Introduction to Point Set Topology, (Part I)
Prof. Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture - 51
Regularity and Normality

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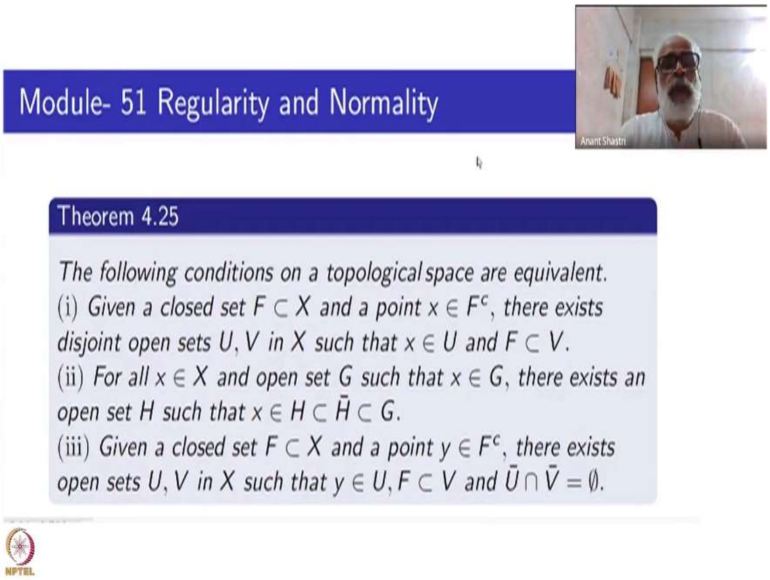

Welcome to module 51 of Point Set Topology, Part 1. So, as promised last time let us now start the study of regular spaces and normal spaces. As before in the case of Frechet and Hausdorff, I will first state a theorem which gives you a number of equivalent conditions and then make the definition that a space which satisfies any one of the conditions will be called regular or normal and so on that is the general plan.

So, this theorem says that the following conditions on topological spaces are equivalent. What are the 3 conditions? Given a closed set F and a point x away from F, in the complement, there exists disjoint open sets U, V in X such that x is inside U and F is contained inside V. Just like in a Hausdorff space any two points are separated, here a point and a closed set are separated by open sets.

So, you can see that this is a one step generalization of regularity ok?

Next, for all x belonging to X and open set G such that x belongs to G, there exists an open set H such that x is in H which is contained inside $\bar{H}$ and is contained inside G. So, in other words, you can say that every neighborhood of a point contains a closed neighborhood.

Remember a neighborhood should be such that there is an open subset contained inside that one containing the point ok. So, this $\bar{H}$ will be a closed neighborhood G was an arbitrary neighborhood, it need not be open actually. Here I have taken an open set G. That is enough because once it is a neighborhood, this G can be replaced by a subset, which is open that is all.

The third condition is: given a closed set F inside X and a point y in outside of F(similar to the conditions in (i), there exist open sets U and V in X such that y is in U and F is inside V; all this is same as number (i), but the last part is here it is only disjoint there, and here $\bar{U} \cap \bar{V}$ is empty.

Look at the third condition is much stronger apparently, but the claim of the theorem is that all the three are equivalent to each other.

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Proof: (i) $\Longrightarrow$ (ii) Take $F = G^c$ and apply (i) to get open sets U and V. Take $H = U$. Since $F \subset V$, V open and $U \cap V = \emptyset$, it follows that $F \cap \bar{U} = \emptyset$. This implies that $\bar{H} \subset F^c = G$.
(ii) $\Longrightarrow$(iii) Taking $G = F^c$ we get $y \in H \subset \bar{H} \subset G$ where H is an open set. Applying (ii) for each $x \in \bar{H}^c$, we obtain an open set H_x such that $x \in H_x \subset \bar{H}_x \subset \bar{H}^c$. Put $V = \cup\{H_x \ : \ x \in F\}$. Then V is an open and contains F. Also, $H_x \cap H = \emptyset$ for each x and hence $V \cap H = \emptyset$. Hence, $y \in \bar{V}^c$. Applying (ii) once again, we get an open set U such that $y \in U \subset \bar{U} \subset \bar{V}^c$. This means that $\bar{U} \cap \bar{V} = \emptyset$.
(iii) $\Longrightarrow$ (i) is obvious. ♠

So, let us prove (i) implies (ii): All that I do is take F equal to G^c ok. G was open G^c is closed x is in G. So, x is in now F^c. Now apply (i) to get open sets U and V such that x is inside U and F is inside V and U and V are disjoint.

Now, take $H = U$ you want to get something here since F is inside V, V open $U \cap V$ is empty, it follows that $F \cap \bar{U}$ is empty ok, actually $F \cap \bar{U}$ is empty whatever this implies $\bar{H}$ is contained inside F^c and F^c is G. So, this means $F \cap \bar{H}$ is empty ok.

(ii) implies (iii): Take $G = F^c$, we get y in the inside H, containing $\bar{H}$ contained inside G, by applying property (ii), where H is an open set ok. Now applying (ii) for x belonging to $\bar{H}^c$, H bar is closed and so, $\bar{H}^c$ is open right? And x is inside that. So, what we get? We obtain an open set H_x such that x belongs to H_x contained in $\bar{H}_x$ contained in $\bar{H}^c$ ok? These H_x are different from H.

Now, take V equal to union of all these H_x's, where x belongs to F. That V is an open set containing F. Clearly $H_x \cap H$ is empty for all x and so, $V \cap H$ will be empty. So, $V \cap H$ is empty, hence y is inside $\bar{V}$, the closure of V. Applying (ii) again we get an open set U such that y is in U contained inside $\bar{U}$ contained inside $\bar{V}$.

Whenever you have an open set ok then y belongs to U contained inside $\bar{U}$, there is such a thing. So, $\bar{V}$ is closed complement will be open. So, you have to apply (ii) at least three times here. For the last one, for each point here you have applied (ii) here once you have applied this one here also ok this means that $\bar{U} \cap \bar{V}$ is empty.

So, the so called stronger condition is obtained ok. And (iii) implies (i) is obvious.

So, condition (i) is taken as the definition usually, though (ii) and (iii) are equivalent you can use whichever one you like. Because (i) is the easiest to verify when you want to test whether a given topological space is regular or not you want to test the simplest thing, the easiest thing.

So, for that matter I will also take (i) as a definition namely given a closed set and a point outside it, there are disjoint open subsets containing them: x and F respectively, which is similar to Hausdorffness, only thing is instead of y, I have a closed set F ok? There is no condition on x, x could be any point not in F, that is all ok?

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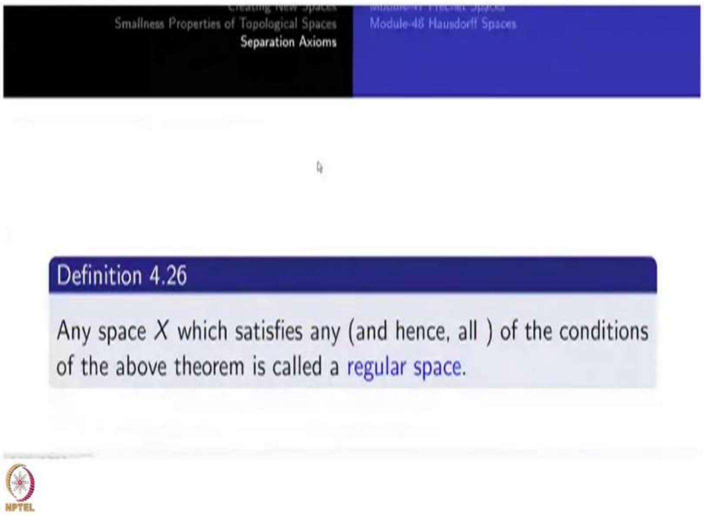

Any space which satisfies any of the three conditions above and hence, all the conditions above in this theorem that will be called a regular space alright?

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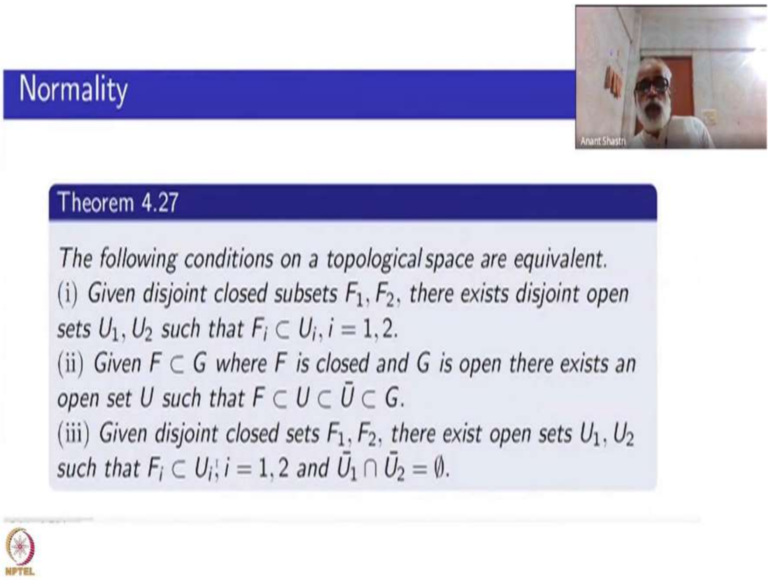

Now, I will introduce the normality. The following conditions on a topological space are equivalent. Again there are three of them almost parallel to (i), (ii), (iii) of the previous theorem, but the difference is now that given disjoint closed subsets F_1 and F_2, there exist disjoint open sets U_1, U_2 such that F_1 is contained inside U_1 and F_2 is contained inside U_2.

So, you know immediately the difference: instead of a point at a closed set I have taken two different closed subsets disjoint closed subsets. So, that is the difference ok? Then (ii) and (iii) imitate, just like conditions (ii) and (iii) of the previous theorem. Given F inside G where F is closed and G is open, there exists an open set U in between them such that F is contained inside U contained inside $\bar{U}$ contained inside G, ok? So, this is the condition (ii).

Given disjoint closed sets F_1 and F_2, there exists open sets U_1 and U_2, such that F_i's are contained inside U_i's and $\bar{U_1} \cap \bar{U_2}$ is empty. So, this is similar to property (iii) in the previous definition. Again the proofs are also similar. So, let us go through the proofs again that these three conditions are equivalent.

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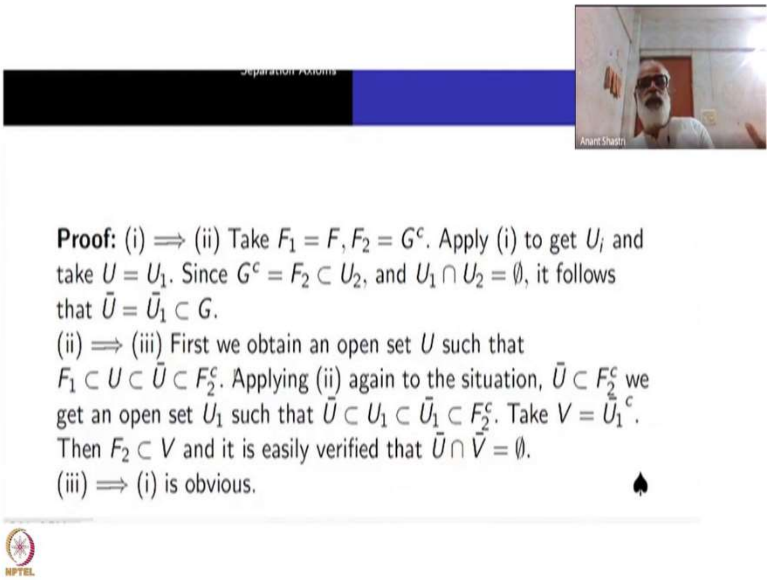

So, how to get (i) implies (ii)? Take F as F_1 and F_2 as G^c. You are given F closed and G open are given, F is contained inside G, right? Take $F_2 = G^c$ that will be a closed subset disjoint from F. Apply (i) to get U_1 and U_2, ok? Then take U equal to U_1. Now G^c, which is the complement of G is F_2 and it is contained inside U_2. This implies $U_1 \cap U_2$ is empty. Therefore, $\bar{U_1}$ which is $\bar{U}$ is contained inside G, ok? So, that is (i) implies (ii).

Similarly (ii) implies (iii), this proof is similar. First obtain an open set U such that F_1 is inside U contained inside $\bar{U}$ contained inside F_2^c. F_2^c is open and F_1, F_2 disjoint means F_1 is contained inside F_2^c.

So, in between I can put an open set U, lets say, such that U is contained $\bar{U}$ contained inside F_2^c. Apply (ii) again ok? To this situation namely with $\bar{U}$, $\bar{U}$ contained inside F_2^c. In between these two, you can put one more open set, we get an open set U_1 such that $\bar{U}$ contained inside U_1 contained inside $\bar{U_1}$ contained inside F_2^c.

Now, you take the complement of $\bar{U_1}$. $\bar{U_1}$ is closed. So, the complement will be open that as V. Then this F_2 will be contained inside V, ok? See here this is contained inside F_2^c. So, when you take the complement this will be contained inside the complement of that. That is all. And it is easily verified that $\bar{U} \cap \bar{V}$ is empty ok.

So, proofs are identical to the proofs for the regularity that is why I have put them together in one single place.

(iii) implies (i) is obvious. Because (iii) is a stronger condition right? Go back here, disjoint closed subsets are contained in disjoint open sets. $\bar{U_1} \cap \bar{U_2}$ itself is empty implies $U_1 \cap U_2$ is also empty and that is all we need in (i). So, (iii) implies (i) is obvious alright.

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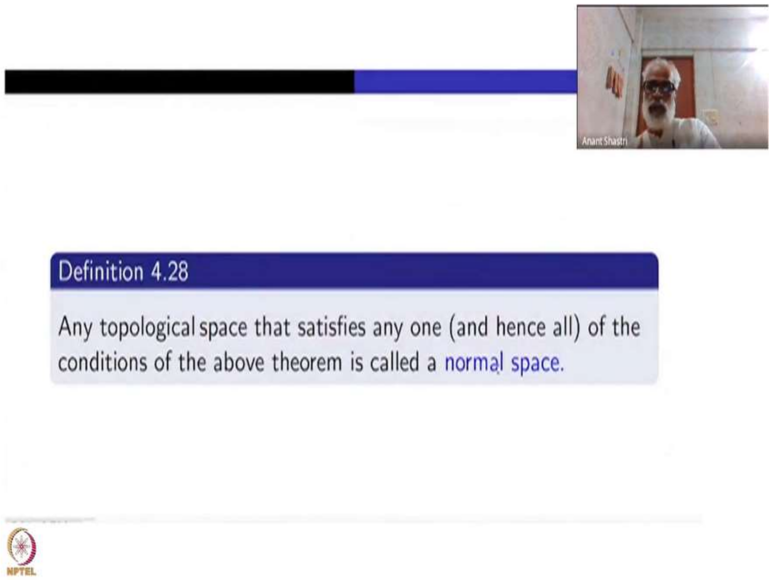

Now what to do with this regularity and normality ok? So, what is normality anything any topological space, which satisfies (i), (ii), (iii) of the above theorem any one of them and hence all of them ok? Normally you can take the first one as a definition, but in application you can use any one of them.

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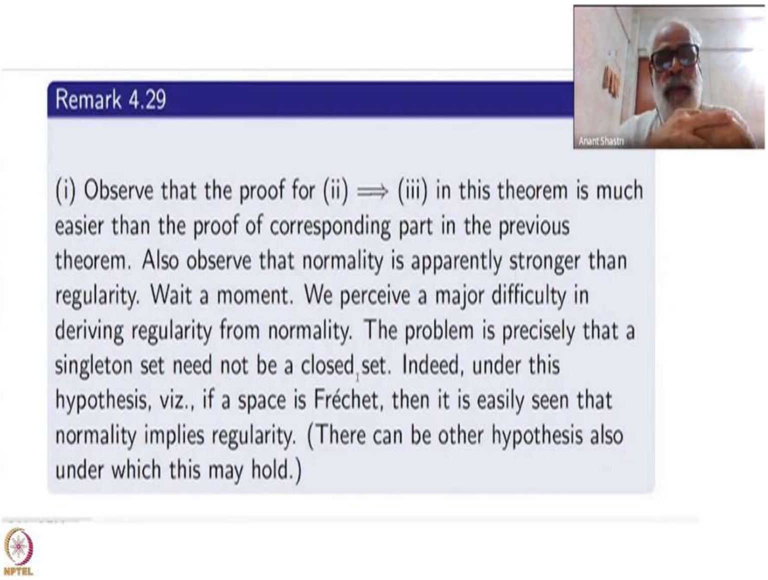

Proof that (ii) implies (iii) in this theorem is much easier than the proof for the corresponding part of previous theorem, because you have to do things point by point and so on and then take the union and so on. So, for normality it was easier actually.

Also observe that normality is apparently stronger than regularity, `apparently' why? Because point and closed sets are separated in regularity. Here any two closed sets are separated, but you have to be careful both F_1 and F_2 must be closed here whereas, there x could be any point and the other one is closed so; obviously, we do not have the hypothesis that singleton sets are closed ok? We do not have that therefore, normality may not give you regularity ok?

So, we perceive a major difficulty in deriving regularity from normality. The problem is precisely that singleton sets need not be closed sets. Indeed under this extra hypothesis namely, singletons are closed which is that X is Frechet right? If all the singletons are closed that is called a Frechet space, so, if you have Frechetness, then it is easily seen that normality implies regularity. There can be other hypothesis also under which this may hold.

But under Frechet, we know that normality implies regularity ok? Regularity of course, may not imply normality that is too much to expect, but if you say it is not then you have produce a counter example. That is the only way. So, you have to give a counter example. Counter examples are not all that easy alright. So, we will work on that. One those things now ok?

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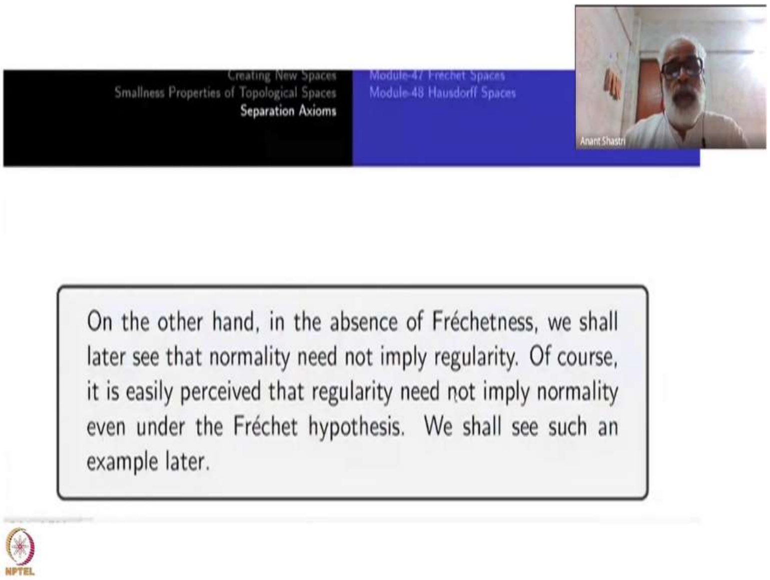

In the absence of Frechetness, we shall later see that normality need not imply regularity. Which just means that we have to give a counter example. Of course, it is easily perceived that regularity need not imply normality even under Frechet condition. It may not. We shall see such an example later.

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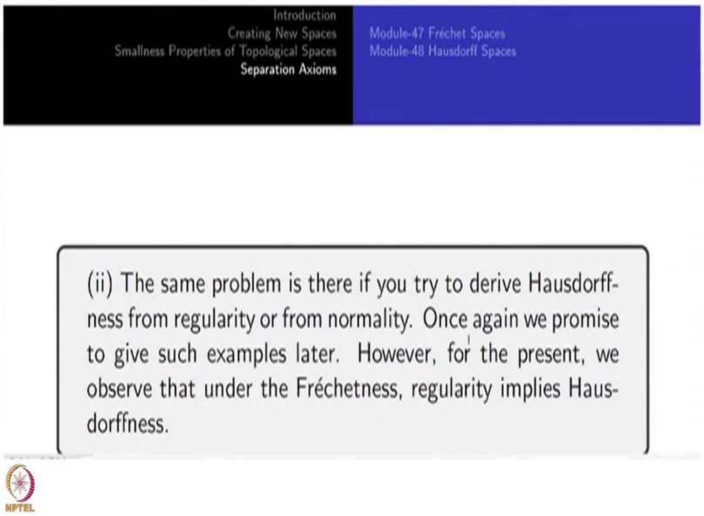

Same problem is there if you try to derive Hausdorffness from regularity. Because it looks like I started with the comment that regularity is a kind of generalization of Hausdorffness right? Any two points are separated by open sets, that is Hausdorffness. Here point and a closed set are separated, right? So, why cannot we separate two points? Once again the second point may not be a closed set right?

So, second point is arbitrary. So, all points must be closed in oreder to apply regularity to get Hausdorffness. That means, Frechetness again. If you have Frechetness, regularity automatically gives Hausdorffness. Now you see the importance of Frechetness. Under Frechetness, regularity implies normality, normality implies regularity and normal and Hausdorffness also and so on ok. I repeat, under Frecheness, Regularity implies Hausdorffness, and normality implies regularity. Therefore, it will imply Hausdorffness also.

So, that may be one of the reasons why Frechet, even in his definition of topology itself put that condition ok.

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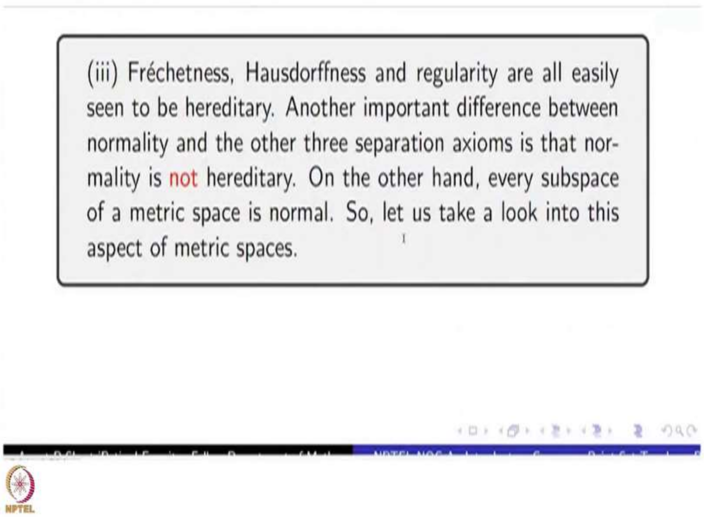

Now, let us work out these things, our habit of checking whether a property is hereditary co-hereditary and so on. Frechetness, Hausdorffness and regularity are all seen to be hereditary.

Hereditary means what? Take a subspace it should have same property. ok? So, let us look regularity. Take a subspace and take a point and a closed set. A closed set in the subspace is what? What is the closed subset in the subspace? It is some closed subset in the larger one intersected with the original subspace. The point is already in the subspace and not inside the smaller set.

So, it is not in the larger one also. Therefore, you can apply the regularity of the larger space to conclude that there are disjoint open subsets as needed. Now we intersect them with the subspace. So, that will give you disjoint open subsets containing the point and the closed set ok.

So, that was the hardest one. Hausdorffness and Frechetness you can do it easily ok. So, every subspace of a regular space is a regular space; however, you try to do the same thing for normality you will have problems, why? Because starting with two closed subsets inside the subspace, there are closed subsets in the larger one. The problem is they may not be disjoint.

You start with disjoint closed subsets in the subspace. That means what? There are closed subsets in the larger one when you intersect with the subspace, they will give you the original sets, which are disjoint. But why these larger ones, new ones inside the larger space, should be disjoint? Nobody guarantees you that. In fact, that can happen and that way the normality may break down. It may go may not go down to the subspace ok?

In fact, it happens that normality is not hereditary. So, again we have to construct some example for that ok? A counter example ok? On the other hand, where are these things coming from you know, metric spaces. A metric space is Frechet, Hausdorff, regular and normal as well. Very easy to prove.

In fact, every subspace of metric space is a metric space. Therefore, every subspace of a metric space is also normal whereas, in general, subspace of a normal space may not be normal. So, that makes us think about what is going wrong? So, we have to take a fresh look at the metric spaces itself ok?

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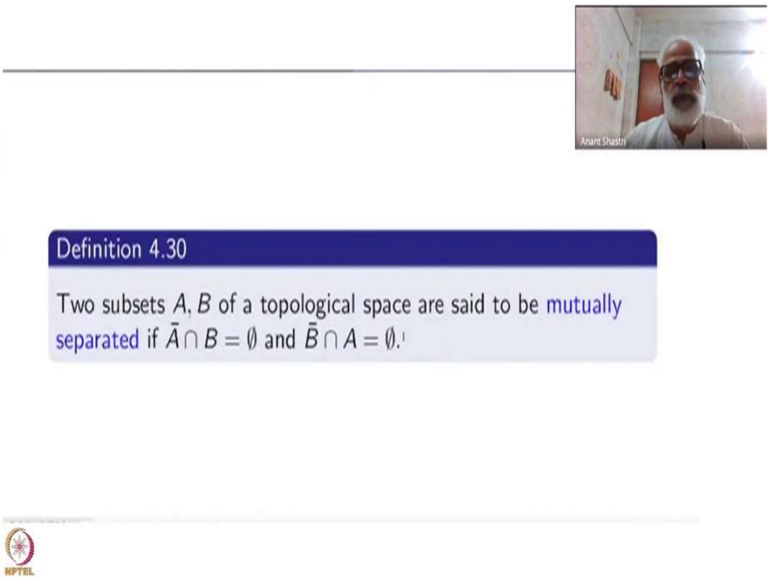

So, let us take a look at metric spaces, namely subspaces are also normal why ok? So, that is the aspect. Two subsets A and B of a topological space are said to be mutually separated if $\bar{A} \cap B$ is empty and $\bar{B} \cap A$ is empty.

So, this is slightly you know, a generalized concept of two closed subsets being disjoint. If A and B are closed subsets then saying they are separated is the same thing as they are disjoint that is all. Because $\bar{A} = A$ and $\bar{B} = B$ right? So, there is nothing more than that ok? Instead of taking closed subsets, start with any two subsets. Saying they are disjoint is weaker than saying that they are separated. You see $A \cap B$ may be empty, but $\bar{A} \cap B$ may not be empty.

So, this is a stronger condition on arbitrary subsets, than saying that they are just disjoint ok. So, this definition is made in the light of the third condition in normality you will see why now you see.

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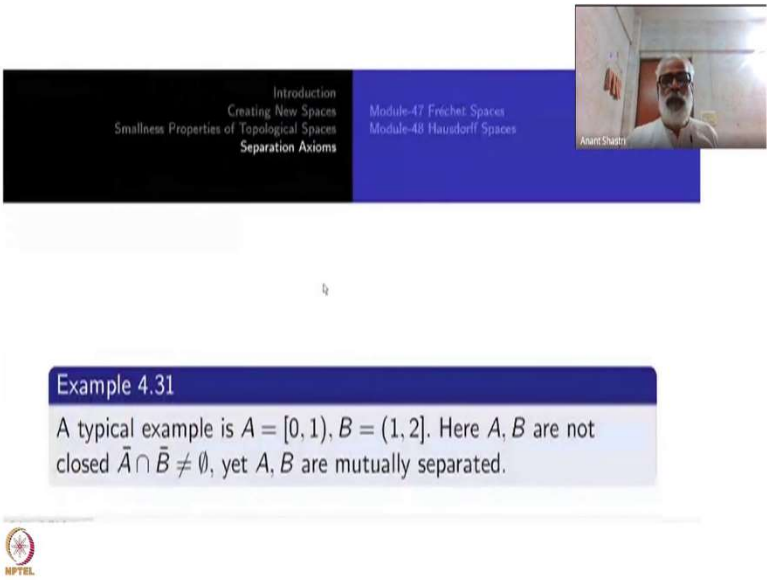

A typical example of mutually separated subsets is: $A = [0, 1)$ and $B = (1, 2]$, if you take $\bar{A}$ and $\bar{B}$, they are not disjoint. They will have common point viz., 1. Also, A and B are not closed.

But A and B are mutually separated, why? $\bar{A} = [0, 1]$ and 1 is not there in B. Similarly, $\bar{B} = [1, 2]$ and 1 is not here in A. So, they are mutually separated ok? So, this is a typical example inside $\mathbb{R}$. You can construct many many such examples. Now, what is that good for? Let us see now.

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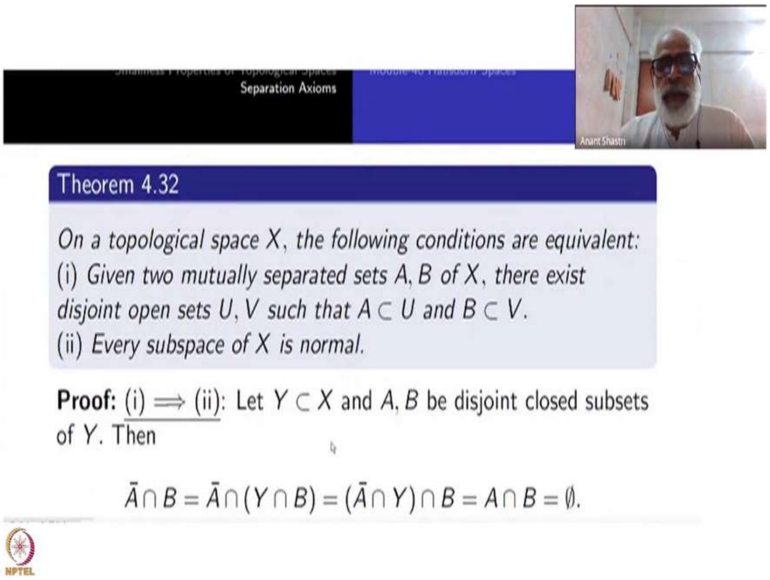

On a topological space X, the following conditions are equivalent. Given two mutually separated subsets A and B of X, there exist disjoint open subsets such that A is contained inside U and B is contained inside V.

Now, you see this is definitely a stronger condition than normality why? Because in normality I started with the A and B are closed subsets, right? Then automatically they are mutually separated, this condition is satisfied. But if you start with arbitrary subsets A and B which are mutually separated, they may not be disjoint, they may not be closed and so you cannot apply normality to get disjoint open sets U, V such that A contained inside U, B contained inside V. So, this statement is definitely stronger than normality. It implies normality I have shown you. because if start with A and B are closed they are automatically disjoint. So, you will get this one. So, condition (i) is stronger than normality.

But this theorem says condition (i) is equivalent to every subspace of X is normal, X is normal is fine. That is that (i) implies normality.

But more than that. Condition (ii) is obviously stronger. What it says every subspace of X is normal ok. So, let us prove this one. Every subspace of X is normal then you have to show this one also ok. So, both ways we have work to do here.

So, let Y be a subspace of X, A and B be disjoint closed subsets of Y. Then I have to produce disjoint open subsets of Y containing A and B respectively. that is my aim ok?

Now, what happens? A and B are disjoint closed subsets, closed inside Y not inside X. So, $\bar{A} \cap B$ is $\bar{A} \cap (B \cap Y)$, because B is a subset of Y. So, you can rewrite it as $(\bar{A} \cap Y) \cap B$, you know, by associativity of the intersection. Now what is $\bar{A} \cap Y$? Can you tell me what is this? All closure points of A which are already inside Y.

Therefore they will be inside the closure points of A inside Y. Therefore, they are in the closure of A inside Y, but A is closed inside Y. So, it is A. So, this $\bar{A} \cap B$ and $A \cap B$ is empty is the starting hypothesis, A and B are disjoint closed sets ok. Similarly $\bar{B} \cap A$ will be also empty, exactly similarly.

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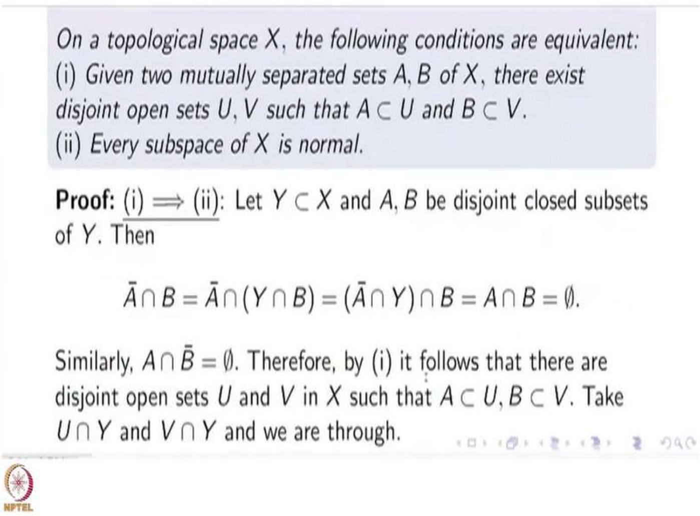

Therefore when you pass on to the larger space they will be separated ok. Therefore I can apply (i) to get disjoint open sets U and V in X, such that A is inside U and B inside V. Now you take intersection with Y, we are through alright.

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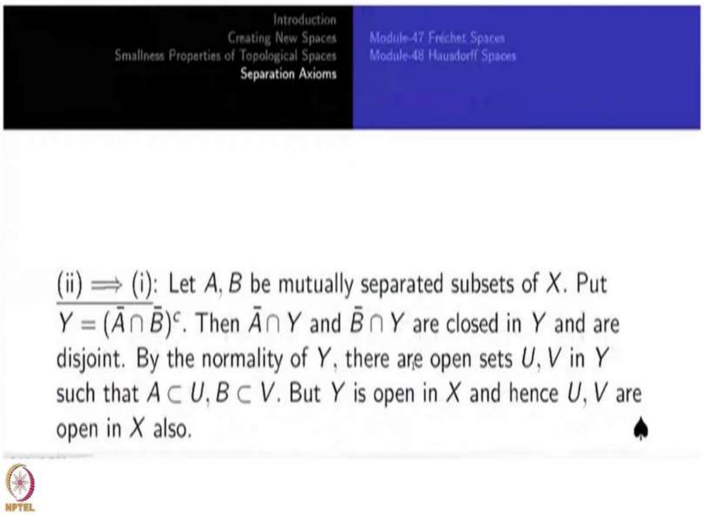

Now let us proof the converse. Let A and B be mutually separated subsets of X. Put Y equal to $\bar{A} \cap \bar{B}$ and the complement of that.

Remember, $\bar{A} \cap \bar{B}$ may not be empty if its empty you are in a good shape. There is nothing to bother about. So, it may not be empty but $\bar{A} \cap B$ is empty, $\bar{B} \cap A$ is empty.

So, look at Y equal to the complement of $\bar{A} \cap \bar{B}$. So, that is the subspace. Then $\bar{A} \cap Y$ and $\bar{B} \cap Y$ are closed subsets in Y. After all, $\bar{A}$ is closed inside X itself. So, intersection with Y will be closed inside Y, ok? And they are disjoint, because $\bar{A} \cap Y$ intersection with $\bar{B} \cap Y$ is equal to $(\bar{A} \cap \bar{B}) \cap Y$, but Y is just the complement of this one right? They are disjoint because our choice of Y. There is nothing more than that. By the normality of Y, this is the hypothesis, that every subspace is normal, there are open sets U and V inside Y, such that this A is inside U, B is inside V, ok? But what is $\bar{A} \cap \bar{B}$, this is a closed set.

So, its complement is an open set, that is our Y. U and V are open inside Y, they will be open inside X also. So, you do not have to fatten them. The same U and V will be open inside X also right therefore, U and V are open inside X also A contains inside U, B contains inside V. So, we were arrived at number (i) alright.

So, the proof was not at all difficult, but you have to think of, you know, this clever step here. Instead of arguing with some points here points there etc., and get confused ok. So, you have to think about this what subspace should I take. So, apply to the right subspace you get the answer very easily.

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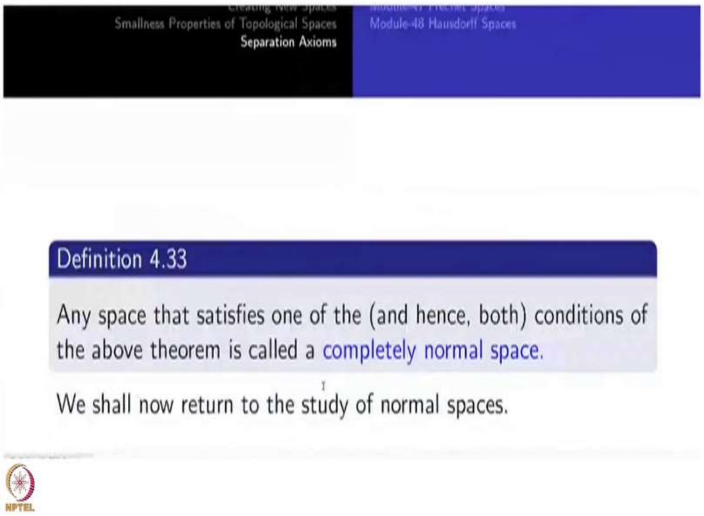

Following this theorem we can now make a formal definition. We have observed that this condition is stronger than normality.

So, we just call it completely normal space. A space that satisfies one of the conditions and hence both the conditions of the above theorem is called a completely normal space ok?

So, we shall now return to the study of normal spaces. Maybe it is time now. So, let us stop here and take up the normal spaces next time rigorously.

Thank you.