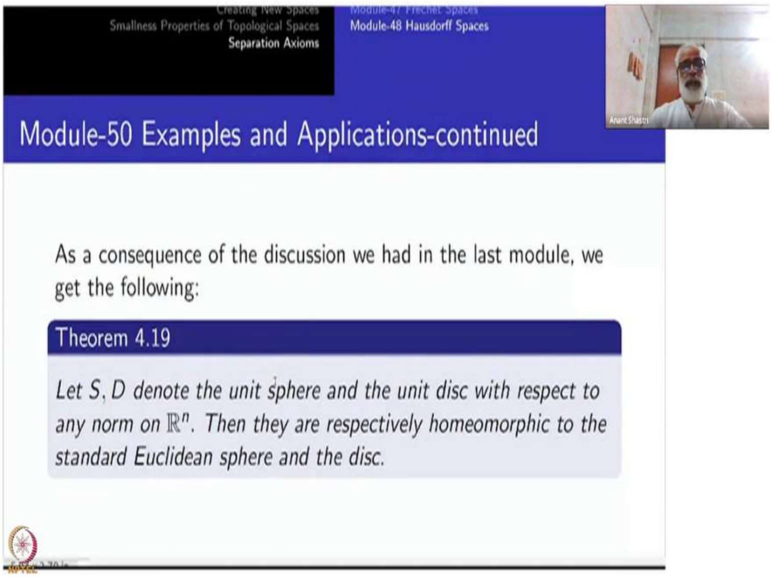
**Introduction to Point Set Topology, (Part I)**
**Prof. Anant R. Shastri**
**Department of Mathematics**
**Indian Institute of Technology, Bombay**

**Lecture - 50**
**Examples and Applications Continued**

(Refer Slide Time: 00:16)



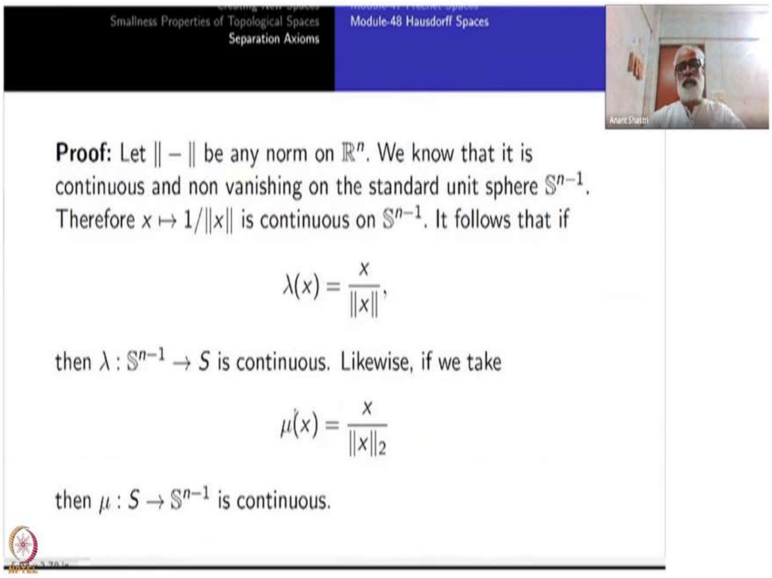

Welcome to module 50 of Point Set Topology, part 1. So, let us continue the study of Haudorffness along with compactness and so on. Last time we gave some applications to function analysis, normed linear spaces, in particular. We showed that every normed linear space in which the unit sphere is compact is finite dimensional.

Of course, we did many other things, like we showed that the entire of $\mathbb{R}^n$, no matter which norm you take is a cone over the unit sphere. When you change a norm the units will change but the cone over the unit sphere is always homeomorphic to the underlying linear space $\mathbb{R}^n$. So, this what we have seen.

So, let us continue that as a consequence of the discussions last time we also get the following theorem. Let $S$ and $D$ denote the unit sphere and the unit disc respectively with respect to any norm on $\mathbb{R}^n$. Remember if the norm you have taken is $\ell_2$ norm then I have a different notation, I have special notations for the sphere and the disc. This is a general notation when the norm on $\mathbb{R}^n$ is arbitrary.

$S$ and $D$ are respectively homeomorphic to standard Euclidean sphere and the standard disc ok. So, this is a consequence of the description last time.

(Refer Slide Time: 02:36)



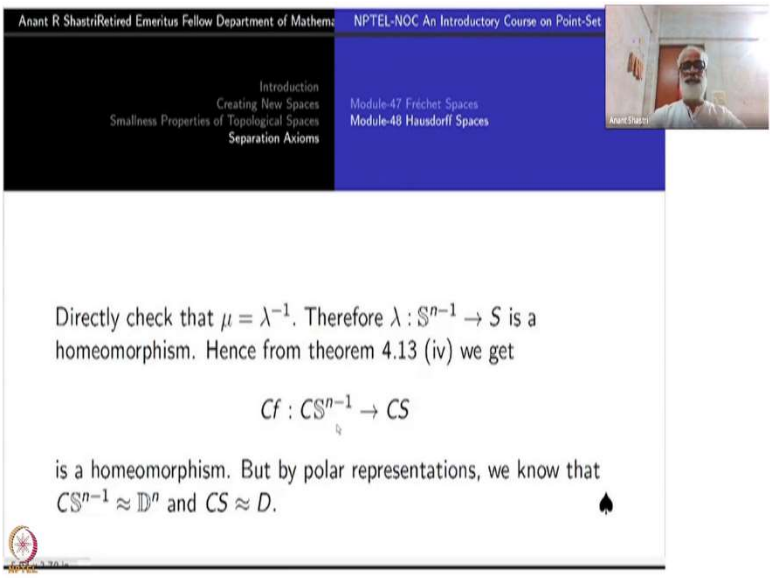

However, let us just go through the proof carefully, and see what exactly is involved in it ok? Fix any arbitrary norm on $\mathbb{R}^n$.

We know that it is continuous and non vanishing on the standard unit sphere $\mathbb{S}^{n-1}$. A norm is a function into $\mathbb{R}$, right? $\mathbb{R}^+$ actually. Restricted to $\mathbb{S}^{n-1}$, we already established that it is continuous. Now, what I mean by continuous? Now, I am taking the standard $\ell_2$ norm on $\mathbb{R}^n$. With respect to that this function which is a norm on $\mathbb{R}^n$ is always continuous. This what we have seen.

In particular, since on the sphere it is non vanishing also, therefore, $1/\|x\|$ is also continuous ok. So, it follows that, if I put $\lambda(x) = x/\|x\|$, then this $\lambda$ is from $\mathbb{S}^{n-1}$ to $S$ is continuous. You see, here you have to be careful, on the codomain we have the new norm and on the domain we have standard norm ok.

So, $\ell_2$ norm is 1 does not mean that this norm is 1. So, I have to divide by this so that I get a function into this $S$ here ok? So, $x$ going to $x$ is continuous, $1/\|x\|$ is continuous, the product would be continuous ok? Likewise, we take $\mu(x) = x/\|x\|_2$. So, the role will be reversed now, domain and co-domain, then $\mu$ will be a continuous from $S$ to $\mathbb{S}^{n-1}$. Why? What is easy to verify is that $\lambda$ and $\mu$ are inverses of each other ok?

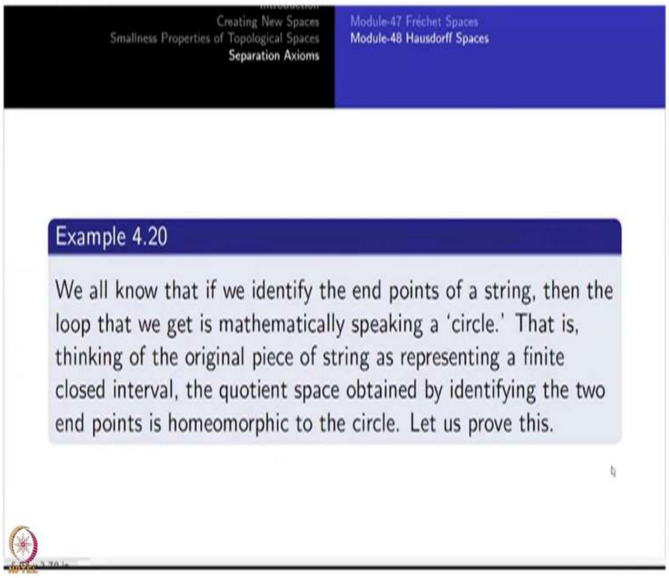
(Refer Slide Time: 04:45)




Therefore, each of them is what a homeomorphism, one is the inverse of the other. Once that is the case you can take the cone $C\mathbb{S}^{n-1}$ which is $D$. It is homeomorphic to the cone $CS$, ok? This is $\mathbb{D}^n$, here this will be the disc $D$ here.

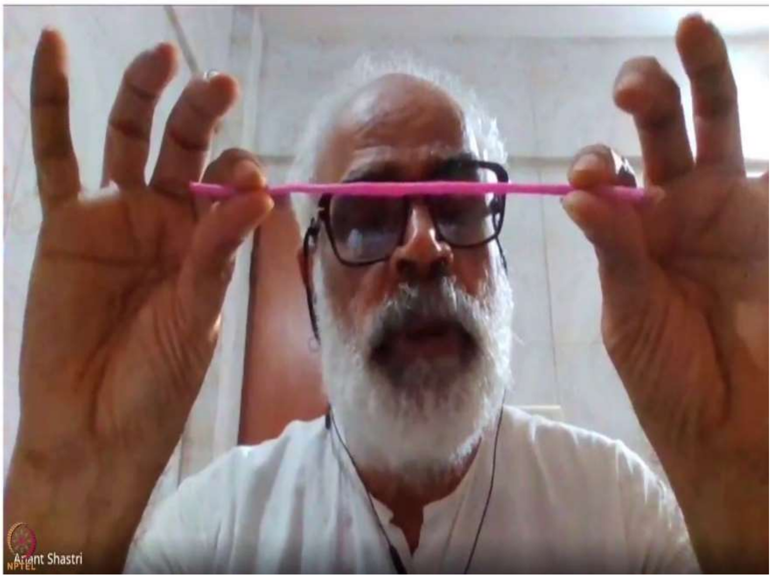
So, this also you have seen last time ok? The first one is $C\mathbb{S}^{n-1}$ and hence is $\mathbb{D}^n$, the second $CS$ is the unit disc with respect to the new norm. So, once two spaces are homomorphic, the cones over them are also homeomorphic is one of the theorem that we saw last time.

(Refer Slide Time: 05:42)




So, now let me take genuine example here, which you may all be interested in. Maybe I may have to stop this presentation and then do something.

(Refer Slide Time: 06:07)




I wanted to show you a piece of string, can you see that? What does it represent? I mean how do you represent this by a mathematical object?

So, you can either say that this is a closed interval when you include the end points of the string or an open interval or a half open interval. There is no other way to represent a half open interval or open interval or closed interval all of them are represented by a piece of string. I have to tell you what it is, ok.

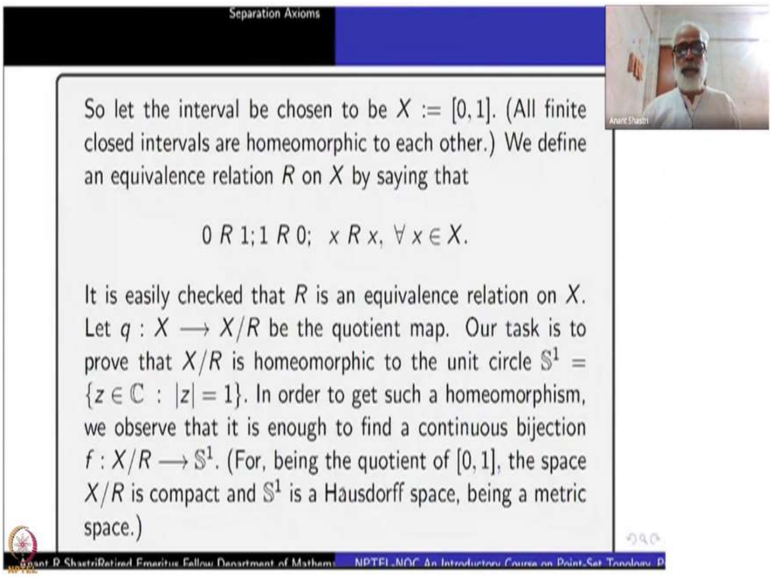
(Refer Slide Time: 06:56)

So, you can say it is the interval $[0,1]$. When you identify just $0$ with $1$ and nothing else what do you get? You will get some object like this, which is a model for $\mathbb{S}^1$ ok? Actually the mathematical object $\mathbb{S}^1$ is the model, and this physical thing is my hand is the object whatever. It does not matter whether it is like this or like this and so on up to homeomorphism this $\mathbb{S}^1$.

So, all this a layman will understand that this is a circle. Now, mathematically we want to rigorously say that identifying the end points of $0$ and $1$ gives you $\mathbb{S}^1$, right. So, that is what I want to rigorously prove now. That was my main interest yeah you know ok?

So, let me go back to the slides now ok. So, so let us do this business, namely, let us prove that the quotient of the closed interval $[0,1]$ by the identification namely the end points $0$ and $1$ are identified is actually $\mathbb{S}^1$, ok?

(Refer Slide Time: 08:21)

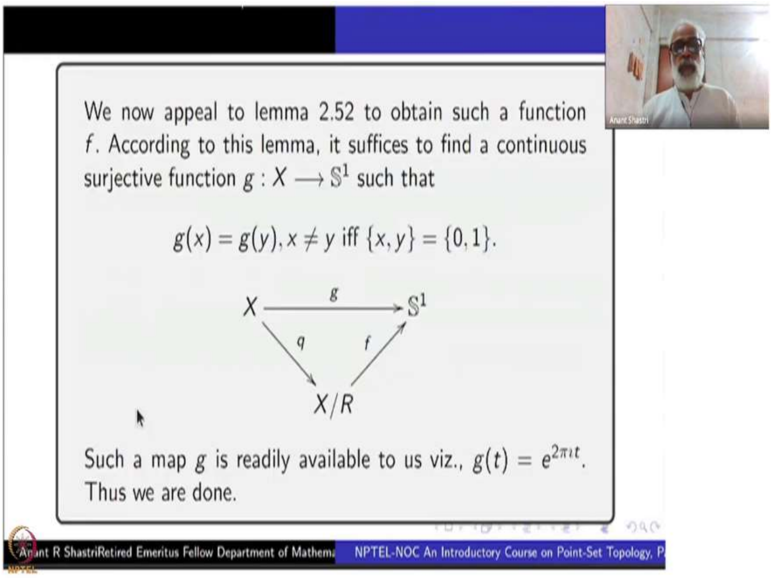



So, I will denote by $X = [0, 1]$, ok? I do not have to take any other closed interval because they are all homeomorphic to $[0, 1]$. Any closed interval is homeomorphic $[0, 1]$ that we know already.

So, define a relation in $X$ : 0 is related to $1, 1$ is related to $0, x$ is related to $x$ for every $x \in X$. So, this is an equivalence relation ok? There is no other rule here ok. Now, let $q$ from $X$ to $X/R$ be the quotient map, this $X/R$ which has the quotient topology, we want to show that is homeomorphic to $\mathbb{S}^1$. Our task is to show that it is homeomorphic to the unit circle, ok.

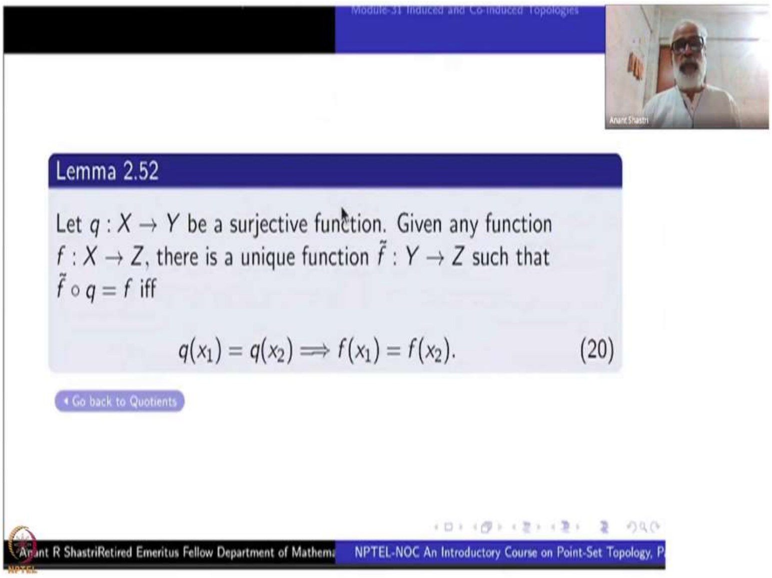
To get a homeomorphism you observe that whenever you have a quotient space of a space $X$, what you do is you construct the function on $X$ itself, on the mother space $X$ itself ok? But now what we observe is $X$ is compact therefore, $X/R$ is compact and $\mathbb{S}^1$ is Hausdorff. Then one of the theorem that we have tells you that if you have a continuous bijection, then it will be automatically a homeomorphism.

(Refer Slide Time: 09:53)




So, that is what we are going to expect here ok. But now to construct a function from $X/R$ to $\mathbb{S}^1$, we appeal to this lemma which we proved long back. So, what was it?

(Refer Slide Time: 10:08)

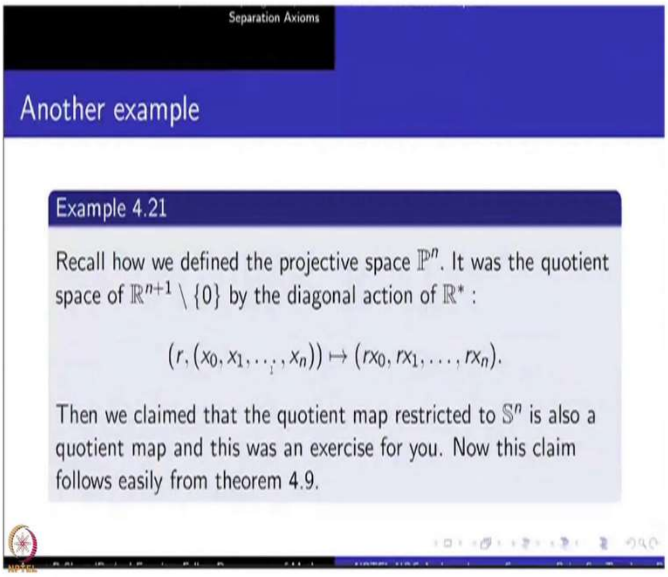



When you have a quotient map $X$ to $Y$ ok? given any function $f$ from $X$ to $Z$, there is a unique function $\tilde{f}$ from $Y$ to $Z$ such that $\tilde{f} \circ q = f$ if and only if this happens ok?

Namely, whenever two points are identified by $q$, the same thing should happen to the function $f$ that you are interested in. That function $f$ should also identify those two points, this is the condition. Of course, now our notations are slightly different. $X$ to $X/R$ is the same, that is $q$. I want to get a homeomorphism here ok. To get a map here, I should have a map $g$ here first of all such that $g(x) = g(y)$, whenever $q(x) = q(y)$. Here this implies either $x$ is 0 and $y$ is 1 or $y$ is 0 and $x$ is 1, that is the unorderd pair is the same ok.

Why I have put this one? Because finally, I do not want any more identifications here, I want this one to be injective ok. I want first of all this function, so it must send 0 and 1 to the same point here. So, $g$ must have that property. $g(0)$ must be $g(1)$, ok. Otherwise you know, otherwise there is no identifications by which I mean if $q(x) \neq q(y)$, then $g(x)$ will be not equal to $g(y)$. So, such a map is readily available to us. You do not have to work hard for this. Take $g(t) = e^{2\pi i t}$, restricted to $[0, 1]$. If $t$ is 0 or $t$ equal to 1, it is the unit of $\mathbb{S}^1$ and everywhere inside the interval, it is injective. So, when you come down here, you get a continuous function which is injective, but g is already surjective therefore $f$ is also surjective. So, this $f$ becomes a homeomorphism because $X$ is compact, $X/R$ is compact and $\mathbb{S}^1$ is Hausdorff.

(Refer Slide Time: 12:40)



Let us go to some other example now, namely, the projective space $\mathbb{P}^n$. We have not studied this one much, but we have we are familiar with its definition. However, I will recall the definition.

So, projective space $\mathbb{P}^n$, the real projective space it is called; there is a complex version also as you may anticipate. It is defined as a quotient of the space of non zero vectors in $\mathbb{R}^{n+1}$ by the diagonal action of $\mathbb{R} \setminus \{0\}$, namely $(r, (x_0, x_1, \ldots, x_n))$ goes to $(rx_0, rx_1, \ldots, rx_n)$. So, it is also scalar multiplication you may say diagonal action or scalar multiplication ok? We have then claimed that this quotient map when you restricted to $\mathbb{S}^n$, is also a quotient map. That time it was an exercise for you. Now, this claim easily follows from our earlier theorem.

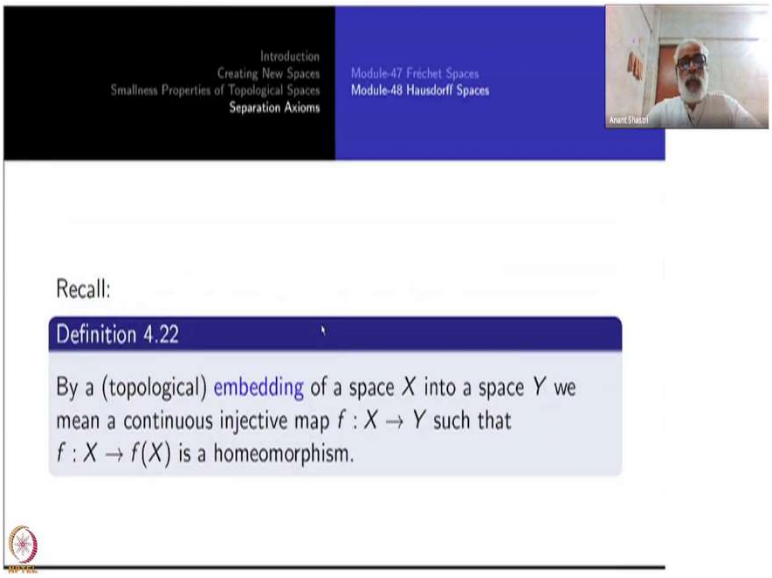
Why? Because, $\mathbb{S}^n$ is compact ok; and what we have shown is that quotient map is a closed map ok. So, this restriction will be also a quotient map now from $\mathbb{S}^n$ to $\mathbb{P}^n$. It is easily seen that it is surjective, surjective continuous map. But now because $\mathbb{S}^n$ is compact the function will be a closed map.

Every closed subset if a compact space is compact and the image of a compact set is compact, and compact subset of a Hausdorff space is closed. So, that was the theorem ok. So, you can use that here. It follows that $\mathbb{S}^n$ to $\mathbb{P}^n$ is also a quotient map. Of course, it is not a bijection. So, it is not a homeomorphism. Here, antipodal points $x$ and $-x$ are mapped to the same point ok?

This quotient is easier to understand, namely it is under the antipodal action, $x$ goes to $-x$. Both of them go to same point under $q$, ok? So, in particular, why I took this example? Now, it will follow that $\mathbb{P}^n$ is compact ok? We could not say that $\mathbb{P}^n$ is compact earlier.

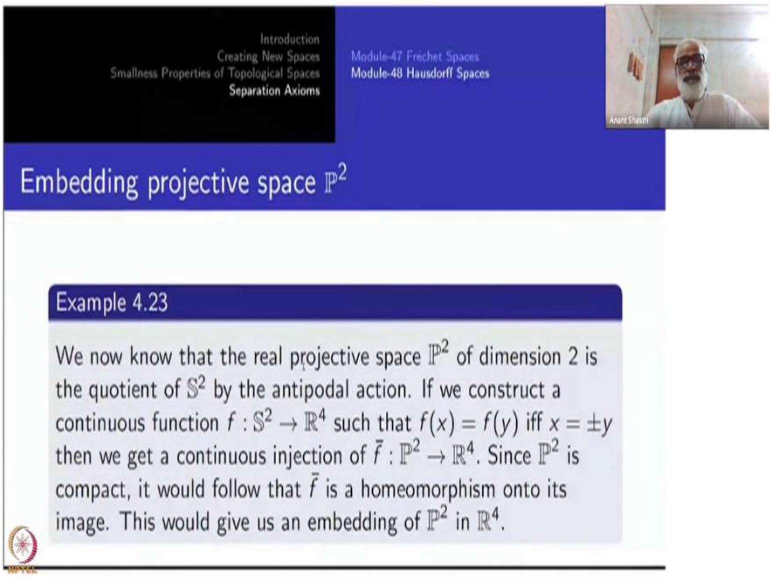
The only thing we have to bodily verify is that $\mathbb{P}^n$ is Hausdorff, which is not very difficult to verify. The entire exercise here you do not have to worry, you have to just show that $\mathbb{P}^n$ is Hausdorff, that much you have to verify. Then only this theorem can be applied ok?

(Refer Slide Time: 15:56)




So, now I would like to play a different game here. Recall that we have defined an embedding long long back maybe, of a topological space $X$ into another space $Y$. What is the meaning of an embedding? It means a continuous injective map such that, when you restrict $f$ from $X$ to $f(X)$, not the whole space $Y$ this is a homeomorphism, where $f(X)$ is given the subspace topology from $Y$. So, that was the definition ok? Beyond the definition we have not done much.

(Refer Slide Time: 16:43)




Now, at least we will have some examples here. So, now, I specialize to the case wherein $n = 2$. So, look at the projective space $\mathbb{P}^2$ of dimension 2 which is quotient of $\mathbb{S}^2$ by the antipodal action ok. What I want to do is, I want to explicitly write down an embedding of $\mathbb{S}^2$, inside $\mathbb{R}^4$, sorry embedding of $\mathbb{P}^2$ inside $\mathbb{R}^4$.

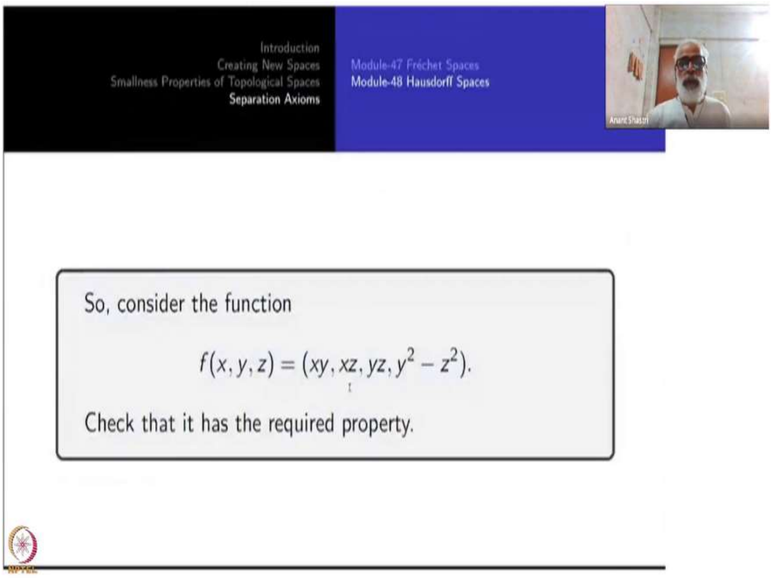
$\mathbb{S}^2$ is embedding in $\mathbb{R}^3$, $\mathbb{S}^2$ is a subspace of $\mathbb{R}^3$ right by definition. But now $\mathbb{P}^2$, I would like to embed inside $\mathbb{R}^4$; why? Because for some reason I am not able to embed it inside $\mathbb{R}^3$.

Actually, a deeper theorem in algebraic topology will tell you that you cannot embed $\mathbb{P}^2$ inside $\mathbb{R}^3$, ok? Notions such as orientability etc have to be studied to understand that result ok? So, to get such a function, what I should do? I should construct a function from $\mathbb{S}^2$ to $\mathbb{R}^4$ such that $f(x) = f(y)$, if and only if $x = \pm y$.

Then I will get a map $\bar{f}$ from $\mathbb{P}^2$ to $\mathbb{R}^4$, ok. Indeed, this is if and only if. When you come here it will be already injective mapping. If $f(x) = f(y)$, if and only if $x = \pm y$, then the corresponding function $\bar{f}$ from $\mathbb{P}^2$ to $\mathbb{R}^4$ will be injective. Once again this is compact that is Hausdorff.

So, $\bar{f}$ from $\mathbb{P}^2$ to $\bar{f}(\mathbb{P}^2)$, that will be a homeomorphism which means $\bar{f}$ is an embedding. The task is to find a function $f$ from $\mathbb{S}^2$ to $\mathbb{R}^4$, which has this property: points are mapped to the same point only if they are antipodal, otherwise they are mapped to distinct points. This is what I have to do ok?

(Refer Slide Time: 19:21)




So, hunting around various examples, it turns out to be a pleasant surprise that we can do it with a quadratic embedding. Namely, you know, whenever you have $\pm x$ going to same point. So, the natural thing is to look for quadratic functions $x^2, y^2, x^2 + y^2, xy + y^2$ and so on, tand their combinations.

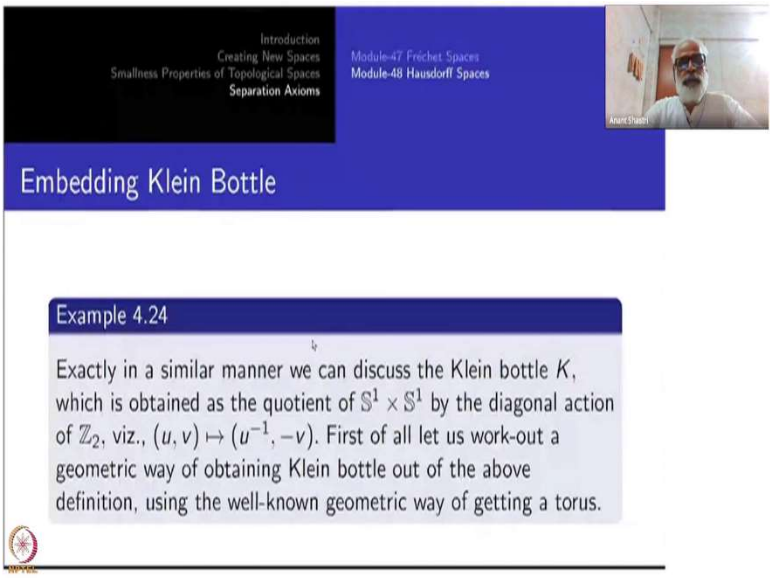
So, they will have this property right? Homogeneous quadratics. And a hunt like that is actually giving you the result. So, naturally you can try this out. $x, y, z$ are three coordinates here. Remember, I am only interested inside $\mathbb{S}^2$ ok; not the whole of $\mathbb{R}^3$. This will make sense in the whole of $\mathbb{R}^3$ no problem because we have polynomial functions.

Look at $xy, xz$ and $yz$. Obviously, three easiest function. Of course, I could have taken $x^2, y^2$ and $z^2$, but there will be a problem. If I take $x^2, y^2, z^2$, would it work? But then even $xy, xz, yz$ also does not seem to work but I am allowed to take one more coordinate function, namely I pick $y^2 - z^2$, ok?

After that it is a matter of checking that when you pass down to $\mathbb{P}^2$ that the map is injective. Obviously, if you replace each $(x, y, z)$ with $(-x, -y, -z)$ on the left side, the right side remains unchanged ok? Do not just change $x$ to $-x$, that is not the action, that is not required. When you change it is $(-x, -y, -z)$. So, that is the antipodal point of $(x, y, z)$. That will also represent same point in $\mathbb{P}^2$. Therefore, this will give you a function $\bar{f}$ from $\mathbb{P}^2$ to $\mathbb{R}^4$.

Now, you have to see that this function is injective that is all ok? So, that part I am going to leave you as a pleasant exercise. Verify it ok?

(Refer Slide Time: 21:57)



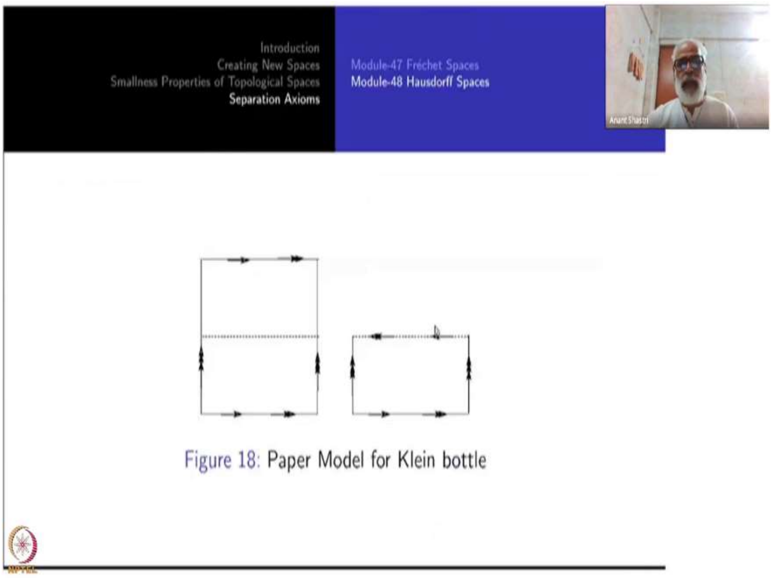

I will go to another interesting example. Once again, this is also something like $\mathbb{P}^2$, a non-orientable surface. $\mathbb{P}^2$ was one such. This also cannot be embedded in $\mathbb{R}^3$. But similar approach to that we did above seem to work here also and we will get an embedding in $\mathbb{R}^4$, ok?

So, first of all I have to explain what is this Klein bottle $K$ ok? It is the quotient, a double quotient, under $2-1$ mapping (just like $\mathbb{P}^2$) of the torus $\mathbb{S}^1 \times \mathbb{S}^1$, ok? What is the action? Action is important here, by the diagonal action ok of the group $-1, 1 = \mathbb{Z}_2$, ok? So, diagonal action has to be taken very carefully here: $(u, v)$ going to $(u^{-1}, -v)$. So, one coordinate you take inverse, you know multiplicative inverse another coordinative you take additive inverse.

On the second factor it is just the antipodal action. But on the first factor, this is the multiplicative inverse ok?

First of all, let us work out a geometric way of obtaining Klein bottle out of the above definition. Suppose I take this as the definition ok? This not a standard definition, but now let us say use the well-known geometric way of getting a Klein bottle out of the torus. The torus $\mathbb{S}^1 \times \mathbb{S}^1$ is defined as the quotient of a rectangle, $I \times I$ wherein the $4$ sides are identified in a particular way. The $[0,1] \times \{0\}$ is identified with $[0,1] \times \{1\}$. So, that is the opposite side right?, in the same oriented fashion. Similarly, $\{0\} \times [0,1]$ will be identified with $\{1\} \times [0,1]$, ok? Again the opposite sides.

(Refer Slide Time: 24:43)




That is the identification here, this is the square $I \times I$. So, this bottom thing goes to the top thing here, see the double arrows are indicating that. You see the triple arrow on vertical sides they are identified, this is the identification.

The arrow tells you how the identifications are done. For example, here an element looks like $(t, 0)$. So, where does it go? It will be identified with $(t, 1)$. Similarly, $(0, s)$ will be identified with $(1, s)$ here ok. So, the quotient space under these identification is the torus.

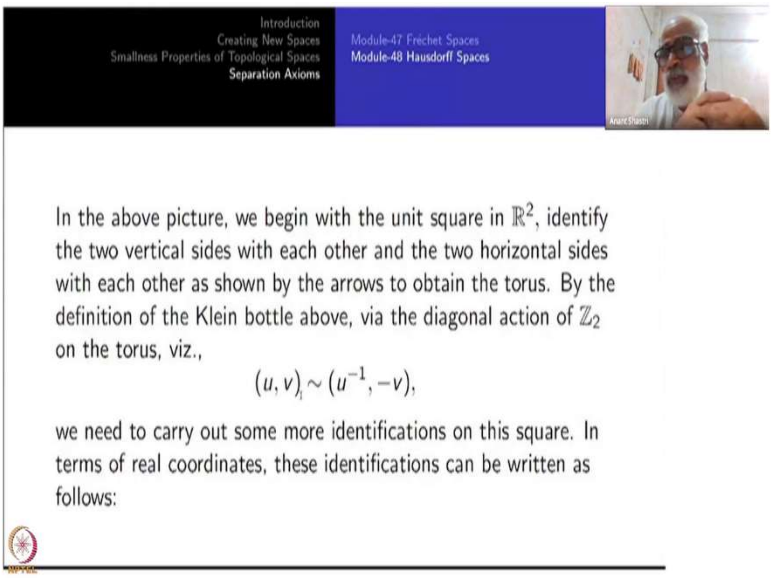
But to get the Klein bottle, I have to do some more identifications. Because on the torus, I have an action $(u, v)$ going to $(u^{-1}, -v)$, right? So, that is what I have tried to express here ok. So, what happens is if you look at this arrow, the bottom line here, this will be get identified with the middle one dot dot dot dot in the opposite direction. These elements will be there all the time there is no identification there ok?

But what further identification? what happens is this part will be identified with that part ok? This is getting identified this way alright and this getting identified that way. All these points here they will be identified with corresponding points here, something here will come here and so on in the opposite direction.

In the interior of half of this rectangle, there will be no identification. Everything above will be identify with some point below. Therefore, the upper half part of this rectangle is unnecessary. So, I have cut it off and taken the lower half rectangle here of the original one.

Now, the identifications of this one will give you the Klein bottle. So, this is the geometric way ok, to justify all these things rigorously you have to write down formulae, that is the only way ok?

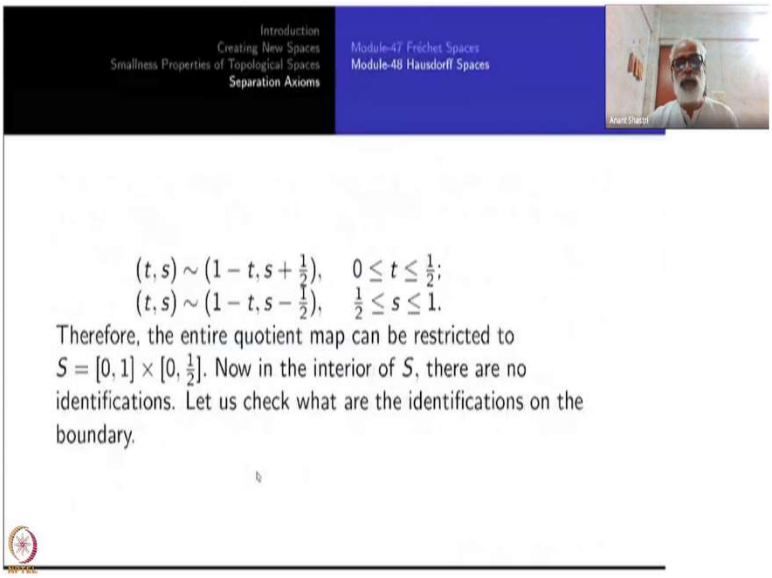
(Refer Slide Time: 27:41)



So, that is what we have to do ok. So, this is the idea diagonal action $(u, v)$ going to $(u^{-1}, -v)$.

Since, $u$ and $v$ are now representing elements of $\mathbb{S}^1 \times \mathbb{S}^1$, but I am representing them on the plane, only after identification, named $t$ going to $e^{2\pi it}$, you will get an element of $\mathbb{S}^1$. So, what is the corresponding identification of this one in terms of real coordinates?

(Refer Slide Time: 28:21)



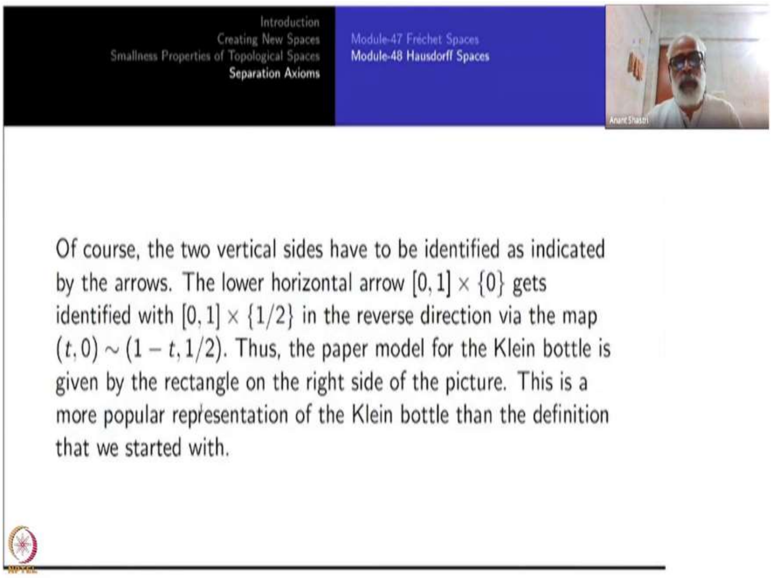

So, I have to convert that and that amounts to the following thing: if you represent the element $u$ by $t$, that means what? $u$ is equal to $e^{2\pi it}$, ok? Similarly, this $v$ is represented by $s$ means, $v$ equal to $e^{2\pi is}$, that is the meaning of this. So, $u^{-1}$ will be represented by $1 - t$, ok?

Whereas, $-v$ will be represented by $s + 1/2$,ok? This is very easy. You apply $e^{2\pi it}$, what happens, $e^{2\pi i(1-t)}$ is just the inverse of $e^{2\pi it}$ and the other one is a minus of that. Because half of $2\pi i$ will be $\pi i$. So, $e^{\pi i}$ is multiply by $-1$, ok.

So, that is all I have done. And this is first half the bottom half. In the second half, $(t, s)$ will be $(1 - t, s - 1/2)$, because plus half goes away out of that is 1. So, I have to take the half less than equal to $s$; less than equal to I have to take.

So, these two have the same effect. But the actual map will be $s - 1/2$, because I have to be within a $0$ less than $1$. If $s$ is bigger than half, $s + 1/2$ will go out of $1$ right. So, that is why I have to write like this one, the entire quotient map can be restricted to $S$ equal to $[0, 1] \times [0, 1/2]$; you do not need a second part at all, this part. Now, in the interior of $S$, there are no identifications let us check what are the identifications on the boundary ok?

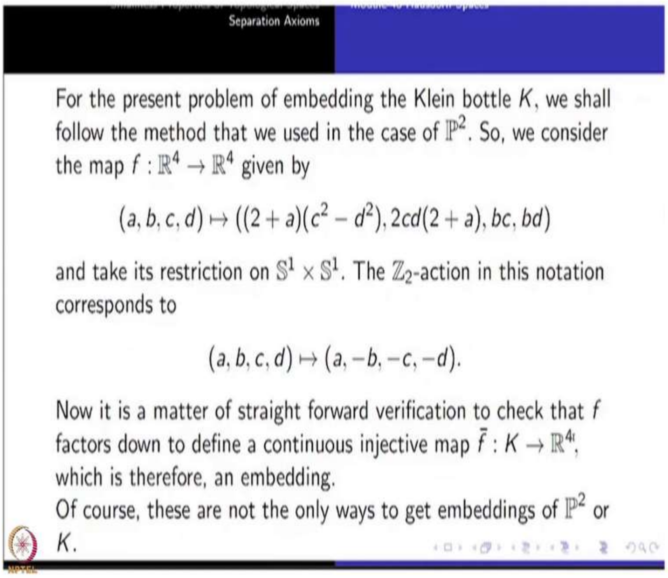
(Refer Slide Time: 30:24)




So, this is what I have already done. The two vertical sides have to be identified as indicated by the arrows. The lower horizontal arrow $[0, 1] \times \{0\}$ gets identified with $[0, 1] \times \{1/2\}$, in the reverse direction via the map $(t, 0)$ going to $(1 - t, 1/2)$. I have to just understand what is happening here, when you put $s = 0$ and $s = 1/2$, ok.

So, this is what happens. Thus the paper model of Klein bottle is given by the rectangle on the right hand side here, in the picture which I have shown. Paper model means you have to indicate how you are going to identify the boundaries that is all ok. So, this is the explanation of how to construct a paper model of Klein bottle alright.

(Refer Slide Time: 31:34)

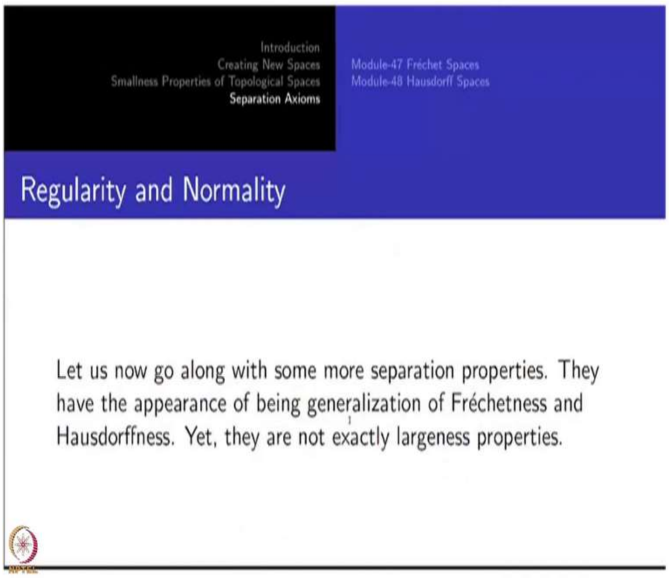



But now I go back to my definition of the Klein bottle as a quotient of $\mathbb{S}^1 \times \mathbb{S}^1$ by the identification namely $(u, v)$ identified with $(u^{-1}, -v)$, ok. Once again this time I am denoting this $(a, b, c, d)$ etc inside $\mathbb{R}^4$ itself. Because $\mathbb{S}^1$ is embedded in $\mathbb{R}^2$, so, $\mathbb{S}^1 \times \mathbb{S}^1$ is embedded in $\mathbb{R}^4$. From $\mathbb{R}^4$ to $\mathbb{R}^4$m I want to take a map such that, whenever two points are identified they are going to same point, if and only if. So, that is the same technique as before.

So, what I do take $(a, b, c, d)$ in $\mathbb{R}^4$. Remember these satisfy the extra conditions: $a^2 + b^2$ is $1, c^2 + d^2$ is 1. That is the condition in $\mathbb{R}^4$ for this to represent $\mathbb{S}^1 \times \mathbb{S}^1$, ok. Let it go to $((2 + a)(c^2 - d^2), 2cd(2 + a), bc, bd)$; ok? And take its restriction on $\mathbb{S}^1 \times \mathbb{S}^1$, this map is completely defined on the whole of $\mathbb{R}^4$, but I am only interested in $\mathbb{S}^1 \times \mathbb{S}^1$. That means put $a^2 + b^2$ equal to 1 and $1, c^2 + d^2$ equal to 1; that is all.

So, once you put that condition, you can think of $(a, b, c, d)$ as some $(\cos\theta, \sin\theta, \cos\psi, \sin\psi)$ . Something like that also you can try ok. Then $\mathbb{Z}_2$ action in this notation corresponds to $(a, b, c, d)$ goes to $(-a, -b, -c, -d)$. Now, it is a matter of straight forward verification to check that $f$ factors down to define a continuous injective map $\bar{f}$ from $K$ to $\mathbb{R}^4$, ok? By the similar argument as in the case of $\mathbb{P}^2$.

(Refer Slide Time: 34:01)




How I got this map? If you look at the term $((2+a)(c^2-d^2), 2cd(2+c))$, if you write as $(\cos\theta, \sin\theta)$ and $(\cos\psi, \sin\psi)$ up to here if you put the third coordinate equal to $b$, it gives you `embedding' of $\mathbb{S}^1 \times \mathbb{S}^1$ in $\mathbb{R}^3$ obtained by rotating the circle of radius $1$ and centre $(2,0,0)$ in the $xz$-plane about the $y$-axis. So, you use one more coordinate here to get an embedding $\mathbb{R}^4$ of the quotient, the Klein bottle ok. So, this $(2+a)$ corresponds to you know the center of the circle is shifted to $(2,0)$.

See, and this is $2+a$. And then this is $(\cos 2\theta, \sin 2\theta)$. So, you just take instead of $\theta, 2\theta$ so that that will help to give you a map of $K$. That is a slight modification there ok yeah.

So, so we have done something nice today. Let us stop here, next time we will take two more properties which are very very important again, the separation properties regularity and normality.

Thank you.