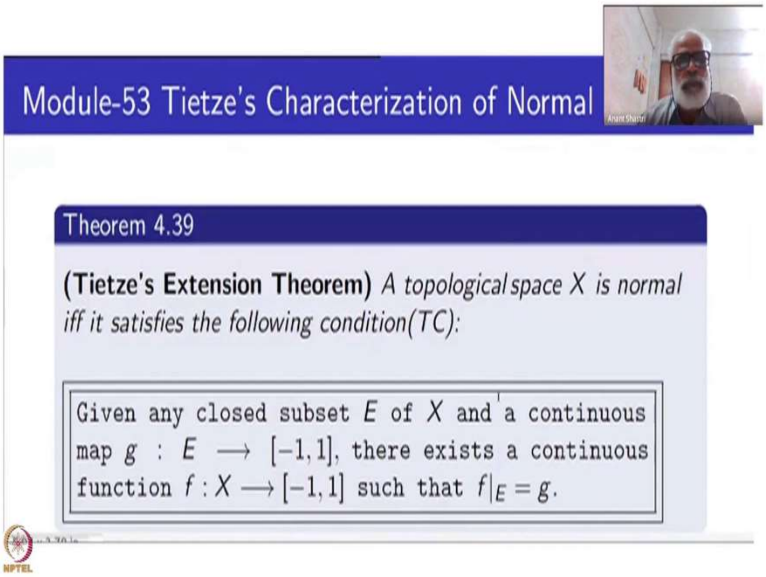
Introduction to Point Set Topology, (Part I)
Prof. Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture - 53
Tietze's Characterization of Normal Spaces

(Refer Slide Time: 00:17)



Welcome to module 53 of point Set Topology course part 1. Today, we will continue the study of Normal Spaces. Characterization of normal spaces. Last time we did the characterization due to Urysohn. Now, we will do the characterization due to Tietze. Tietze's characterization is another landmark, but it actually uses, you know, Urysohn's characterization, construction of continuous functions.

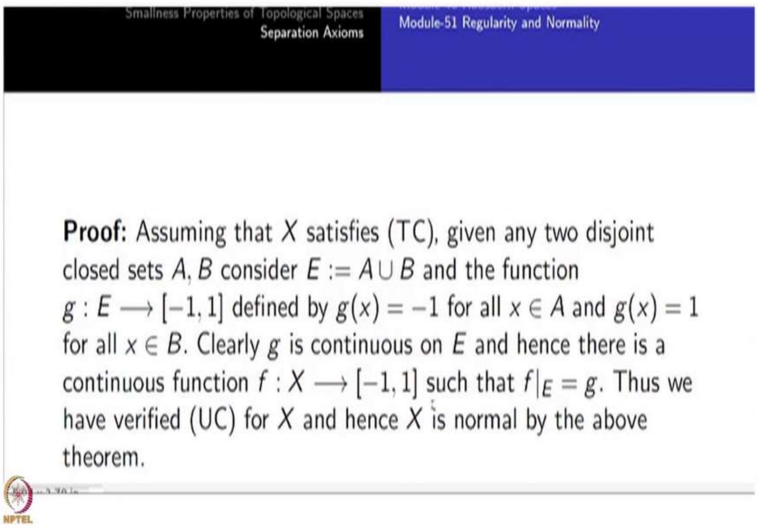
But it is quite mysterious in a sense that here we have only one closed set. So, it requires quite an ingenious mind to have explored this one. So, it is not at all easy to come up with this kind of idea. So, let us look at the statement here. A topological space X is normal, if and only if it satisfy the following condition which I have put as Tietze condition.

So, what is the condition? Given any closed subset E of X and a continuous function g from E to $[-1,1]$, there exists a continuous function f from X to minus $[-1,1]$ such that f restricted to E is g, ok.

Obviously, here E is a nonempty subset ok? That is otherwise we do not have any continuous function g from E to $[-1,1]$. So, this is the hypothesis. Given a closed set and a continuous function ok. So, there is no need to worry about that, E being empty and so on. So, every continuous function defined on a closed set can be extended to a continuous function, retaining the co-domain as it is namely any closed interval.

So, we are using the model interval $[-1,1]$. I have already told you that you can always change the co-domain to any closed interval each time ok. So, that is not very crucial, but now instead of $[0,1]$ as in the previous theorem, we will use $[-1,1]$ which is more convenient for the writing down the proof ok.

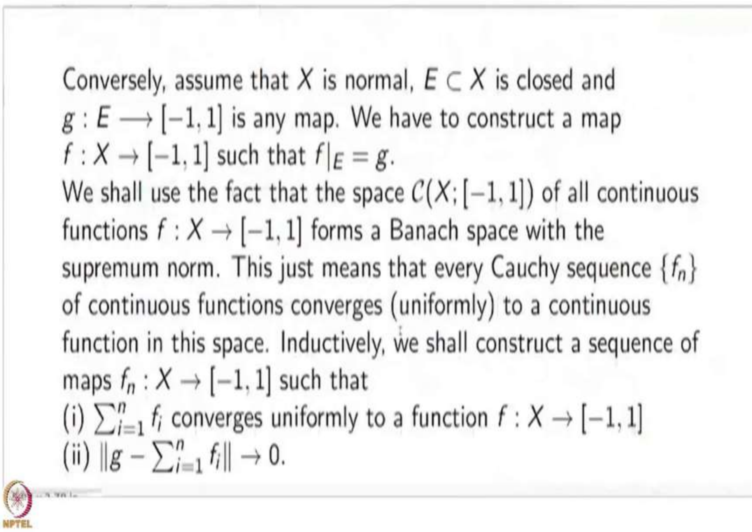
(Refer Slide Time: 03:41)



Assume that X satisfies the TC, given any two disjoint closed subsets A and B, put $E = A \cup B$. And apply this condition to get a function g from $[-1,1]$, namely $g(x) = -1$ for $x \in A$ and $g(x) = 1$ for $x \in B$ ok? g is continuous because A and B are disjoint closed sets ok. So, this makes sense. Only two values you have taken, on A it is -1, on B it is 1. But now g will get extended to a continuous function from X to $[-1, 1]$.

So, there is some f here f restricted to E is g ok. So, that is the condition for the Urysohn's characterization UC. Therefore, X is normal by the above theorem alright. So, we have taken a shortcut here, used Urysohn's characterization, instead of trying to prove open subsets etcetera alright? But in the converse proof also you are going to use Urysohn's theorem. Anyway, one way is done.

(Refer Slide Time: 05:23)



Now, let us proof converse again, this converse is taking up some time. Assume X is normal and E containing x is closed and a map g is given ok. g is defined only on E and its continuous there, we have to construct a map f from X to $[-1, 1]$ such that f restricted to E is g. So, here we shall use the fact that set of all continuous functions from X to $[-1, 1]$ ok, they form a Banach algebra, Banach space remember that with the supremum norm.

Remember that we had introduced the set of all function from X to $\mathbb{R}$ or $\mathbb{C}$, whatever, and then we took the subspace of bounded functions, on which we put the supremum norm. Now, if we have continuous functions, continuous functions on X ok? They may not be bounded, but here I am taking they are bounded by actually $[-1, 1]$, the values are inside $[-1, 1]$.

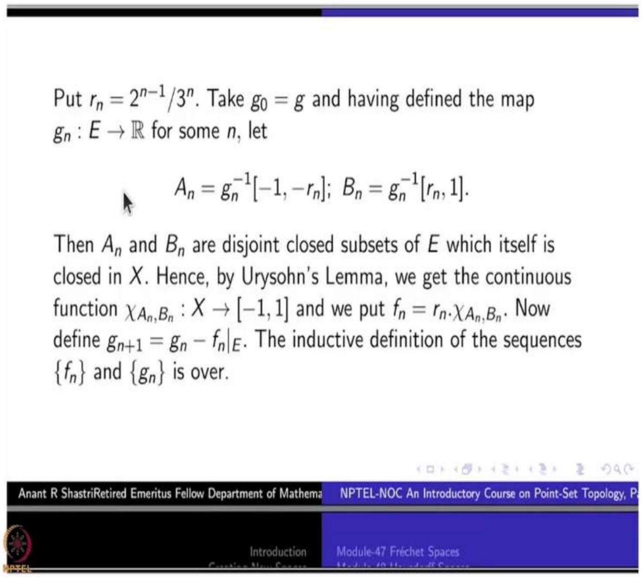
So, this will be a subspace of that Banach algebra alright. This norm makes sense because now all functions are taking values to $[-1,1]$ ok. We also saw that the subspace continuous functions is a closed subspace of this Banach algebra and therefore, this itself as as submetric space, (it is non-linear space) it is complete.

That means, Cauchy sequences will converge inside this one, which just means that if you take a Cauchy sequence of continuous functions from X to $[-1,1]$, you can take the limit in that larger space. But that limit is continuous. Therefore, it is in the smaller space ok. So, this also you have seen the convergence, with respect to the supremum norm is nothing but uniform convergence ok? So, this is what we are going to use. Essentially if you do not want to use all these terminologies, all that means that if you have a sequence of continuous, real valued functions ok which converges uniformly, then the limit function is continuous.

So, this is the fact that comes out of ordinary analysis. So, which we are going to use it now here ok. How are going to use? Inductively, we shall construct a sequence f_n, from X to $[-1,1]$ such that summation f_i's, these are partial sums, that is a sequence, not the sequence f_n, the partial sums that sequence converges uniformly to a function f from X to $[-1,1]$. And it has this property that norm of $g - \sum_{i=1}^{n} f_i$ converges to 0. Because g is only defined only on E, this norm should be taken on E. So, the same norm restricted to E, ok.

So, all these f_i's restricted to E. Of course, f makes sense on the whole of X, but g makes only on E. So, that is the meaning of this one. So, this is what we want to do now. So, I will explain these things more carefully in what comes here.

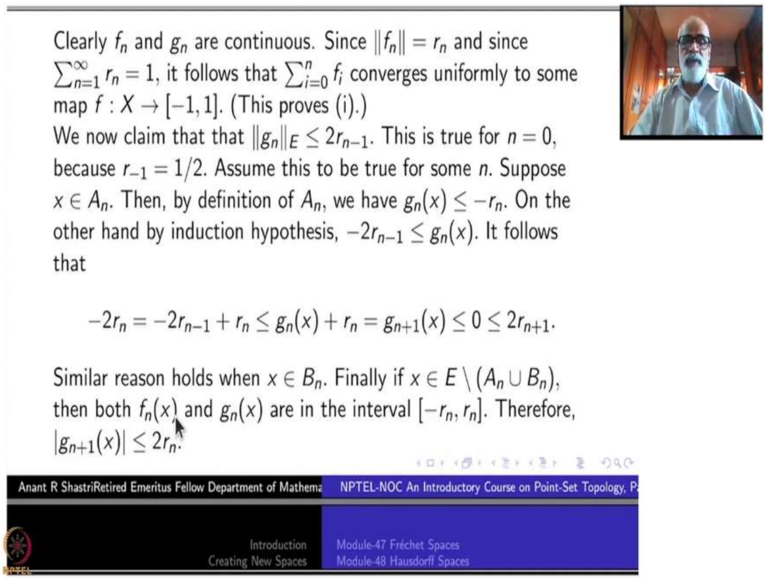
(Refer Slide Time: 09:41)



Put $r_n = 2^n/3^{n+1}$. Take g_0 equal to g. Having defined the map g_n from E to $\mathbb{R}$ for some n, inductively we are going to construct the next g_{n+1}. So, for that put A_n equal to $g_n^{-1}[-1, -r_n]$ and $B_n = g_n^{-1}[r_n, 1]$. Then clearly A_n and B_n are disjoint closed subsets of E ok, which itself is closed in X. Therefore, A_n and B_n are disjoint closed subsets of X. Hence, by Urysohn's Lemma, we get a continuous function χ_{A_n,B_n} from X to $[-1, 1]$, remember this χ_{A_n,B_n} is a continuous function which has the property that on A_n it is equal to -1 and on B_n it is equal to 1, that is all we are going to use it here ok.

After that we put $f_n = r_n \chi_{A_n,B_n}$. Finally, we define g_{n+1} as $g_n - f_n$ restricted to $E.g_n$ is defined on E already, f_n is defined on the whole of X, but we take f_n restricted to E and take g_{n+1} to be $g_n - f_n$ restricted to E on the subset E. The inductive definition of the sequences f_n and g_n is over.

(Refer Slide Time: 11:46)



Next task is to show that these things converge to whatever we wanted to. Namely, first of all g_n and f_n are both continuous. Norm of χ_{A_n,B_n} is 1. Therefore, norm of f_n is r_n. And $\sum_{n=1}^{\infty} r_n$ is actually 1. This just means that $\sum_{i=1}^{n} f_i$ converges uniformly to some one function f. Because of uniform convergence that function is automatically continuous. And it takes value between $[-1,1]$ ok? I repeat. This function f is continuous, because it is a uniform limit of sum of finitely many f_i's here, each f_i is continuous. As n tends to infinity, the convergence is uniform convergence. Therefore, f is continuous.

We now claim the second part (ii) namely, norm of g_n on E, that is, supremum $|g_n(x)|$ where x ranges over E, right? That is the definition, we want to say that this is less than or equal to $2r_{n-1}$. Again the proof will be by induction. When $n = 0, r_{-1}$ is nothing but $1/2$ ok? And $2r_{n-1}$ is 1 and g_0 is our function g, which is taking value between $[-1,1]$.

Therefore, norm of g_0 is less than equal to 1 ok. So, assume that this statement is true for some n, then we shall prove it for $n+1$ ok. So, let us examine what happens to the function g_{n+1}. Suppose x is inside A_n, by the very definition of A_n, $g_n(x)$ will be between -1 to $-r_n$. So, it is less than equal to $-r_n$.

On the other hand, the induction hypothesis says that -$2r_{n-1}$ is less than equal to $g_n(x)$, the modulus of this 1 is less than equal to $2r_{n-1}$ this is the induction hypothesis. So, if you put them together, now you subtract f_n, f_n on A_n is $-r_n$ right, so we have to add r_n.

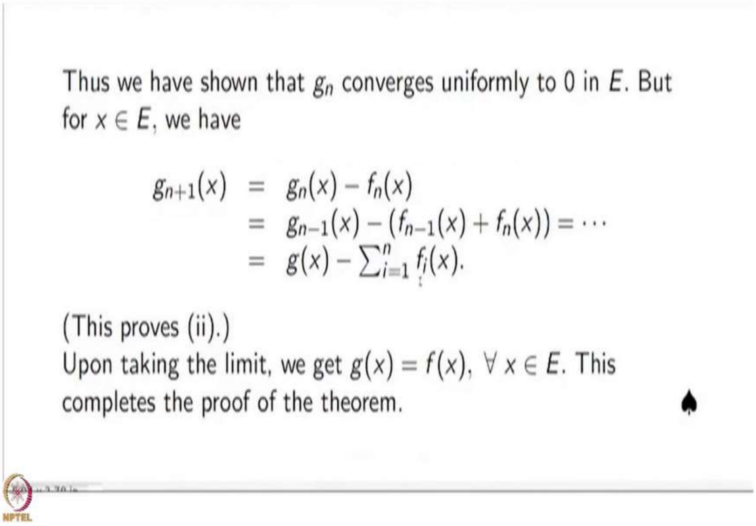
So, what I get is $-2r_n$ which you can verify is same thing as $-2r_{n-1} + r_n$, that is less than or equal to this is $g_n(x) + r_n$, but this is the same thing as $g_{n+1}(x)$ now because $+r_n$ is same thing as $-f_n$ ok. So, g_n is less than equal to $-r_n$ here $r_n \leq 0 < 2r_{n+1}$, ok.

Exactly similar reason, not exactly same reason when you take x is inside B_n this becomes now $-f_n$ becomes $-r_n$, etc, finally, you will get a similar thing namely what you get is that $-2r_n \leq g_{n+1}(x) \leq 2r_n$.

So, what remains now? Look at a point which is nor in A_n nor in B_n ok, then both A_n and g_n are in the interval $[-r_n, r_n]$ by definition, because if it is not in A_n, it is bigger than $-r_n$ not in B_n it is less than or equal to r_n ok. So, such a point automatically satisfies $g_{n+1}(x)$ ok less than or equal to r_n and bigger than or equal to $-r_n$.

So, modulus is less than or equal to $2r_n$ because both f_n and g_n are inside this one, f_n is always between $-r_n$ to r_n. So, this completes the proof of inductive claim that norm g_n on E is less than equal to $2r_{n-1}$. For the same reason, because $\sum r_n$ is convergent and the sequence is is dominated by this $2r_n$, it will follow that $g_n(x)$ converge this to 0 as n tends to infinity because these thing converge to 0 ok.

(Refer Slide Time: 17:55)



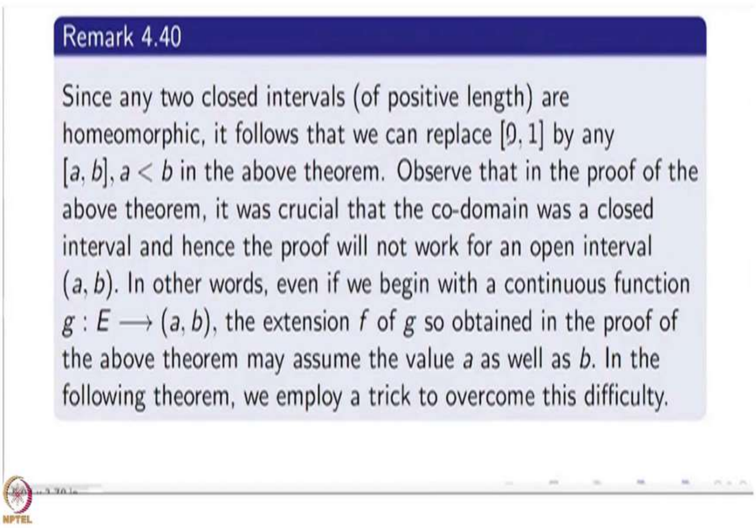
Now, what is this g then? Mysterious thing? Why these g_n's? That is clear because g_n is nothing but the remainder after n terms ok. I could not say that one because, before that I have to show that the series is convergent. So, we have shown that g_n converge uniformly to 0 inside E, but now, if x belong to E, then g_{n+1} is $g_n(x) - f_n(x)$ by definition. What is $g_n(x)$? $g_{n-1}(x) - f_{n-1}(x)$.

So, you club them together, it will become $g_{n-1}(x) - f_{n-1}(x) + f_n(x)$, but again we repeat this process to g_{n-1} to $g_{n-2} - f_{n-2}$ plus all this. So, go on doing that till you hit g_0, g_0 is g and this term will become $\sum_{i=1}^{n} f_i$. First f n, then f_{n-1} etc up to f_1 all these terms will come.

So, g_{n+1} is nothing but $g(x) - \sum_{i=1}^{n} f_i(x)$, this is the second statement we wanted to show that norm of this converges to 0, norm of this same thing at g_{n+1} of norm, norm of this one. And we have shown that this converges to 0, ok. So, you see that this is just a partial sum and this is remained after n terms that is the whole idea.

Upon taking the limit, this is 0, this will become f now. Because i ranging to 1 to infinity is our f alright. So, $g(x) - f(x)$ norm is 0 it just means that $g(x) = f(x)$. So, these are all happening for every x inside E. So, that completes the proof of Tietze's theorem.

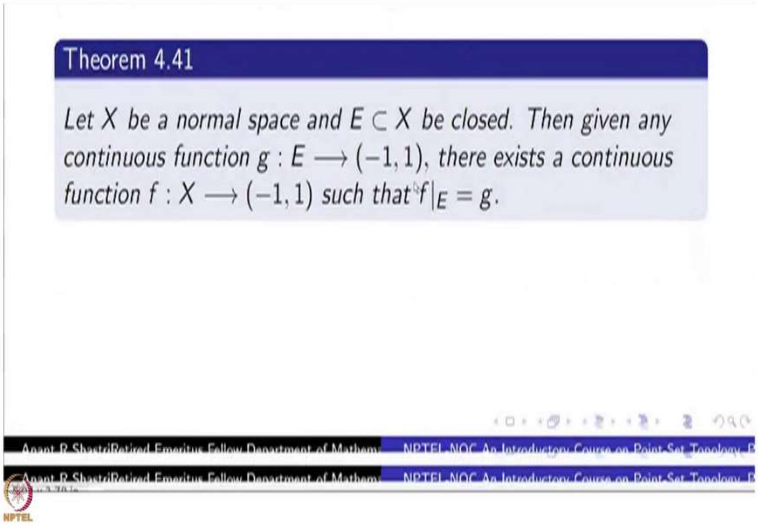
(Refer Slide Time: 20:27)



So, let me make a few comments here yeah. Since any two closed intervals, I am making this comment again, of positive length are homeomorphic to each other. We can use, instead of $[-1, 1]$ or $[0, 1]$ we can use any interval $[a, b]$ ok. But in the proof of the above theorem it was crucial that the co-domain was a closed interval, you are going to take limits of certain sequences, for each point you are taking limits. So, they are sequences of points of $\mathbb{R}$. So, the limit point could be in at end points of the interval, I mean that is possible right? So, even if you start with the function, suppose the given function g is taking values in the open interval. Extended function may not be taking values inside the open interval. It may hit a or b that is possible ok? Yes or no? So, because when you take open intervals, you know open interval is not complete. So, you cannot use all these completion results that is the whole idea ok.

But you look at the statement of the Tietze theorem, this point will not afffect it. In other words, I want to have a theorem like this. Suppose g is a function from E to an open interval. Then I can extend it from X to the open interval itself. So, that is the statement ok? That is not the part of the statement in the theorem as given above but it is extra statement we have to work out. And there is a trick which will help us to prove such a theorem.

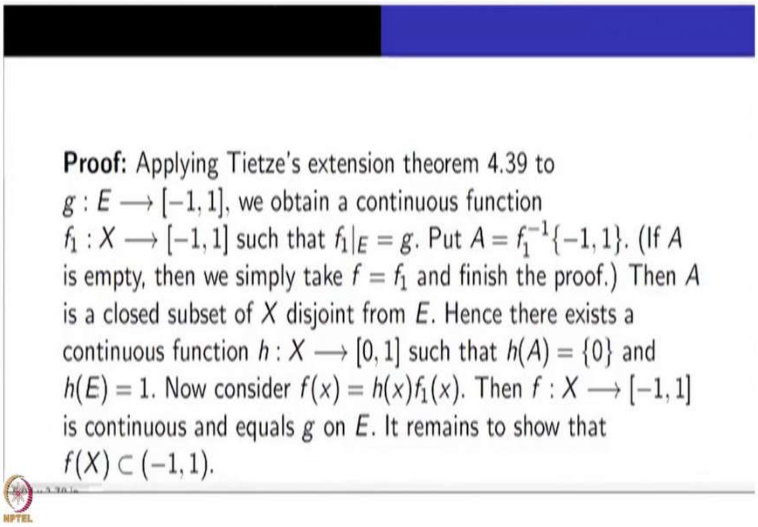
(Refer Slide Time: 22:48)



So, let us do that. So, this is the statement here. So, I have stated a separate theorem that X be a normal space, E contained in X is closed, then given any continuous function g from E to the $(-1, 1)$, there exists a continuous function f from X to $(-1, 1)$, such that f restricted to E is g ok? Directly from the theorem that we have proved this one does not follow, that is the remark I made ok?

What follows you can go to the closed interval, taking inclusion of this one into closed interval, then you will get an extension f_1 of g, from the theorem. So, that much is true. Now, how to come back to the open interval here is a trick.

(Refer Slide Time: 23:48)



Applied Tietze extension theorem 4.39 to the map g treated as a function from E to $(-1,1)$, we obtain a continuous function f_1. Let us call it f_1, because this is not going to be f from X to $(-1,1)$ such that restricted to E it is g, what may happen is the following: Look at $f_1^{-1}(-1,1)$, just the two points. Put that as A, that is a closed subset. This may be empty, if A is empty then we are happy; that means, $f(X)$ is inside $(-1,1)$ so we can take $f = f_1$. If A is not empty then we have to work harder ok?

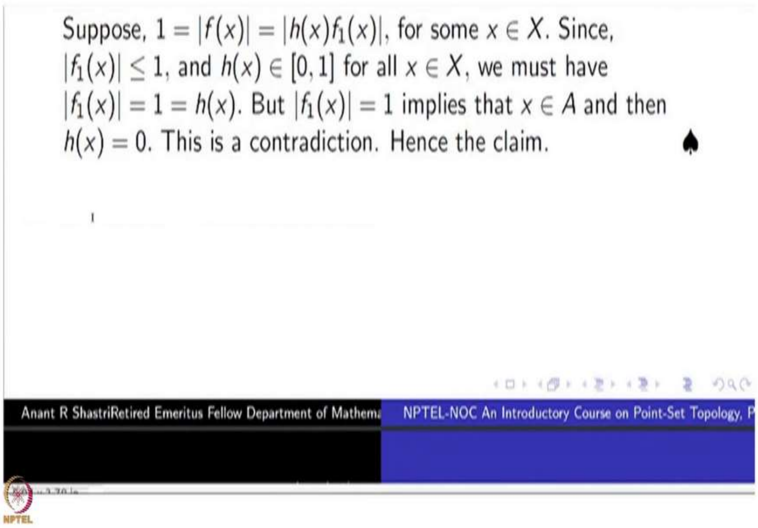
In any case A is a closed subset of X disjoint from E; why? Because to begin with we have started with a g such that g is taking values strictly inside $(-1,1)$ and f is restricted to E is g. So, this A and E are disjoint subsets and both of them are closed. Hence, you can apply Urysohn's lemma here ok? To get another function h from X to $[0,1]$, such that $h(A)$ is 0 and $h(E)$ is 1.

So, you see here I am using again $[0,1]$ as the domain instead of $[-1,1]$. That is deliberate, it is not just arbitrary ok. So, $h(A)$ is 0 and $h(E)$ is 1. So, whole idea is you do not want to disturb the E part. So, you have put this 1 here, you do not want this A part. So, you have to kill it, $h(A)$ is 0. That is the whole idea. Now, you multiply the original f_1 with h, take $f(x) = h(x)f_1(x)$. Ok?

This is a product of two functions, both of them taking values in $[-1, 1]$ right? So, this will take values in $[-1, 1]$ there is no problem and it is continuous. And when you take x inside E this $h(x)$ is 1. So, $f(x)$ is $f_1(x)$ which is $g(x)$, the original g.

Therefore, all that you have to show is that the new function f takes values inside $(-1, 1)$, right? Then the theorem is over. Why it does not take the value in $(-1, 1)$? That is what I should ensure now.

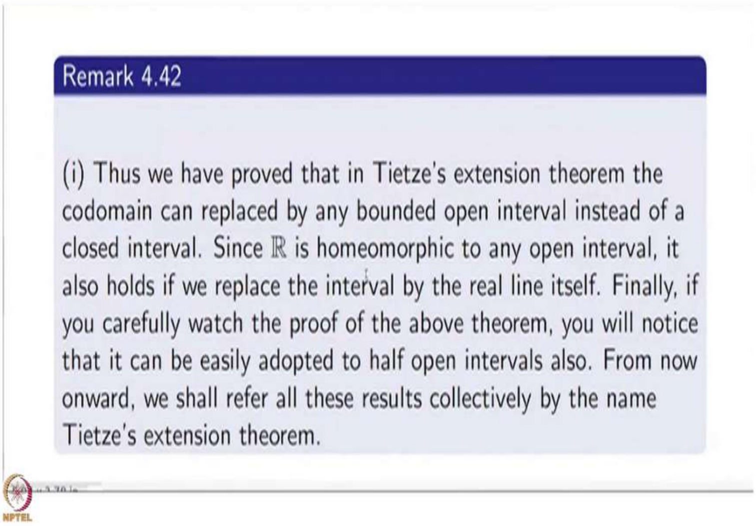
(Refer Slide Time: 27:07)



So, suppose modulus of $f(x)$ is equal to 1, but what is modulus of $f(x)$? What is $f(x)$? $f(x)$ is $h(x)f_1(x)$ ok. Suppose for some x, it is equal to 1, either it is -1 or 1 then $|f_1|(x) \leq 1$ and $h(x)$ is inside $[0, 1]$ for all x, remember that. We must have modulus of $f(x)$ actually equal to 1, even if any one of them is smaller than 1 then the modulus of the product will not be equal to 1 ok. So, both the modulus must be equal to 1, Modulus of $h(x)$ is already $h(x)$ because this is already non negative, so $h(x)$ it is actually equal to 1.

But then after multiplying by $h(x)$ what we get? 1 is equal to $|f|(x) = |f_1|(x)$. But $|f_1|(x)$ is equal to 1 implies x is inside A, remember that A was defined as $f_1^{-1}(-1)$ as well as of 1, but then h is 0 and $h(A)$ is 0. But just now we have $h(A)$ is 0 means right hand side would be 0 this cannot be 1. So, we have shown as h is also 1. So, that contradiction proves that modulus of $f(x)$ is never equal to 1, ok?

(Refer Slide Time: 28:51)



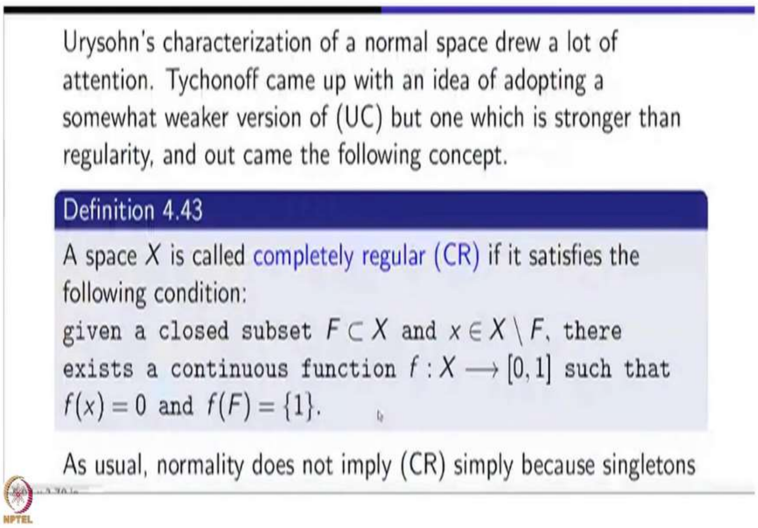
So, I have studied this is trick, this kind of tricks are used quite often in algebraic topology also. So, in summary what we have proved is Tietze extension theorem the co-domain can be replaced by any bounded open interval instead of a closed interval. But $\mathbb{R}$ is homeomorphic to any open interval. Therefore, you can take the whole of $\mathbb{R}$ also.

Finally, if you carefully watch the proof of the above theorem you will notice that it can be easily adopted to half open intervals also. If you do not want to avoid both the end points, you have to work only for one of them. So the set A which you have taken to be f inverse of minus 1 plus 1, you take it to be inverse of inverse one of the end points only. So, it will work for half open intervals also ok.

Therefore, all these statements can be collectively called Tietze's extension theorems for normal spaces. Given continuous function on a closed set, taking values in open (a, b) half open $(a, b]$ or closed $[a, b]$, whatever, you take the co-domain ok, with the same codomain there is a continuous extension of the function g to a function f on the whole of X.

So, that is the way you have to understand Tietze extensions theorem at various places, whichever form is necessary for you can use that alright.

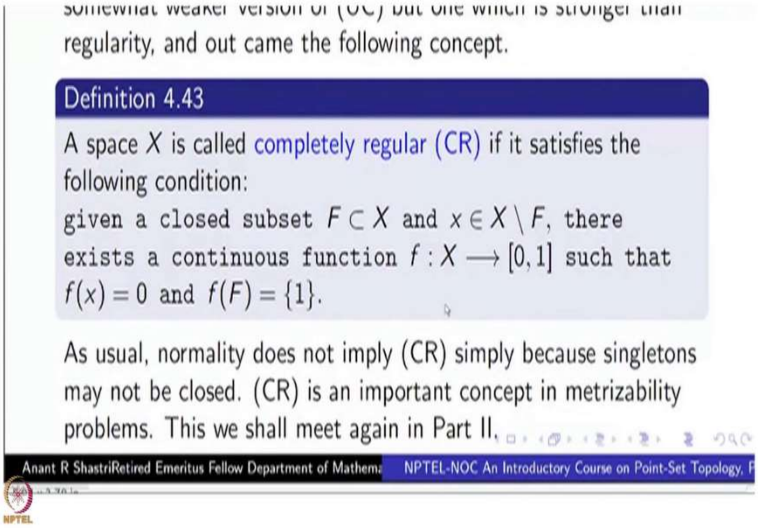
(Refer Slide Time: 31:02)



So, that is the way you have to understand Tietze extensions theorem at various places, whichever form is necessary for you can use that alright.

Urysohn's characterization of normal spaces drew a lot of attention. Tychonoff came up with an idea of adopting a somewhat weaker version of the Urysohn's characterization, but one which is stronger than regularity. So, another concept came in between, which is called complete regularity.

(Refer Slide Time: 31:32)



A space X is called completely regular, if it satisfies the following condition: Given a closed subset F of X and a point x belonging to $X \setminus F$, (remember this much was the hypothesis for a regularity, but instead of two disjoint closed subsets, one closed subset and a point outside it ok? Then instead of open sets and so on now), you have function i.e., There exists a continuous function f from X to $[0, 1]$, such that $f(x) = 0$ and $f(F)$ is 1. So, this is a perfect mixture of regularity condition and Urysohn's criteria. A mixture. Such a space is called a complete regular (CR) space. As usual normality does not imply CR, simply because singletons may not be closed.

This complete regularity is an important concept in metrizability problems. I think Urysohn may have tried to prove some immedability results through which to get a metrization result. That may be thre motivation for him to explore and come up with the result which is now known to us by his name.

Now ok, so this complete regularity is important in metrizability problems, that will be discussed and taken up in part II. So, I am just giving you a glimpse of that here, even if you forget it, it is ok. So, I think we are now convinced that why Urysohn's lemma is so important alright. So, let us ah meet next time.

Thank you.