Introduction to Point Set Topology, (Part I)
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Lecture - 52
Characterization of Normality

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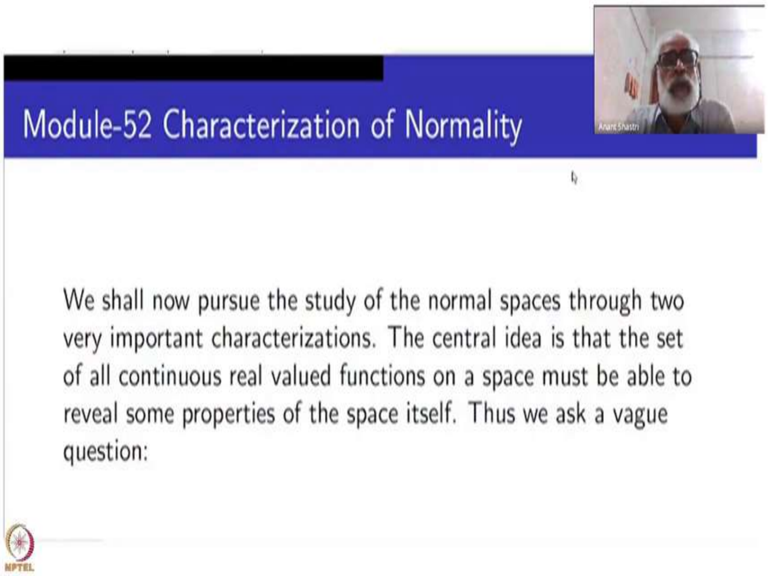

Welcome to module 52 of Point Set Topology course Part I. So, today we shall take up the study of normality again, Characterization of Normality; characterization in terms of continuous functions. In fact, we will have two different characterizations which are closely related to one another. One is due to Urysohn and then using that another one is due to Tietze.

The central idea is that the set of all continuous real valued functions on a space must be able to reveal some properties of the space itself. Often in algebraic geometry and sometimes in algebraic topology also this is the central theme. Look at the set of real valued or complex valued functions or those which have some extra properties and so on, declare them as what is known as coordinate space coordinate ring. And then the ring will dictate all the geometry and topology.

So, this is the theme they follow. We do not go into that depth here, but the idea of why we need such a thing like characterizations only for that reason I am telling you so where it is much more important than to us.

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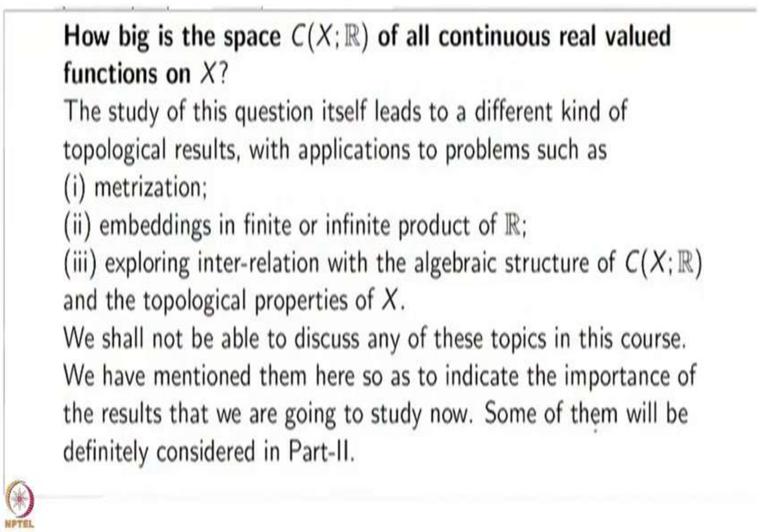

So, we can ask a vague question, how large is the set of all continuous real valued functions on a given space X. The study of this leads to a different kind of topological results altogether, with applications to problems such as I have just mentioned a few of them here--metrization problem, embedding problem. Embedding problem just means that take a space whether it can be embedded inside some Euclidean space of finite or infinite dimension.

And exploring the inter-relationship between the ring structure, the algebraic structure of $C(X, \mathbb{R})$ with the topological properties of X. So, these are the few things. For example, when you go to function theory, it is not all continuous functions they take. They will take analytic functions ok. So, that is what is important for them in complex analysis for example.

If you go to algebraic geometry they will only take polynomial functions ok. So, it depends. For us, if you want to study the entire topology you must better take all continuous functions ok. In differential topology you will take differentiable functions, smooth functions and so on. So, we shall not be able to discuss any of these topics, I mentioned three of them in this course more than what I have mentioned.

And I have mentioned it only because why the Urysohn's characterization or Tietze characterization is important, but some of these problems and applications will be taken up in Part-II of this course, ok. So, let us proceed with these characterizations.

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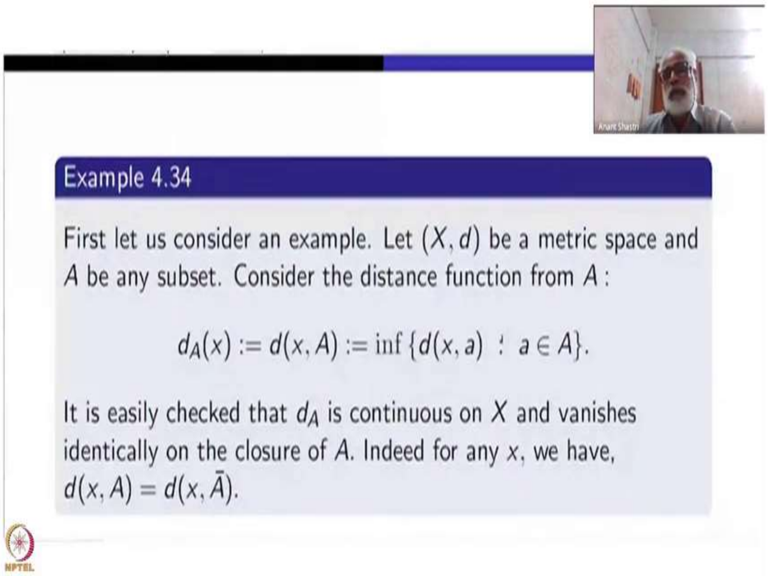

And I have mentioned it only to show why the Urysohn's characterization or Tietze characterization is important, but some of these problems and applications will be taken up in Part-II of this course, ok. So, let us proceed with these characterizations. How one might have, you know, come up with this kind of thing? I would like to know. I cannot read Urysohn's mind nor I can go back in history, go back in time an ask him. So, I am guessing that this may be the one which led him to consider such a thing.

So, again go back to metric spaces. Take any subset A. Consider the distance function from A . So, distance function is defined on $X \times X$ ok? Gere d_A or $d(x, A)$, you can write it in two different notations, which is nothing but the infimum of all the distance between x and a, where a ranges over A, x is fixed. So, that is the distance from A ok, the infimum of all these numbers.

For the point x distance between x and A is defined by this formula. It is easily check that d_A is continuous ok on the whole of X. And clearly if a is inside A, then d of you know, we can put $x = a$. So, that will be 0. So, infimum will be 0. So, it will vanish right? But it will vanish on $\bar{A}$ also. In fact, $\bar{A}$ is precisely the set wherein this function will be identically 0, ok? So, that is easy to check.

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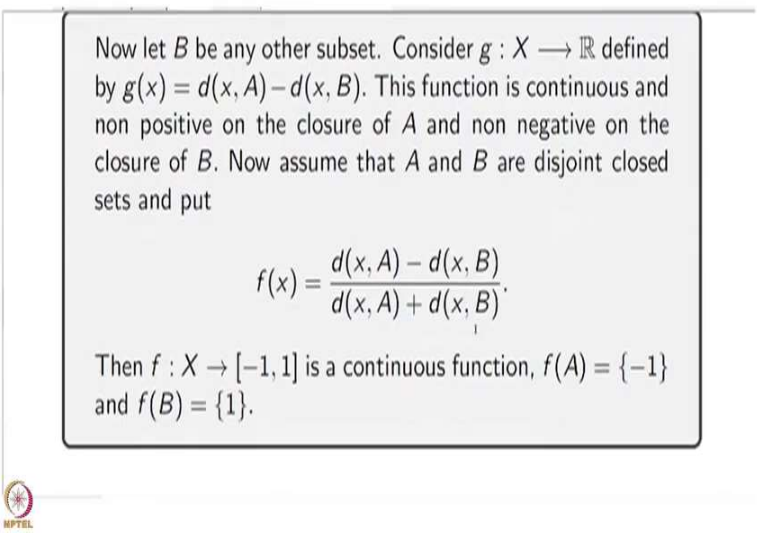

Now, let B be another set. Consider the function g from X to $\mathbb{R}$ defined by $g(x) = d(x, A) - d(x, B)$. See for A and B, we have defined $d(x, A)$ and $d(x, B)$. Take the difference. A, B nothing special here. First I took A and then I am taking B, that is why I am writing, otherwise I could have taken $d(x, B) - d(x, A)$ also as good as this one, one will be the negative of other.

This function is again continuous because it is the difference of two continuous functions and it is non-positive on the closure of A because the first term will be 0. And on the closure of B it is non-negative because the second term will be 0, ok?

Now, you take a special case when A and B are disjoint closed sets. When A and B are disjoint closed sets at least one of them must be non-zero for all the points because A and B

are disjoint. So, if $d(x, A)$ and $d(x, B)$ are both $0, x$ will be inside $A \cap B$. So, there is no intersection right? That is it.

Therefore, the sum function will never vanish $d(x, A) + d(x, B)$. What does that mean? I can divide by that function ok, sum and difference are both continuous, sum is never 0. Therefore, this quotient function is also continuous. I am calling it f, ok. So, $f(x)$ is the difference function divided by the sum function.

Why one would think of this one is itself say a moot question, but I think, since I could think about this one, so, Urysohn also must have thought about this one. How I thought about it also is a mystery. I could come up to this one after long thinking. This must be the reason.

Now check that f is from X to $[-1, 1]$, because the numerator is never bigger than the modulus of the denominator, ok. Denominator is already non-negative, actually positive. So, this is always true. The image of f will be between -1 and 1, ok?

Moreover, A and B are closed subsets, we know that if x belongs to A, $d(x, A)$ is 0. So, f is $-d(x, B)$ divided by $d(x, B)$ which is -1. Similarly, $f(B) = \{1\}$. So, you see what we have produced a continuous function from the whole of X, which is -1 on A and 1 on B, ok.

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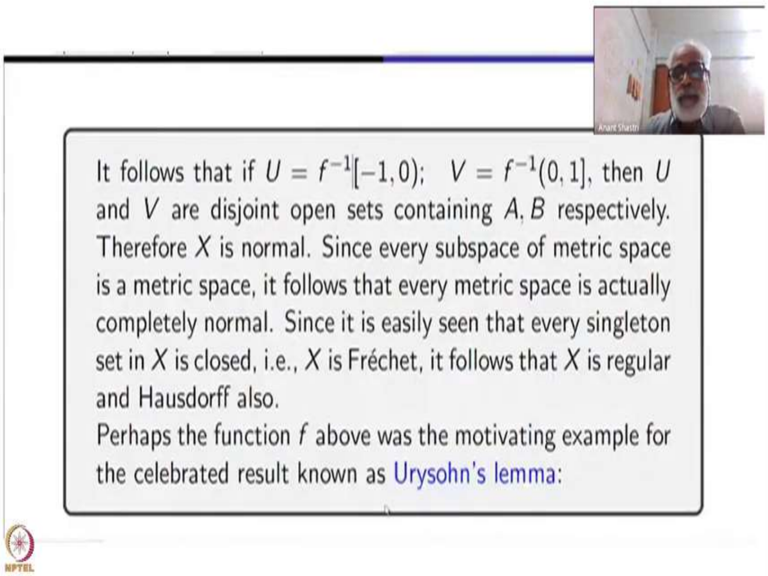

Now, you can take just a little a small neighborhood of -1 and a small neighborhood of 1. I have taken sufficiently large namely $[-1, 0)$ and here $(0, 1]$. Only thing is I have been careful that they are disjoint open subsets in $[-1, 1]$. Take the inverse image. Call them U and V. They will be disjoint open subsets of X, ok? Clearly they contain A and B respectively ok.

So, suddenly what we have proved is that disjoint closed subspaces of a metric space can be separated by open subsets. Actually they can be separated by continuous functions. So, in particular, every metric space X is normal. Indeed since every metric space is normal and subspace of a metric space is also metric space, it follows that every metric space is completely normal ok? It is completely normal.

Also we have seen that singleton sets are closed in a metric space. That means, they are Frechet spaces. Therefore, complete normality implies regularity as well as Hausdorff-ness also ok. Once they are Frechet, regular implies Hausdorff. Complete normality implies regularity, complete normality implies normality anyway, ok?

So, all these things are true for a metric space. So, why I am guessing is that perhaps this function f was the motivating example for the celebrated result known as Urysohn's lemma. Of course, it is due to Urysohn. There is no mistake in that.

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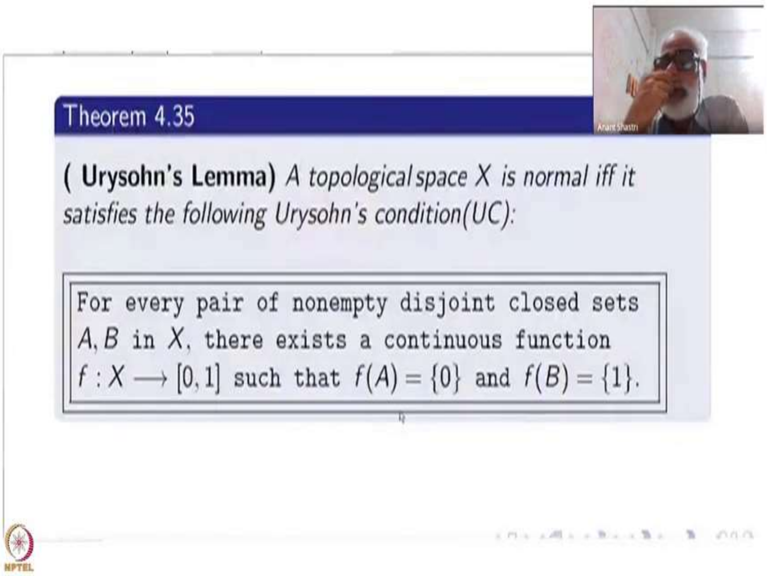

So, this is the lemma. A topological space X is normal if and if it satisfies the following Urysohn's condition UC. So, again and again I will be requiring this condition. So, I will named it UC; Urysohn's Condition. What is this? For every pair of nonempty disjoint closed sets A, B in X, there exists a continuous function f from X to $[0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

See $f(A) = \{0\}$ just means that A has to be nonempty, similarly B has to be nonempty. For that reason we have to assume that they are nonempty disjoint closed sets ok? Otherwise, you know, definition of normality, you could take, in whatever definition you take three of them, you could take A or B empty as well ok. It does not cause any problem there.

But if you want to have a function theoretic characterization here then you have to take A and B are nonempty. Then only you can write $f(A) = \{0\}$ and $f(B) = \{1\}$. You can also do -1 and 1 here by changing the interval, the codomain interval, by a homeomorphism ok. So, that is not so crucial. Getting a closed interval as a co-domain ok? Getting a function f from this one wherein the two sets are going to two distinct points that is the crux of the matter.

It could be that $f(A)$ is equal to some $\{a\}$ and $f(B)$ equal to some $\{b\}$, where a and b are distinct real numbers. So, that is a crucial matter here alright?

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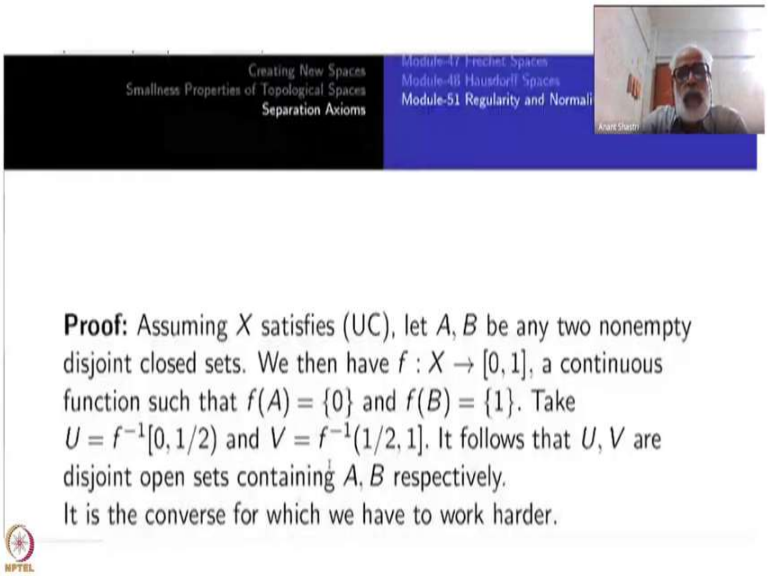

So, one way is obvious which we have seen already in some sense. Assume that UC is satisfied by X. Take A and B any two nonempty disjoint closed sets and take a function f from X to $[0,1]$ continuous function. So, say $f(A)=\{0\}$ and $f(B)=\{1\}$. Then take $[0,1/2)$ and $(1/2,1]$. Take the inverse images, call them U and V. They will contain A and B respectively. So, that is the condition for Urysohn's the normality.

If one of them is empty you can always take that. Suppose A is empty then you can take U empty and B equal to whole of X. So, that is obviously, satisfied. There is no need to worry about that. So, normality is satisfied if UC is satisfied. The converse is where we have to work harder. Well, real hard work is done by Urysohn. We are doing hard work in a different sense we have to learn them properly right, ok.

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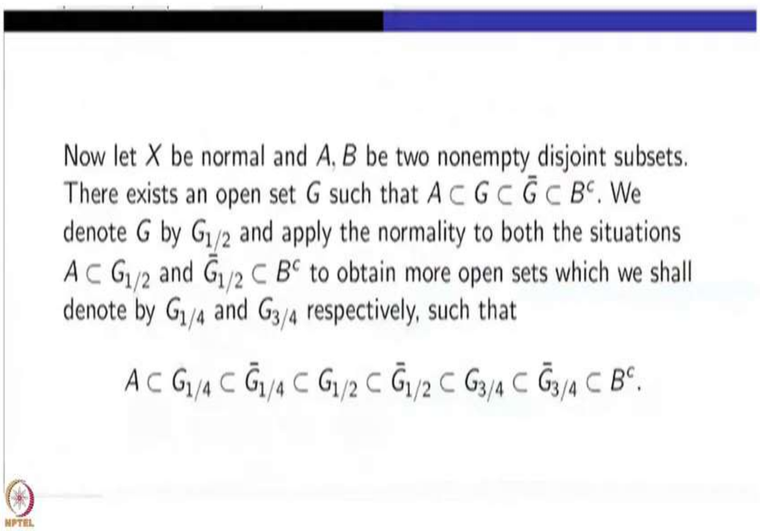

Now, let X be normal and A and B be nonempty disjoint subsets. The second condition in the theorem, tells you something: take A and B^c. A is contained in B^c. A is closed, B^c is open. There will be an open subset G, such that A is contained in G contained in $\bar{G}$ contained in B^c. So, I am using the second condition in the definition of normality.

So, we have started a process here. There is going to be an iteration of this. So, I am denoting the first iteration, this G by $G_{1/2}$ ok. Use of this notation will be clear in a moment. So, you have to wait. Right now it will be better if you write it as G_1 ok. So, but now what I want to do? That is what I have to tell you.

Apply the normality to both the situations. So, this is almost like you know this is one point and there is another point and I have chosen the half of the middle of them. If you have chosen middle from the first point to middle and the middle point to the second point again you can choose middle of them, so, that is the kind of thing that is going on here.

But middle does not make sense here, something in between makes sense, with the relation of inclusion of sets. That is what is being done. So, what I want to do is now between A and $G_{1/2}$ and $\overline{G_{1/2}}$ and B^c ok, I introduce two more open subsets which we shall denote by $G_{1/4}$ and $G_{3/4}$ respectively such that...

From A to $G_{1/2}$; $G_{1/2}$ is open, A is closed. $G_{1/4}$ will sit there contained in $\overline{G_{1/4}}$, contained in $G_{1/2}$. $G_{1/2}$ is contained inside $\overline{G_{1/2}}$ that is already there. We are using that one now. This $\overline{G_{1/2}}$ is closed and B^c once again open. We will have one more open subset in between there, viz., $G_{3/4}$ contained inside $\overline{G_{3/4}}$ contained in B^c.

See what I have done?

From $1/2$, half of that is $1/4$. Between $1/2$ and 1, I have $3/4$. So, keep on cutting down by half. What are those numbers? Next time you will get $1/8, 2/8, 3/8, 4/8, 5/8, 6/8, 7/8$ right. So, those are numbers. So, what are they called?

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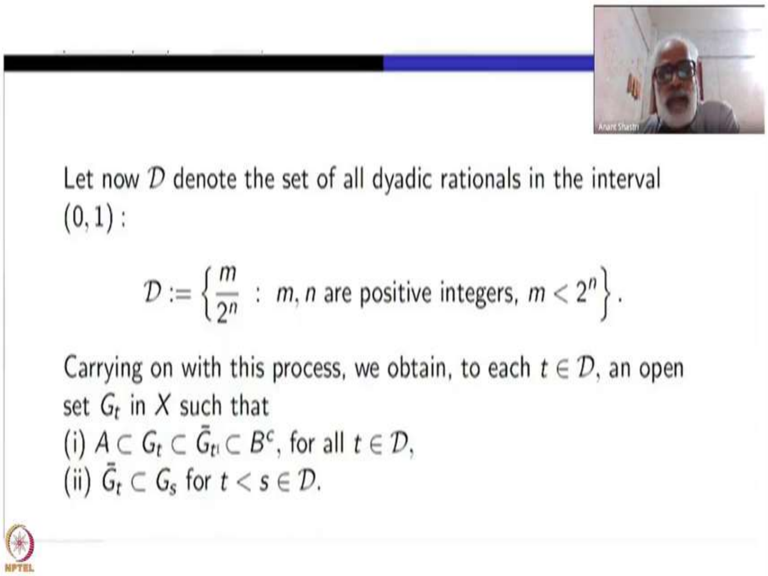

They are called dyadic rational numbers. Rational numbers whose denominator is always a power of 2. But I want only all of them between 0 and 1. I am including 0 and 1. 0 is taking the place of A ok? and 1 is taking the place of B^c. I do not need to write uh any other symbols for them. In between I am going to put all these open sets such that they are of course contained in their own closure and then the closure is contained in the next one and so on.

So, that is what I am going to do. Next step will be an open subset between A and $G_{1/2}$ that is $G_{1/4}$ and one on this side also in everywhere in between you know. $\overline{G_{3/4}}$ is closed and B^c is open. So, between them I need to put one more one more open set and so on. So, go on squeezing open subsets in between ok?

So, let $\mathcal{D}$ denote the set of all dyadic rationals in the interval $[0, 1]$. Namely, integer m divided by 2^n, m and n are positive integers ok? m less than 2^n, you can take m to be only odd numbers if you like. If there is some power you can cancel out, but I want m to be less than 2^n, ok. Carrying on with this process we obtain for each number here an open subset G_t ok.

So, all these new open subsets of X are indexed by this set. So, what is the property? All of them are neighborhoods of $A.A$ is contained inside G_t. G_t is contained inside $\bar{G}_t$. $\bar{G}_t$ is always contained inside B^c. So, this much is obvious, but more than that between G_t and G_s, what is the relation?

As soon as t is smaller than s, $\bar{G}_t$ will be contained inside G_s. So, this is the property ok? For example, it does not depend upon whether I have chosen them first time or second or third time and so on, it depends upon whether the corresponding indices are bigger or smaller. For example, the first one that I have chosen is $G_{1/2}$, in the second stage I chose $G_{3/4}$. So, $\overline{G_{1/2}}$ is contained inside $G_{3/4}$, ok.

So, next I will be choosing you know $5/8$ or $7/8$ and so on. So, you have to compare the indexing numbers first, that can done easily. The corresponding sets also must be compared in a stronger way. Namely, the closure of the smaller indexed one must be contained inside the other open set. So, $\bar{G}_t$ is contained inside G_s, ok? All that we have done is repetition of the normality condition inductively, ok.

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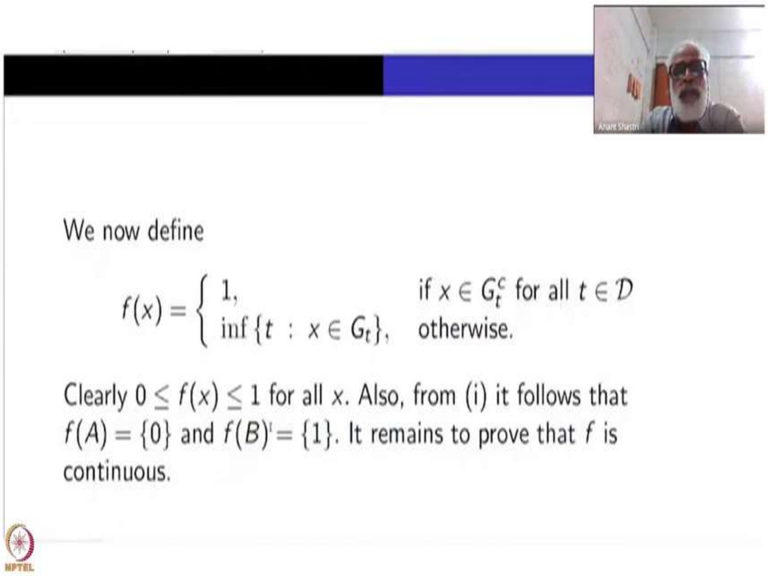


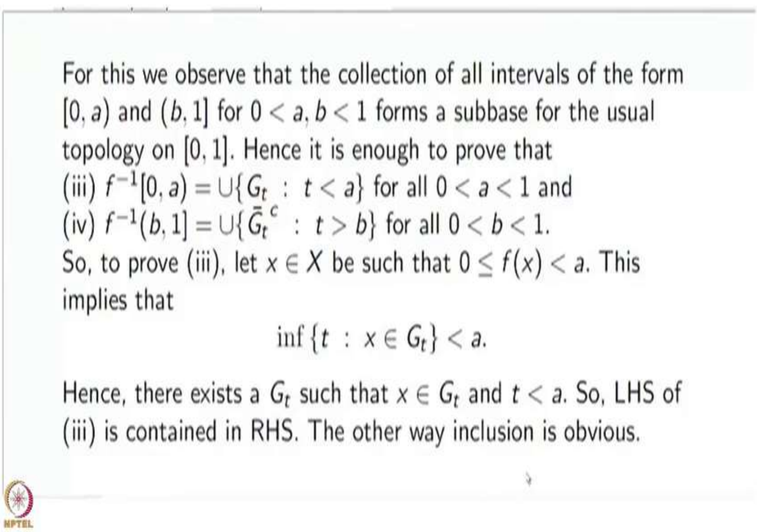
Now, we have produced a continuous function out of nowhere. The crux of the matter is the dyadic rationals are dense inside $\mathbb{R}$ ok. So, once you have some way of nominating these indices, one can cook up functions out of that that is the idea. So, define $f(x)$ equal to 1 ok, if x is in G_t^c for all t. It does not belong to any of G_t's, ok?

See all G_t's are insides what? $\overline{G_t}$ itself is inside B^c. So, if x is in G_t^c, then I am defining $f(x)$ to be 1 ok. Otherwise, I will define it as infimum of all t such that x is inside G_t. See at least this set is nonempty. Therefore, infimum makes sense, ok.

Anyway it is bounded below, all these numbers are bigger than or equal to 0, actually bigger than 0. So, it is bounded below by 0. So, infimum is a finite number. If this set is empty, then you would have a problem. So, whenever a set is empty, define $f(x)$ to be 1. That is the idea. Take the infimum of all t such that x is inside G_t, ok. Since all the G_t all t's are between 0 and 1, infimum whatever this set is it has to be between 0 and 1 and this part is 1. So, 0 is less than equal to $f(x)$ less than equal to 1, this is obvious from this definition.

Now, the first one says what? The closures of all G_t's are contained inside B^c. So, from this, it follows that $f(A)$ will be 0. The infimum of all the dyadics inside $[0,1]$ is 0. And $f(B)$ is 1 because if x in B then x is not in any of the G_t's. So, it is in the complement of G_t for all of them.

So, the set theoretic properties of f are already done very easily. The missing point so far is the important one viz., that f is continuous ok? Producing continuous functions, you know very few such results are there. Often proving some continuous function is a homeomorphism is difficult too but easier. Here producing continuous function itself. So, this is something fantastic that Urysohn has done.(Refer Slide Time: 27:42)



So, first we observe that the collection of all intervals of the form $[0, a)$ and $(b, 1]$ for $0 < a, b < 1$ forms a subbase for the usual topology on the closed interval $[0, 1]$. The usual topology on $\mathbb{R}$ and then I am taking restriction to $[0, 1]$. So, I have to restrict the members of the subbase also. So, I do not want to take anything other intervals which are sub intervals of $[0, 1]$. You have to take open there, but these are half closed intervals. So, $[0, a)$ this will be also an open subset; that is the difference. Similarly, $(b, 1]$ these are also open subsets of the closed interval $[0, 1]$. Suppose you take $(-\epsilon, a)$ and intersect it with $[0, 1]$; what you get? You will get $[0, a)$ only right? So, this is what it is. Hence, once this is a subbase, to prove the continuity of f, it is enough to prove that inverse of $[0, a)$ is open, and similarly, $f^{-1}(b, 1]$ is also open, where a and b are arbitrary points of $(0, 1)$.

For that I am explicitly proving that $f^{-1}[0, a)$ is union of all the G_t's such that $t < a$, ok? Each G_t remember it is an open subset. So, union is also open. For all $0 < a < 1$, I am going to prove this. Similarly, $f^{-1}(b, 1]$, I am going to prove, is union of all $\overline{G_t}^c$. You see closure is closed, the complement is open. Again this is an open subset, but now this time $t > b$, ok. So, if I prove (iii) and (iv), the proof of the theorem will be complete alright.

Let us prove (iii). Take a point x inside X such that it is on the left hand side. What is the meaning of that? $f(x)$ is strictly less than a. Of course, it is always bigger than equal to 0, ok. This implies, remember what is the definition of $f(x)$? Infimum of all t such that x is inside G_t. This $f(x)$ is less than a, ok? What is the meaning of infimum is less than some number?

There must be something here which is less than that, that is there exists G_t such that x is in G_t and this $t < a$, because infimum is taken over all such t. So, I have not done anything other than appealing to just the definition of infimum ok, not very serious also. So, LHS of (iii) is contained inside RHS. For each point x, I have found a G_t here, as needed. alright?

The other way inclusion is obvious. As soon as x belongs to G_t, $t < a$, infimum will be smaller than that t ok? And t is less than a. So, it will be here; $f(x)$ will be less than a, right. Even if one t is such that x is in G_t and that$t < a$, it will yield that infimum will be smaller than a. So, $f(x)$ will be less than a. So, it is here. So, this way containment is obvious alright.

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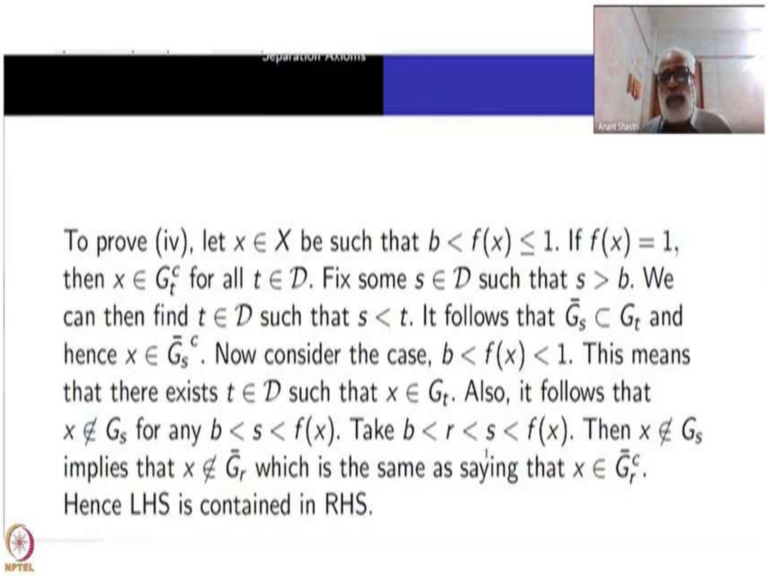

To prove (iv), let us take x to be such that $b < f(x) \leq 1$. If $f(x) = 1$, then remember when it is 1? It is inside G_t^c for all t belonging to $\mathcal{D}$. No G_t will contain x, it is in G_t^c. So, fix some s belonging $\mathcal{D}$ such that such that $s \geq b$; between b and 1. There must be some s. What is s? s is an element of $\mathcal{D}$. So, this is where I am using the fact that the dyadic rationals are dense. So, we can find a t belonging to $\mathcal{D}$ such that this $s < t$. See I have chosen $b < s < 1$. So, between s and 1, I can do another t. So, that is also again by density of $\mathcal{D}$.

So, I can choose actually $b < s < t$ and $t \leq 1$. It follows that $\overline{G_s}$ is contained inside G_t. This was the property (ii), right? Since x is inside G_t^c ok, and G_t^c is contained in $\overline{G_s}^c$. So, one case is over, namely, if $f(x) = 1$, x belongs to one of $\overline{G_s}^c$. Remember what I have to show. I have to show that every point on the left hand side namely point x such that $b < f(x) \leq 1$ is in the union of all $\overline{G_t}$ and so, I have found one such $\overline{G_s}^c$.

Now, it may happen that $b < f(x) < 1$. When $f(x)$ is strictly less than 1, the definition of f uses the second condition. Namely, $f(x)$ is infimum of all t such that x is inside G_t. This means that there is a t such that x is inside G_t, first of all right? And then you have to take that $f(x)$ that $f(x)$ is bigger than b, but less than 1 is, what I assume.

Also it follows that x is not in G_s for any $b < s < f(x)$, because $f(x)$ is the infimum of all such s. So, if s is smaller than $f(x)$, x cannot be in G_s because $f(x)$ is the infimum ok. So, now, you take $b < r < s < f(x)$, where $r, s \in \mathcal{D}$. Again I am using density of $\mathcal{D}$ here ok?

Then x cannot be in G_s implies that x cannot be in $\overline{G_r}$ because all the closure of corresponding things are contained inside G_s ok, which is the same thing as saying that x is in $\overline{G_r}^c$. So, we have found another element which contains the point. Hence LHS is contained inside RHS.

So, you see the proof of (iv) required us to use the density of $\mathcal{D}$ at least 3 times here.

So, proof of (iii) was easier. The proof of (iv) took some time right?

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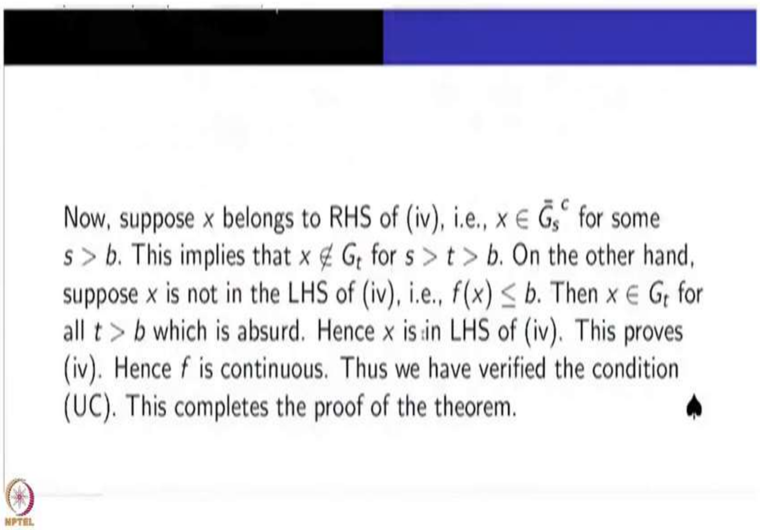

We have not yet completed the proof. What we have to do? RHS is contained inside LHS right? Even that is also not obvious. In (iii) it was very easy. In this part little more you have to say. Suppose $f(x)$ belongs to RHS. RHS is what? Union of all $\overline{G_s}^c$.

So, x is in one of them for some s which is bigger than b; that is the definition of the right hand side. This implies that x cannot be in G_t for any t where $b < t < s$. So, between b and s

if you take another t here, x will not be inside G_t because x is not in $\overline{G_s}$ itself. x is inside $\overline{G_s}^c$, ok.

On the other hand, suppose x is not in the LHS of (iv) ok. I took it is in RHS, I want to show that it is in the LHS. LHS means what? b is smaller than $f(x)$, $f(x)$ is of course, smaller than equal to 1. So, that is the meaning. So, suppose on the other hand x is not in LHS then $f(x)$ must be less than or equal to b.

$f(x)$ is less than equal to b means what? Infimum is less than equal to b ok? Then x must be in G_t for all $t > b$ because the infimum is less than equal to b right. Once x is already inside G_t, r bigger than that one it will be definitely inside G_r also. Because G_t is contained inside G_s for all $t < r$.

So, but that is absurd because we have just shown here that x is not in G_t as soon as t is between s and b, but here it says that for all $t > b$ it should happen. So, that is absurd. So, RHS will be contained inside LHS. You start with a point in RHS, it is contain inside LHS. This is what it is. So, this proves (iv) and hence the continuity of f. Thus the completion of the proof is done, ok.

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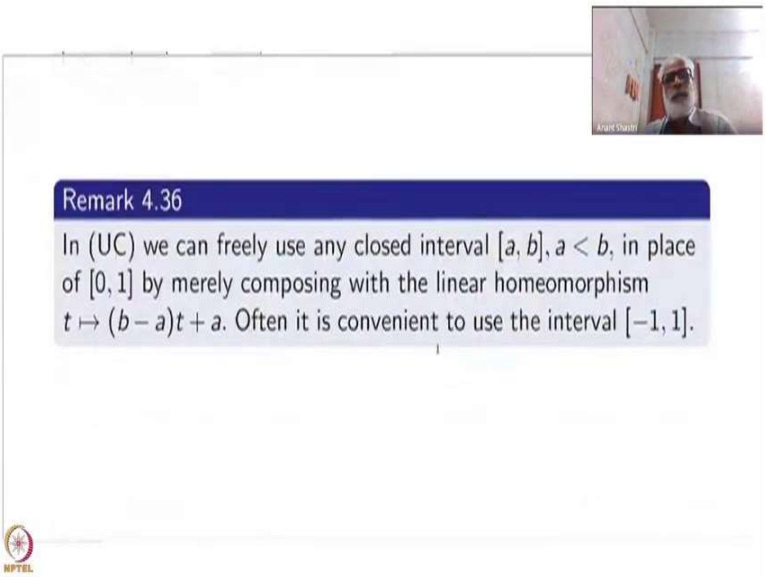

So, I made some remarks here. In UC, I have already done this one. I will repeat it. You can freely use any $[a,b]; a<b$, in place of $[0,1]$ by merely composing with the linear homeomorphism t going to $(b-a)t+a$. Often it is convenient to use the interval $[-1,1]$ instead of $[0,1]$.

Like the metric that we considered you know $d(x,A)-d(x,B)$ divided by $d(x,A)+d(x,B)$; it was between minus $[-1,1]$. So, that is what we are going to do next time, but sometimes you may have to choose some other intervals also. So, any closed interval you can take, no problem, ok.

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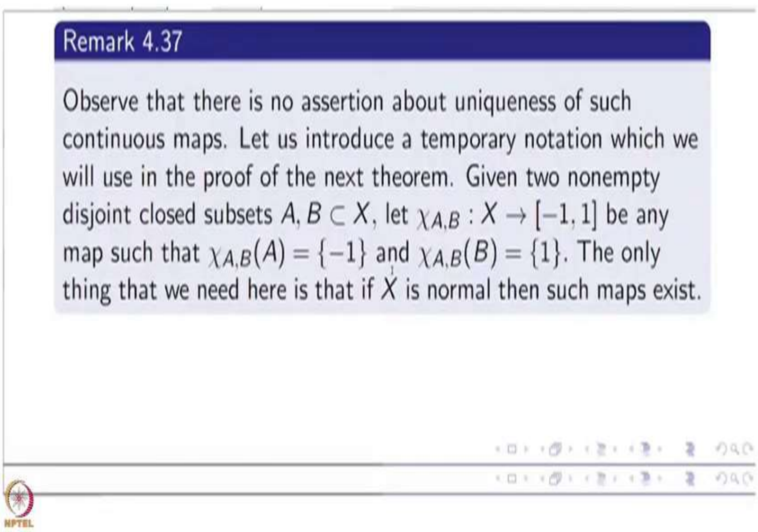

That is the comment here. There is no assertion about the uniqueness of the continuous maps. In the UC there exists some continuous function. There may be plenty of them, indeed, there are lots and lots of function. The difficulty was showing that there exist one, after that you can cook up many many of them ok.

So, one of the special function you know very specific one, I am going to use that. So, I am going to introduce a notation here ok, a temporary notation. Like in measure theory, you have this the characteristic function. Let us introduce a temporary notation, which we will use in the proof of next theorem that we are doing.

Given any two disjoint closed subset A and B of X, let us denote it by $\chi_{A,B}$, (so, it depends upon both A and B) a continuous function from X to $[-1,1]$ such that on A it is -1 and on B it is 1. There are many of them ok. Take any one of them, just call it $\chi_{A,B}$. Depending upon the context. All that I need is that it has this property and it is defined on the whole of X and its continuous. That is why I am just writing this $\chi_{A,B}$. The only thing that we need here is that if X is normal then such functions exist ok. So, choose any one of them and temporarily write it as $\chi_{A,B}$. So, this is what I am going to do next time.

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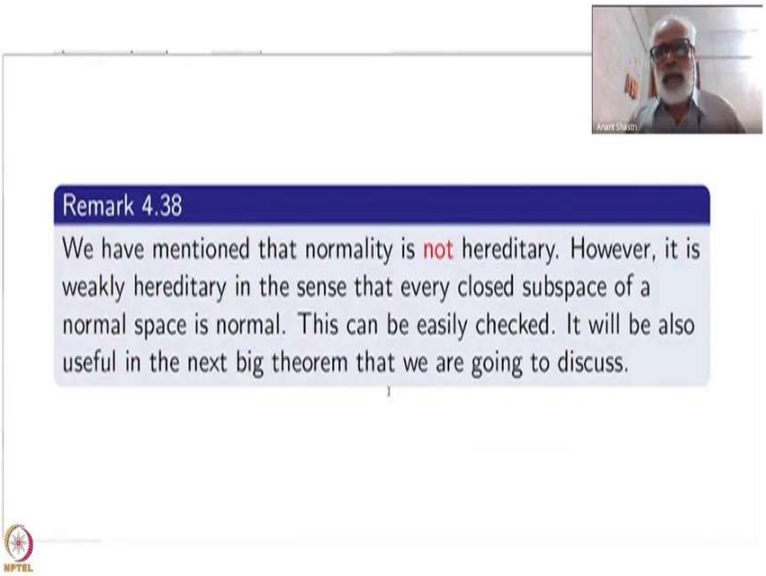

So, this comment also I have made, but I will repeat it. We have mentioned that normality is not hereditary. We have not proved it yet. However, it is weakly hereditary. Namely, every closed subspace of a normal space is normal. That is very easy to prove because closed subspace of a close sub space is closed.

So, start with a start with Y as a closed subset of X. If A and B are closed subsets of Y, then they will be closed in X itself. Therefore, normality of X will produce open subsets in X, which contain A and B disjoint subset. Now, we intersect them with Y, ok? So, Y will be normal. So, closed subspace of a normal space is normal ok? And that information is important and we are going to use that also in the next theorem, ok.

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So, let us stop here. For the next theorem tomorrow.

Thank you.