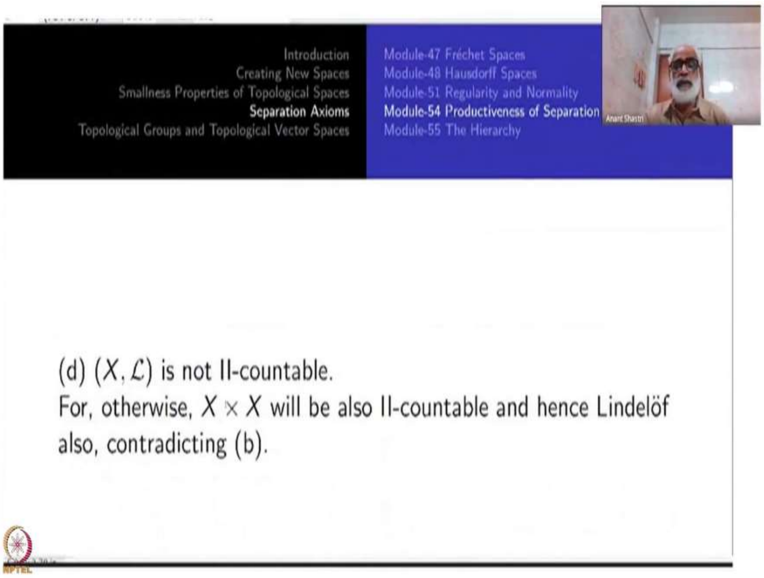

Now, I want to tell you that this semi interval topology is not II-accountable. Second countable of course implies Lindelof. But this an example is anyway Lindelof space. We want to see that it is not II-countable. Why? If it were II-countable, then the product will be also II-countable, because second countability is finite product invariant. Finitely productive. That also you have seen. But once $X \times X$ is II-countable, it will be Lindelof also. So, that will contradict the previous observation that we have done alright. So, this $(\mathbb{R}, \mathcal{L})$ is a space which is Lindelof, but not II-countable ok.

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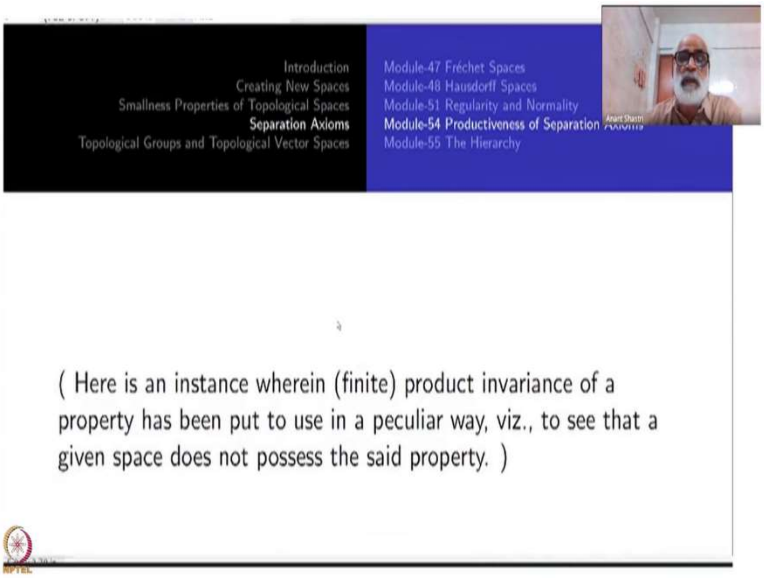

Now, here is a remark, we have used finite product invariance of II-countability in a peculiar way, to see that the space under investigation is not II-countable ok? So, you see you can use certain theorems in a negative way also. So, they also help in this way alright.

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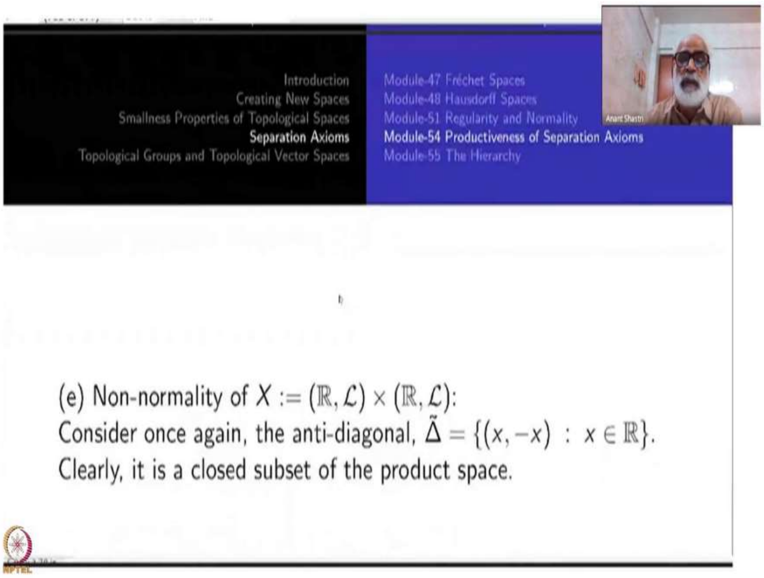

Now, let us continue with this space. Finally, I want to show that the product is not normal ok? Non normality of the product space. Just now, we showed that this is a regular space right and the product of two regular space is regular. So, you will get a exmaple of a regular space, which is not normal ok?

So, how do you show that? This semi-interval topology product with itself is not normal. Once again we go back and see that the diagonal is a closed subset in the product. It is actually discrete. So, that is what we had seen before ok. But now we want to use it very crucially right.

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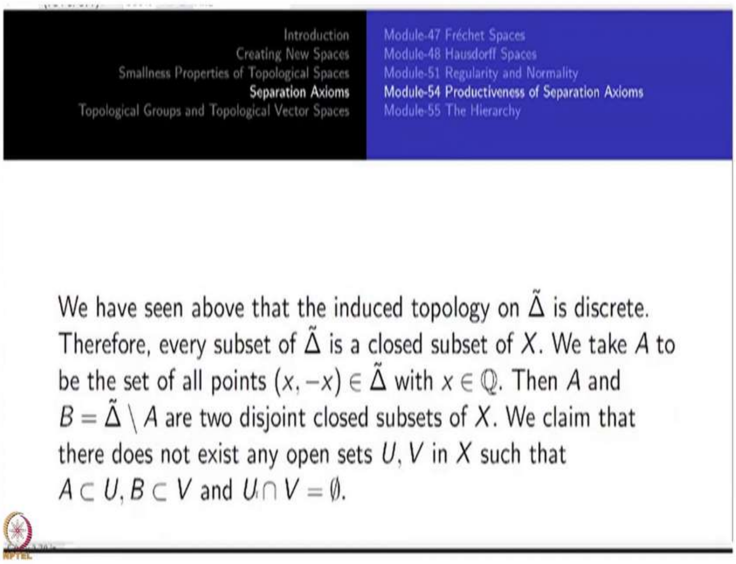

So, we have seen above that the induced topology on $\tilde{\Delta}$ usually Δ will denote the diagonal (x, x). So, this is $(x, -x)$. This is discrete ok why? because you could take half open interval cross half open interval, just touching the point on the diagonal. It is contained in one side, upper side of the whole anti-diagonal.

So, such an open set intersect the antidiagonal in single points. So, that single point is open in the subspace. That is how we have to done it. Now, this will help us in showing that the product is not normal, we take A to be the set of all points $(x, -x)$ with x being rational ok? Any subset of $\tilde{\Delta}$ is closed now ok? Similarly, take B to be the complement of A in $\tilde{\Delta}$. That will be also closed, because the subspace $\tilde{\Delta}$ is a discrete space. And $\tilde{\Delta}$ is closed in the whole space. So, these two are now disjoint closed subsets of X.

A consists of points $(r, -r)$, where r is rational, B consists of $(r, -r)$, r is irrational. Now, we claim that there exist no open sets U and V containing A and B respectively, such that their intersection is empty. In other words we start with U and V open, A contained inside U, B contained in V, then we show that $U \cap V$ is not empty.

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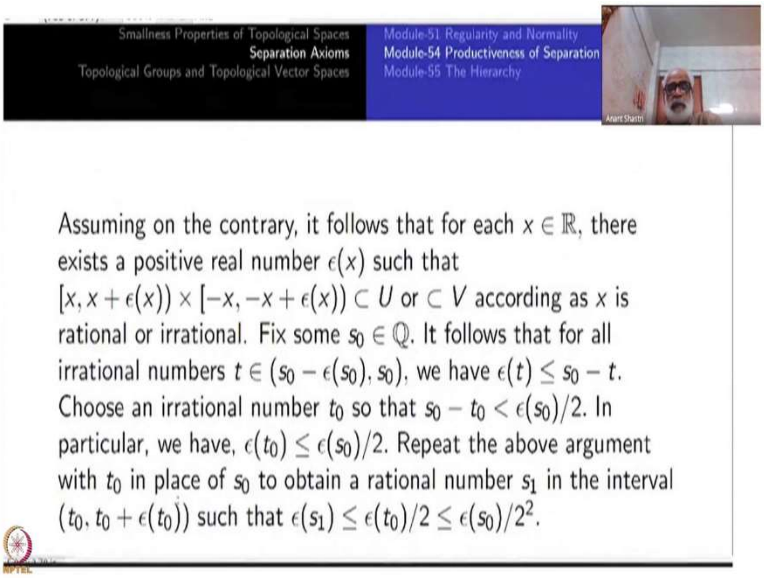

Assuming on the contrary; that means what? Suppose you have two open subsets containing A and B respectively U and V. Since U, V are open it follows that for each x inside $\mathbb{R}$, there exist a positive real number $\epsilon(x)$ such that $[x, x+\epsilon(x)) \times [-x, -x+\epsilon(x))$ is contained in U or V according as x is rational or irrational.

See these are points of our $(\mathbb{R}, \mathcal{L})$ and this is the product topology right. So, I am taking a product neighbourhood which is a basic neighbourhood in the product topology, this is contained inside U or contained inside V according as x is rational or irrational. If x is rational it will be inside U, if x is irrational it will be inside V ok. Of course, at each point the lengths of these intervals x to $x + \epsilon(x)$, this $\epsilon(x)$ will depend upon x alright.

So, now fix a rational number s_0 belonging to $\mathbb{Q}$. It follow that for all irrational numbers t belonging to $(s_0 - \epsilon(s_0), s_0)$, what is this $\epsilon(t)$? We have fixed these $\epsilon(x)$ remember that. I am taking the same thing here. t is in $(s_0 - \epsilon(s_0), s_0)$. If t is irrational then $[t, t+\epsilon(t)) \times (-t, -t+\epsilon(t))$ should not intersect $(s_0 - \epsilon(s_0), s_0) \times (-s_0, -s_0 + \epsilon(s_0))$. That is because U and V are disjoint and these product nbds are contained in V and U respectively. This means $\epsilon(t)$ must be less than $s_0 - t$. Choose t such that $s_0 - t$ is less $\epsilon(s_0)/2$.

So, I am putting further restriction on t. You can show that there is an irrational number t_0 such that $0 < s_0 - t_0 < \epsilon(s_0)/2$. Therefore it follows that $\epsilon(t_0)$ is less that $\epsilon(s_0)/2$. Now, you interchange the role of irrational number and rational number. You have started with s_0, you got a t_0 with all this property, you fix this one. Now, apply the same argument to t_0 to get a rational number, with the same property, but this time you denote it by s_1, repeat the above argument with t_0 in place of s_0 to obtain the rational number s_1, such that s_1 is in $(t_0, t_0 + \epsilon(t_0))$ and $\epsilon(s_1)$ is less that $\epsilon(t_0)/2$ which is less than $\epsilon(s_0)/2^2$.

So, one stage of construction is over. Starting with a rational number, you get an irrational number with some property, that is all and their irrational number you get again a rational number one cycle is over. Now, repeat this cycle, repeat it repeat it. So, what you get? You get a sequence ok?

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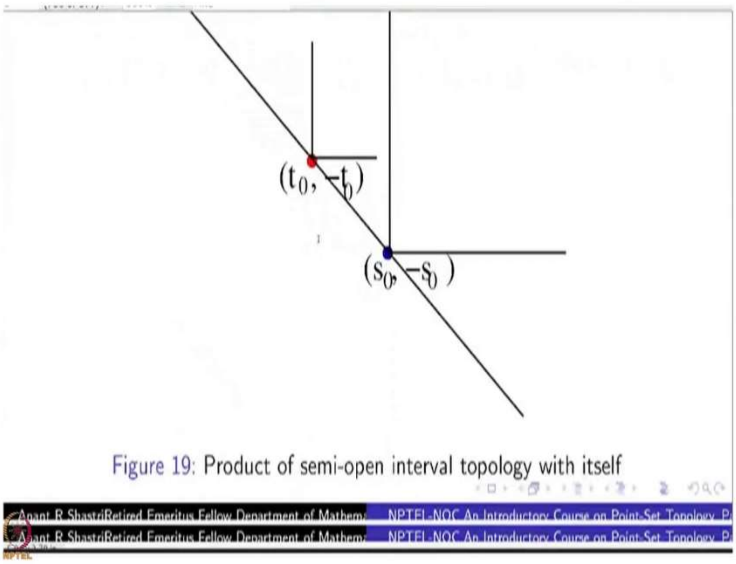

Figure 19: Product of semi-open interval topology with itself

So, I repeat. In the first stage what I have done. Starting with a rational number s_0 here, it has some $\epsilon(s_0)$ and t_0 is just little smaller that s_0, on the left side. Corresponding interval should not come over here. Otherwise these two would not be disjoint. So, the length of the interval has to be at the most this much right. So, this $\epsilon(t_0)$ will have to be smaller than the difference between s_0 and t_0. So, that is all I have got, ok yeah.

Whether I choose it here or here it is the same thing alright, but I have meticulously chosen it behind here, that is all, ok. Now, I can choose the next rational number on the right side of t_0. So, $(s_0, t_0), (s_1, t_1), (s_2, t_2)$ just like in the Liebnitz series ok? Alternatively, they will be between t_0 and s_0 ok. So, that is how I am going to choose those numbers here. For one cycle it is clearly stated and you repeat it ok.

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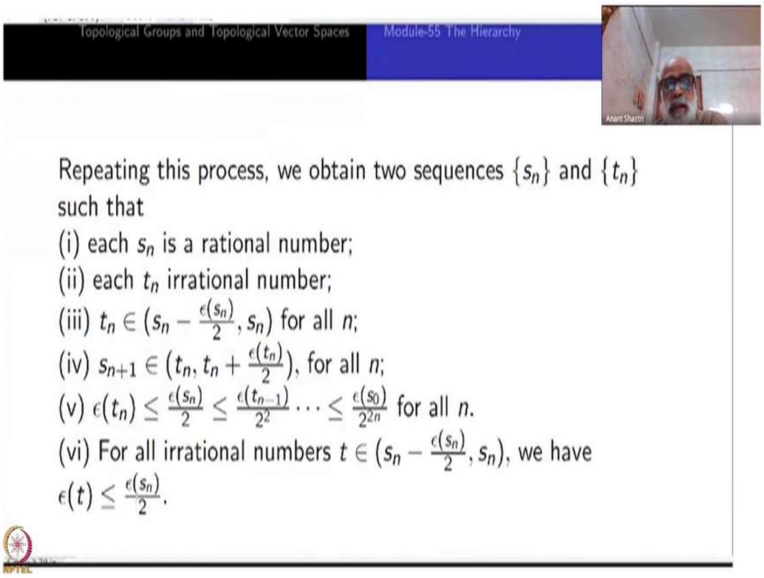

So, what do you get? Repeating this process we get two sequences s_n and t_n, such that each s_n is rational number, each t_n is an irrational number; t_n's are inside $(s_n - \epsilon(s_n/2), s_n)$ and s_{n+1} are inside $(t_n, t_n + \epsilon(t_n/2))$. So, this is this is this minus this, is on the left side, this is on plus side on this side for all n ok. And $\epsilon(t_n)$ will be less than equal to $\epsilon(s_n/2)$ and that is less than equal to $\epsilon(t_{n-1}/2^2)$ and so on, all so on. $\epsilon(s_0/2)$ to the 2^n unit 2^{2n} unit.

By the by at the n^{th} stage you will have what? $2^2,^4$ and so on, 2^{2n}, for all n. For all irrational numbers t between $(s_n - \epsilon(s_n/2), s_n)$, we have $\epsilon(t)$ is less than equal to $\epsilon(s_n/2)$. Once you have chosen that one for everything in between also the length of those intervals which you have chosen has to be short. Otherwise, they will collide with the other you have one which you have chosen for the rational numbers ok.

And the basic assumption is that U and V are disjoint, that is all. Now you see that there is a contradiction here. For all rational numbers for the same reason s belonging to t_n to $t_n + \epsilon(t_n/2)$, we should have $\epsilon(s)$ less than this number.

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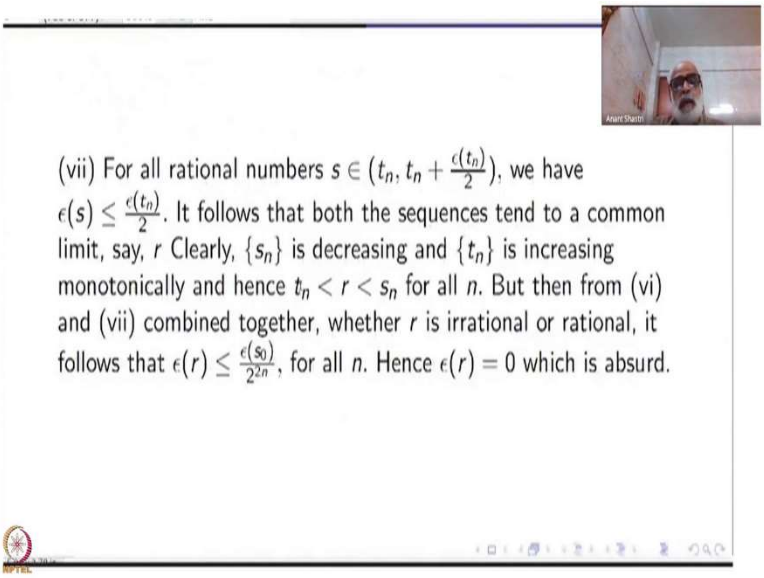

So, it follows that both the sequences tend to a common limit ok say r, that is clearly s_n is decreasing and t_n is increasing. I told you this is similar to the proof of you know alternating series, why it is convergent it is similar to that; s_n is decreasing and t_n is increasing monotonically and hence the limit will be between t_n and s_n, t_n is less than equal to r less than equal to s_n for all n.

But then from 6 and 7 combine together, whether r is irrational or rational it follows that the corresponding $\epsilon(r)$ has to be less than $\epsilon(s_0/2^{2n})$ for all n this r is independent of n. So, when you take the limit of this it will show that $\epsilon(r)$ is 0.

So, the contradiction is to the fact that we can choose you know open rectangles around each point on the antidiagonal such that one set of rectangles are inside U others are inside V, they are disjoint.

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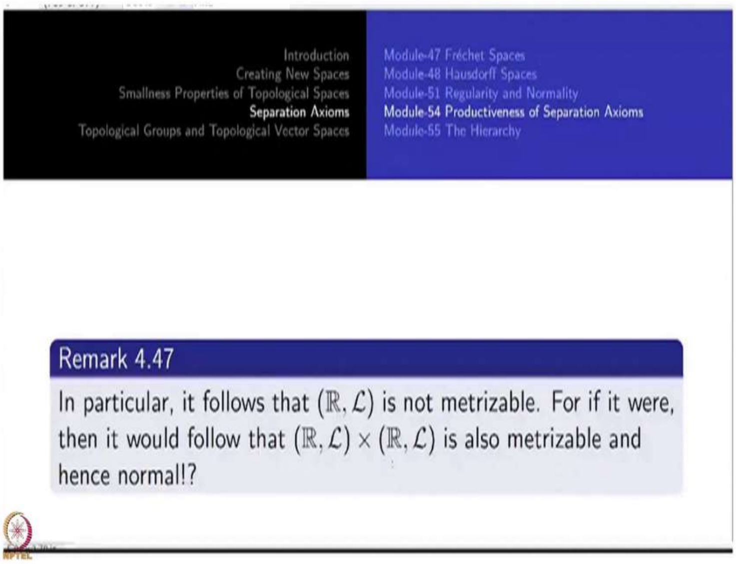

Next, one small observation is that, this semi-interval topology is not metrizable. Why? Because if it is metrizable, its product will be also metrizable. Metrizability means what? There is a metric. The product topology is given by the product metric right. If it is product metric, then it will be normal a contradiction, because just now we prove that it is not normal ok.

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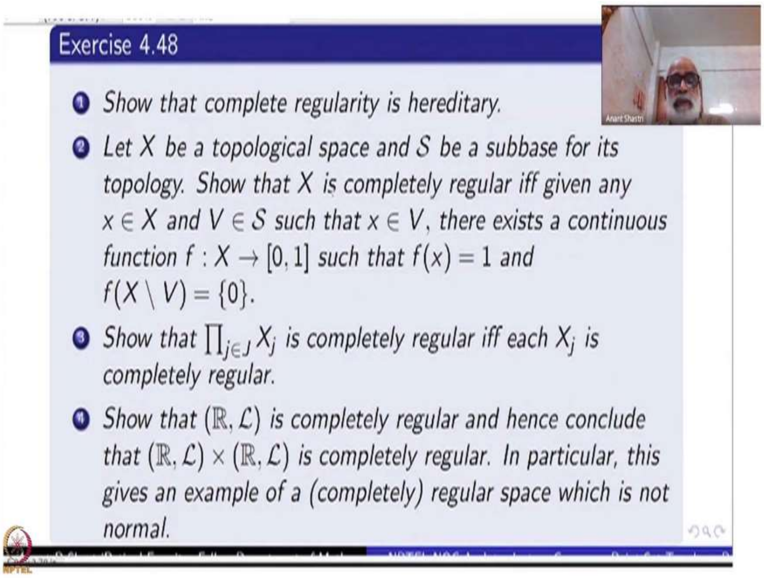

So, that is the contradiction. Here I have put a few exercises to you, try them. They will only illustrate and you will get more and more familiarity with the concepts here. Complete regularity is hereditary ok.

So, Hausdorffness then regularity, complete regularity fine. Normality? It will fail. That is what you have yet to learn.

Next, to check complete regularity of a space X, you have to verify the condition for only for memebrs of a subbase. You have to prove that.

For each point x belong to $X \setminus V$, where V is in a subbbase, there exist a continuous function f such that $f(x)$ is 1 and $f(V^c)$ is 0. Just be careful and be done with it and you can verify it. So, that will be helpful especially while dealing with products. So, I have put that one as an illustrative example, an exercise which will help you to solve the next exercise.

Show that the product of completely regular spaces is completely regular. One way is obvious because of the hereditariness ok. After that you show that the lower limit topological space $(\mathbb{R}, \mathcal{L})$ is completely regular and hence conclude that $(\mathbb{R}, \mathcal{L}) \times (\mathbb{R}, \mathcal{L})$ is completely regular ok. In particular this gives you an example of a completely regular space which is not normal. we just prove that it is not normal ok. So, let us stop here and take up this study next time.

Thank you.