Introduction to Point Set Topology, (Part I)
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Lecture - 05
Examples of Continuous Functions

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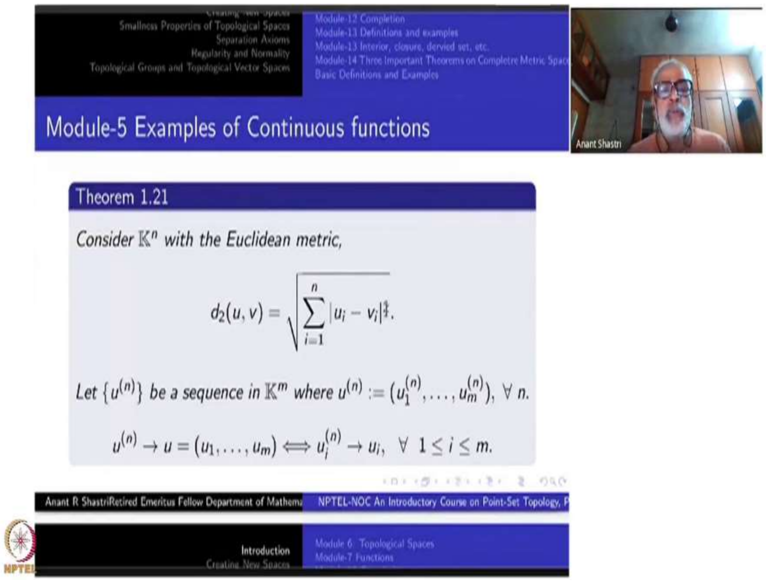

So welcome to Module 5 of Point Set Topology Course. Last time, we introduced the notion of $\epsilon - \delta$ continuity on metric spaces and showed that it is equivalent to sequential continuity. That was the major result last time. So, today we will use that and see, a number of examples, basically. So, here I have stated a theorem which is, again, using sequential continuity ok. So, first of all, you must see what happens to the sequences of vectors, sequences inside $\mathbb{K}^n$, ok.

So, that is the first theorem here. Consider $\mathbb{K}^n$ with the Euclidean metric, which I have denoted by $d_2(u, v)$. And I am just recalling it here, instead of x and y, I am writing u and v; it does not matter; that is what you must be able to do, sum i ranges from 1 to n modulus of $|u_i - v_i|^2$. Then take the square root.

Suppose u_n, now I cannot write this u_n lower because; u itself is a vector $(u_1, u_2, \ldots, u_n)$. So, I am putting a superscript here. This is sequence in $\mathbb{K}^m$ ok. So, I have changed this n

here, that is why I have put m here; now it is in $\mathbb{K}^m$. So this formula is true for all n after all, right? So, I cannot fix that one, this n, it is better, I use for sequences. So, sequencing $\mathbb{K}^m$; where each u_n is $(u_1^{(n)}, u_2^{(n)}, \ldots, u_m^{(n)})$ and so on. So, there are m of them; m tuples of numbers here, real number, complex numbers whatever, so that is n tuple of it is a vector. So, it is a sequence of vector, ok.

So, this is a notation. Suppose, this sequence converges to u; under this metric, ok. And that u , I can write as $(u_1, u_2, \ldots, u_m)$, ok, a vector, if and only if, each coordinate function here $u_i^{(n)}$ converges to u_i, for all $1 \leq i \leq m$, ok. $1, 2, 3$ up to m, ok. So each coordinate function, converges to a point; you take corresponding vector that will be the limit of the entire sequence here, this is what it is.

So, just like a sequence of complex numbers converges if and only if, each real part and imaginary part you get two sequences, both of them converge. So, that has been generalized here for any $\mathbb{K}^m$ ok.

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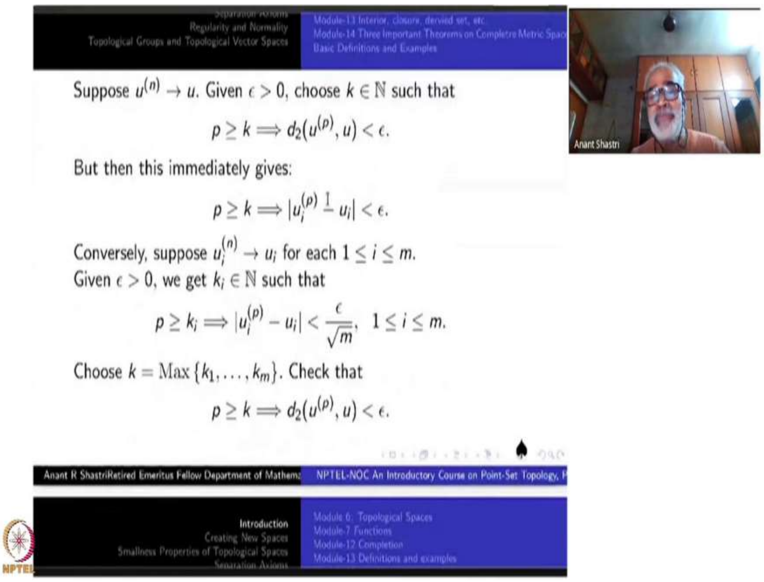

So, let us just go through this one and see why. This is just simple u_n converges to u, ok. Then what does it imply? given $\epsilon > 0$ choose a $k \in \mathbb{N}$ such that $p \geq k$ implies, distance between u_p and u is less than ϵ. What is this distance? This is, the square root of the sum of those squares, ok.

So, that immediately implies that each, each thing here, must be less than ϵ; because this square root of this whole thing is less than ϵ, the square will be less than ϵ^2. So, each must be because these are all non negative numbers. Each must be less than ϵ^2. So, square root should be less than ϵ. So, each $|u_i^{(p)} - u_i|$ must be less than ϵ, ok. So, you have finished, the same p will do the job for all the sequences, ok.

So, what does it mean? u_n converges to u means, the each $u_i^{(p)}$ converges to u_i. So, that is the meaning. Now, for the converse part, you have to work a little harder. So, that is why I have put it here. Conversely suppose, each of them converge, the coordinate functions. Given an ϵ, I should now produce a k such that this happens now, right.

So, what do I do? First of all, what I get is, for each sequence I get a k_i belonging to $\mathbb{N}$ such that, $p \geq k_i$ will imply $|u_i^{(p)} - u_i|$ is less than something, and that something I am choosing not this ϵ, but $\epsilon/\sqrt{m}$. Apply it to the convergence for this number, ok. So, this happens. Now, you choose k to be the maximum of all these $k_1, k_2, \ldots, k_m$ ok. I have each for each i have a k_i, take the maximum of $k_1, \ldots, k_m$. If p is bigger than the maximum, it is bigger than each k_i.

So, all of them will be true, ok. You take the square, that will be less than ϵ^2/m; take the summation there are m of them. So, it will be less than ϵ^2, when you take the square root you will get this one, ok. So, I have, I have not written down that, I have asked you to check this one. But I have told you, how, how it works, alright.

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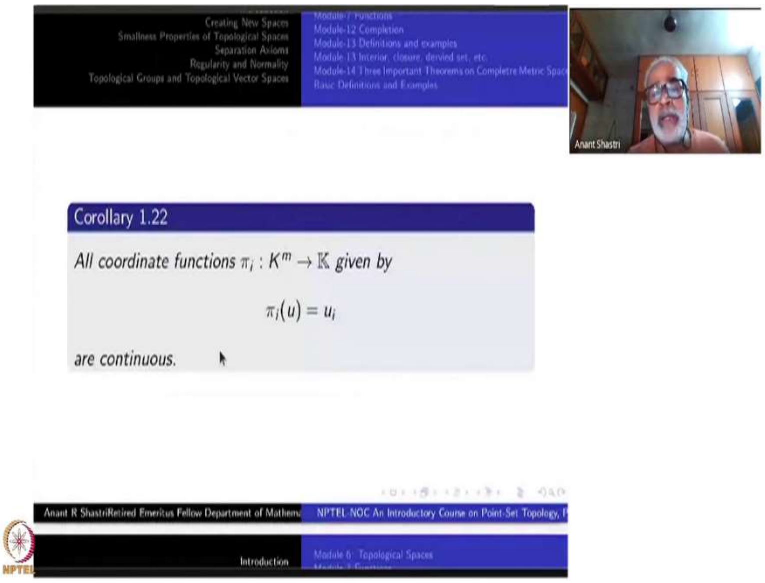

So, as a corollary, what we get is, look at the projection maps π_i, the coordinate projection from $\mathbb{K}^m$ to $\mathbb{K}$, ok.

So, u_i is i^{th} coordinate projection; they are continuous. How do we get it is continuous? If u_p tends to u, $\pi_i(u_p)$ equals to $u_i^{(p)}$ tends to $u_i = \pi_1(u)$. So, that is what we have proved already because this is a part of corollary here, part of that theorem, you do not have to use if and only if, just one part. So, coordinate functions are continuous, ok.

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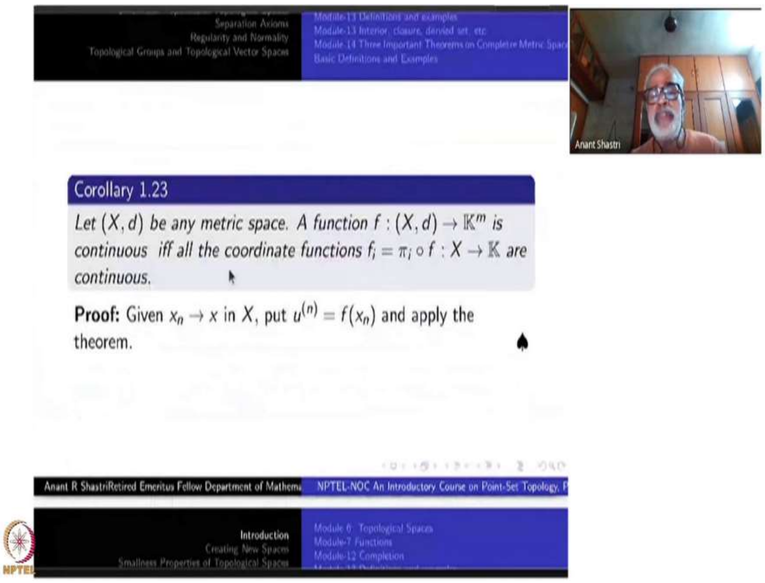

Now, we will improve upon this one. Take a metric space, take a function from X to $\mathbb{K}^m$.

Any function in the product is the same thing as giving you, m functions. An n tuple of, an m tuple of functions $f_1, f_2, \ldots, f_m$, right. So, look at one function, that is gives you m functions here. What are they? They are just $f_1, f_2, \ldots, f_m$ which are just $pi_i \circ f$. First apply f and then take the i^{th} coordinate of whatever you get, ok, that is f_i. Each f_i, so, condition is, if and only if, each coordinate functions, each of them is continuous, ok, all of them are continuous.

So, how do you prove that? Same thing, start with sequential continuity, you see that is easier than, finding $\epsilon - \delta$ and so on. Take x_n tends to $x \in X$ now, put $u^{(n)}$ equal to $f(x_n)$. Then you apply that theorem, ok. $u_i^{(n)}$ will be what? Will be $f_i(x_n)$, alright. So, you see sequential continuity helps you, once that is why we converted this $\epsilon - \delta$ stuff here into sequential continuity in this one, ok.

So, that was the theorem that we had proved last time. So, we can apply that; a function is continuous if and only if each coordinate function is continuous.

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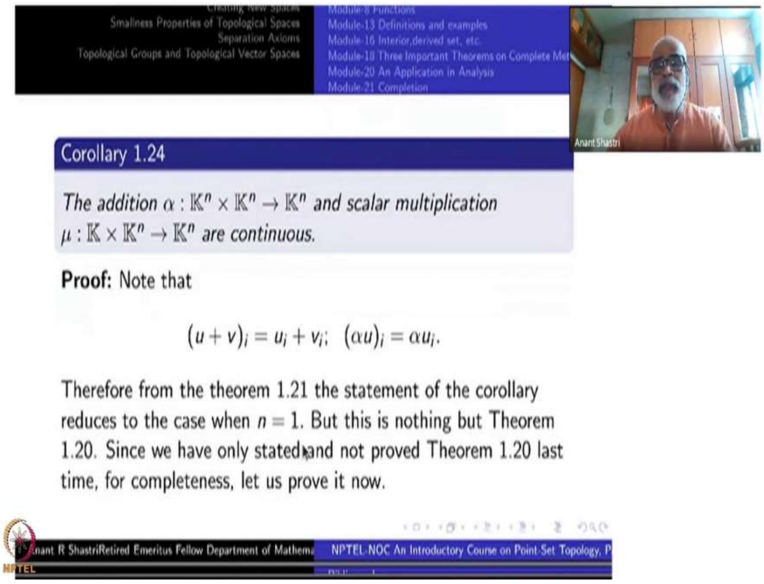

As an additional corollary, let us prove that, the addition in $\mathbb{K}^n$ as well as the scalar multiplication are continuous. Note that, if you take two vectors, u and $v, u+v$, the i^{th} coordinate of $u+v$ is nothing, but $u_i + v_i$, the i^{th} coordinates being added.

Similarly, the scalar multiple αu, where u is a vector; its i^{th} coordinate is αu_i. Therefore, from, our theorem 1.20, the statement of this corollary, reduces to the case when $n = 1$, by taking the coordinates, ok. But, once you put$n = 1$, this is nothing but the theorem 1.20 which we have stated last time. Since we have only stated and not proved it, actually we said that you must have done it in your real analysis course and so on. In any case for the completeness, let us prove it now, ok.

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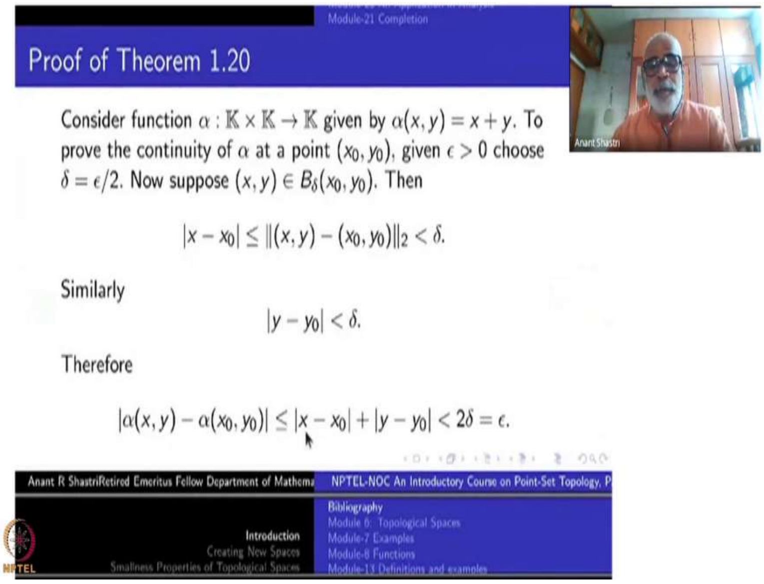

Yeah. Consider the function first, the addition, which I denote by α right. $\mathbb{K} \times \mathbb{K}$ to $\mathbb{K}$, given by $\alpha(x, y)$ is $x + y$. To prove the continuity of α, at any point the (x_0, y_0), given ϵ, choose δ equal to $\epsilon/2$. Now, suppose this (x, y) is in the ball $B_\delta(x_0, y_0)$. Remember what is $B_\delta(x_0, y_0)$ it is all points (x, y) such that their Euclidean distance, from (x_0, y_0) is less than δ, ok.

In particular, if you look at the modulus of $x - x_0$, it is always less than the Euclidean distance of (x, y) from (x_0, y_0). Because this is nothing but, the square root of the sum of $(x - x_0)^2$ and $(y - y_0)^2$. So, it will be smaller than individual ones square and then take the square root. So, this is always true which we have used earlier also.

Similarly, $|y - y_0|$ is also less than δ, right. Therefore, what you get is, $|\alpha(x, y) - \alpha(x_0, y_0)|$ will be less than; what is this one, this is nothing $(x + y) - (x_0 + y_0)$ which is $(x - x_0) + (y - y_0)$. So, modulus will be less than or equal to, $|x - x_0| + |y - y_0|$, each of them is less than δ. Therefore, the whole thing is less than 2δ which is ϵ, ok.

The proof of the multiplication, continuity of multiplication is slightly more complicated, ok.

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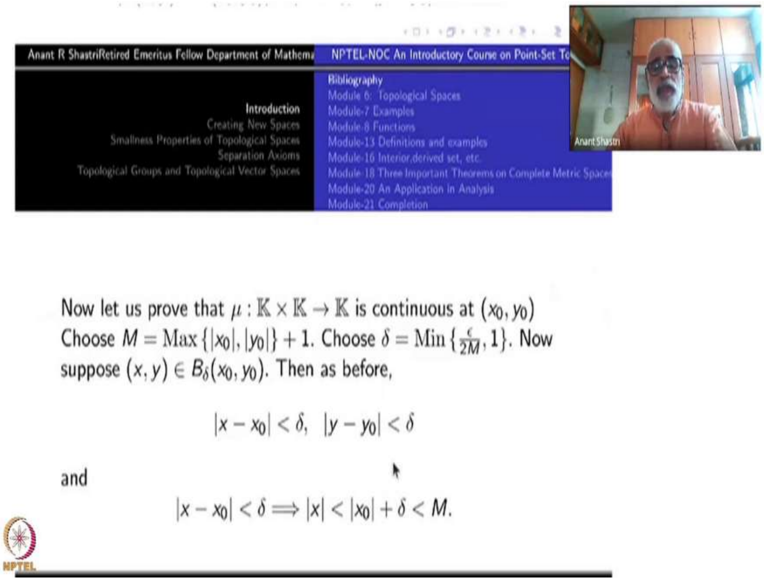

So, let us now go through that. So, I want to prove this multiplication, scalar multiplication on $\mathbb{K}$ is continuous at (x_0, y_0), right. So, given ϵ, I want to choose a δ appropriately. First of all, look at $|x_0|, |y_0|$, take the maximum of the two, ok, add 1, call that number as M; M is maximum of $|x_0|$ and $|y_0|$ plus 1. Then choose δ to be the minimum of $\epsilon/2M$ and 1, ok.

I would like, I want it to be less than $\epsilon/2M$. But I also want it to be less than 1, that is why I am taking the minimum of the two, ok. Now, suppose, $(x, y) \in B_\delta(x_0, y_0)$ as usual. Then I want to show that, α of, sorry, $\mu(x, y) - \mu(x_0, y_0)$ is less than ϵ, that is what I have to show, ok. So, as before, we have, the moment (x, y) is inside this ball, $|x - x_0|$ and $|y - y_0|$ both of them will be less than δ.

Moreover, now $|x - x_0| < \delta$ implies $|x|$ itself is less than $|x_0| + \delta$ and this δ is less than 1, right. So, so this should be less than M, ok. So, this is less than M.

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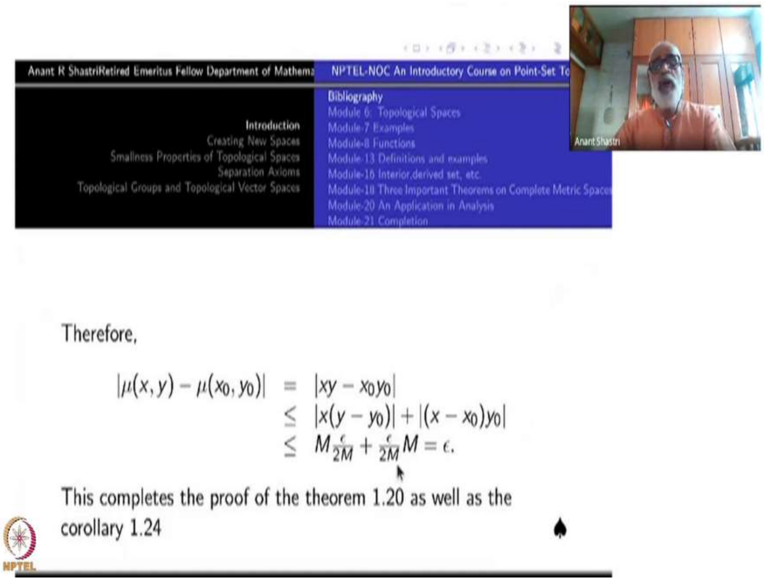

So, if you put all these things together, what you get is now, $\mu(x,y) - \mu(x_0,y_0)$ the modulus of that is modulus of $|xy - x_0y_0|$ we rewrite it as $x(y-y_0) + (x-x_0)y_0; xy_0$ added and subtracted.

So, this is $|x(y-y_0)| + |(x-x_0)y_0|$, ok. And that is less than or equal to, $M\epsilon/2M + \epsilon/2MM$, ok. The M comes out, both for you for both of them here. This is for y_0 and this is for x. So this completes the proof of theorem 1.20 that we had stated earlier and along with that, the corollary 1.24 is also proved, ok. So, we have this very important theorem here, I have generated a corollary that does not matter, ok.

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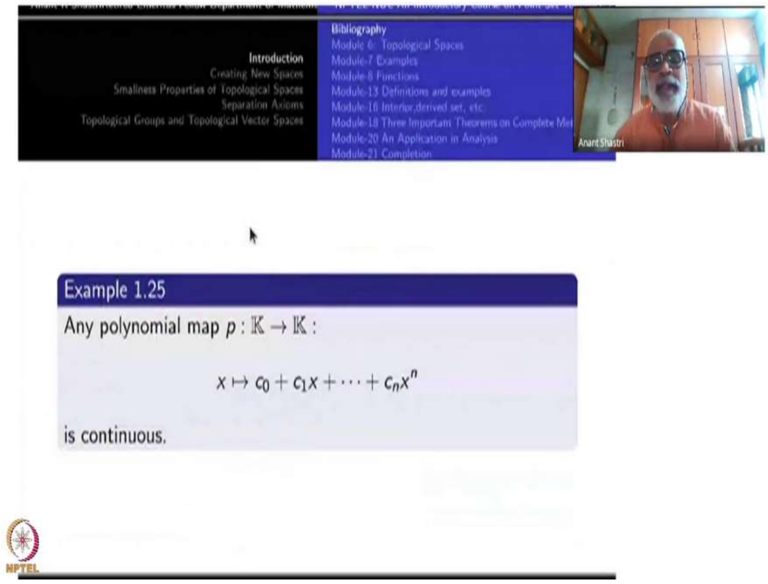

So, as an example of application of this one, any polynomial map p from $\mathbb{K}$ to $\mathbb{K}$, what is a polynomial map? Polynomial is something like $c_0 + c_1x + \cdots + c_nx^n$ where $c_0, c_1, c_2, \ldots, c_n's$ are scalars, right. So, I want to say that, any polynomial map is also continuous; constants are continuous. The first thing I used to use that, apply scalar the, the x going to x is continuous, identity map.

So, multiply the two of them, that will be continuous; which means x^2 is continuous. Multiply again, that to x^3 is continuous; that means, all monomial functions x going to x^n, they are continuous. So, now, we can add two of them at a time. So, c_0 is continuous $c_0 + c_1x$ is continuous. Finitely many of them you have to add, that would be also continuous, ok.

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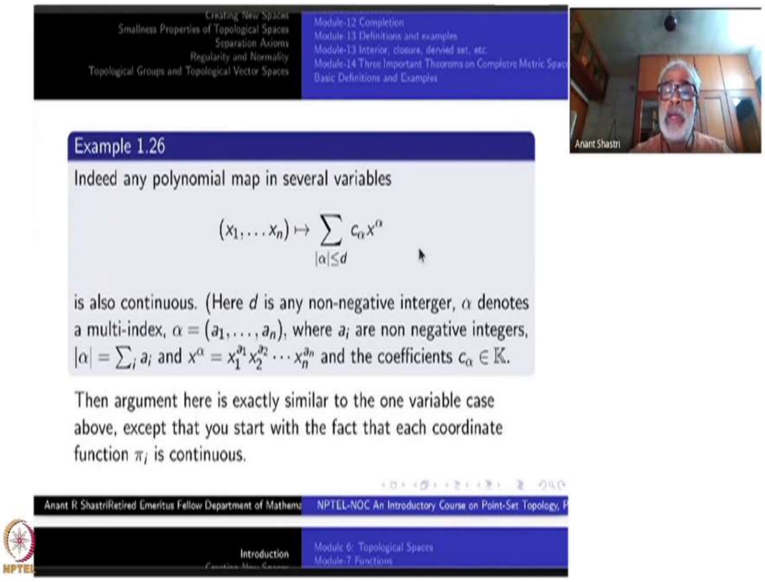

We can go one step further, namely; any polynomial map in any number of variables, finitely many $x_1, x_2, \dots, x_n$ going to summation $c_\alpha x^\alpha$, $|\alpha| \leq d$. This is the notation if d is 0, this is just a constant, there is no choice. If d is 1, you can take x_1, you can take x_2, you can take x_3 and, and all monomials of degree 1 you can take. If degree 2, then you have x_1^2, x_2^2 all of them, not only that, you have $x_i x_j$.

So, those things will be also there. So, how to write all of them. It is a notation here. Take any d, which is a positive integer look at all multi indexes α, which are $a_1, a_2, \dots, a_n$ they are all integers, non negative, ok; n tuples of integers because I have n variables here, ok. Sum total of these, must be less than or equal to d, that is the restriction you are putting because you want to take only finite sum that is all, ok.

Then, what is the meaning of this x^α? It is one single notation for, $x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$. So, that is the notation. Summation of all, $a_1, a_2, \dots, a_n$ that is less than or equal to d, I write as $|\alpha|$, less than equal to d. $|\alpha|$ is this one this is like, your taxicab metric here, ok. And, all the c_α they are inside $\mathbb{K}$, they are coefficients; they may be real or complex, more generally any field will do. But we are concentrating only on real or complex numbers. So, such a polynomial is also continuous, why? Because, first of all, all the x^i's are continuous, this is what we have proved. So, $x_1 x_2, x_2 x_3$ or xy^2 and so, all of them are continuous; multiplying them by constants they are continuous. Then you have to take the sum. So, instead of one variable, you have one more namely, here namely all you have to take various $x_i's$ and then take the product, not just one x and its one powers, ok.

So, that is all the difference. So, here again, the argument is similar, that will give you all polynomials whether one variable, two variable, n variable they are all continuous functions, ok. To be sure, you know that they are also differentiable functions, right. In fact, they are more than that, they are much more than that, they are analytic functions and so on.

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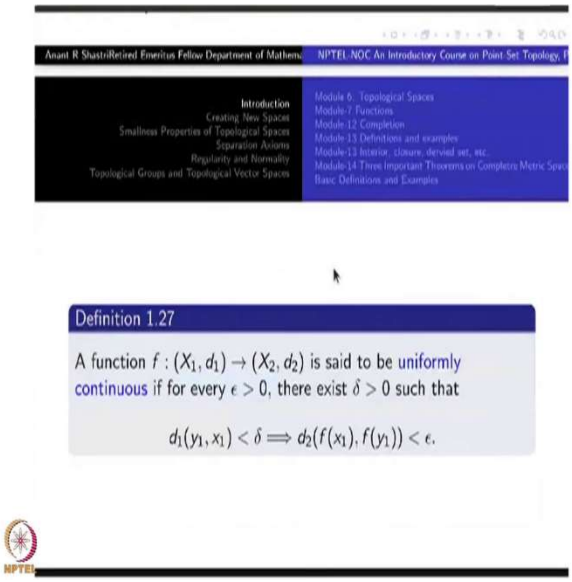

There is one more notion, which we want to recall, once again from, function of one variable, real variable or complex variable.

The definition is exactly the same again. This time, again, you replace the modulus by the distance function, that is all. Take a function from one metric space to another metric space. It is called uniformly continuous, uniform continuity is defined on the entire set A, there is no point in defining at a single point, ok. I can define it on any subset A also, that is possible. But now here, I am defining it for all $x_1 \in X_1$. So, it will be true for any subspaces also, ok.

So, a uniform continuous function, is defined for the entire domain here, ok. Whatever domain you have chosen for the function. What is the definition? For every $\epsilon > 0$, you must have a $\delta > 0$ such that, whenever the points of our domain are close enough, say by δ, $d_1(y_1, x_1) < \delta$, their images must be also satisfy the same relation, but with this ϵ, $d_2(f(x_1), f(y_1))$ must be less than ϵ.

So, you see, there is no reference to, given x and ϵ there exist δ was the definition for ordinary continuity at a point. In this case, there is no reference to a single point here. This delta will just depend upon epsilon, it will serve the continuity for all points, at all points. In particular, uniformly continuous functions are continuous also, in the ordinary sense, that is easy, ok.

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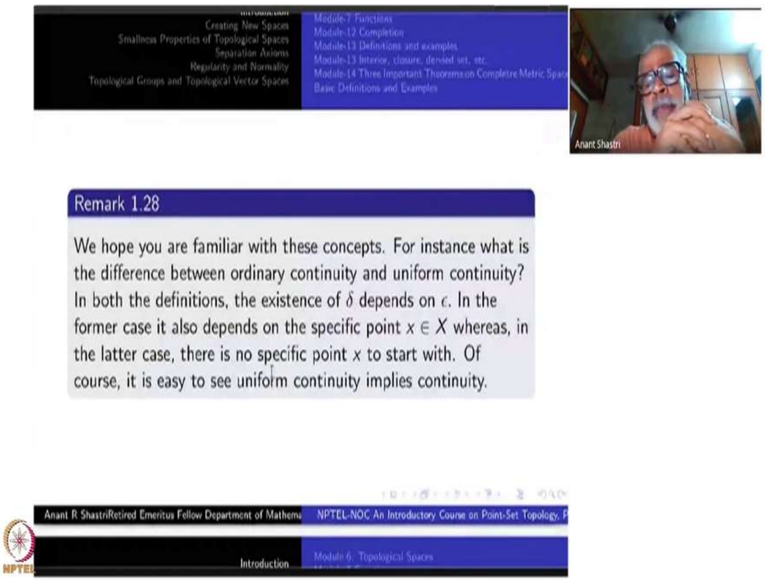

Indeed, I have not said anything other than, replacing the modulus by d_1 and d_2. So, you must be all familiar with these things, ok. So, for instance, what is the difference between ordinary continuity and uniform continuity? In both the definitions, this δ will depend upon ϵ, ok. But, in the ordinary continuity, it also depends because we start with x for each x and ϵ there is a δ ok. In the uniform continuity there is no starting point at all, you do not have to. It starts with for every ϵ, there exists . . ., ok. And then the statement is for all points, all pairs of points. Therefore, uniform continuity implies continuity, there is no problem, ok.

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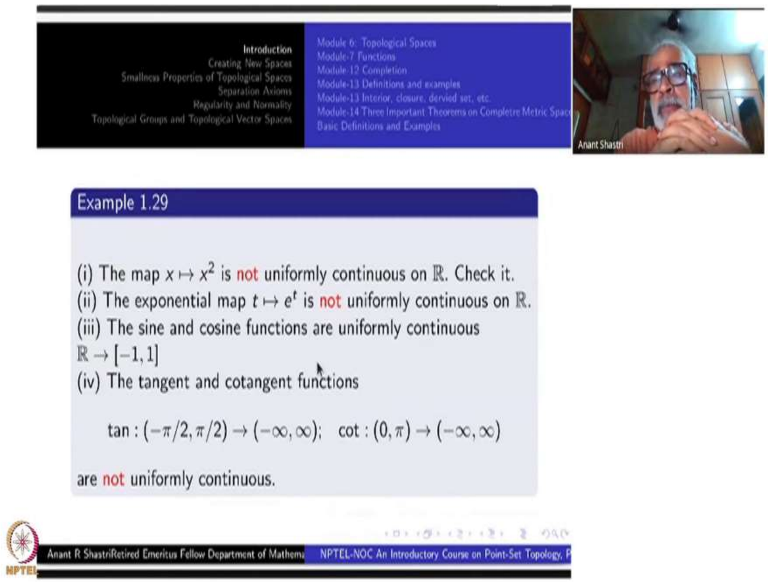

But we will see some examples here, that, generally, ordinary continuous function may not be uniformly continuous. So, the first example is the simplest example, namely $x \to x^2$, on the reals.

You must have checked it, if you have not you should check it right now, alright. Does not take much time. Not uniformly continuous means what again you have to do? Given ϵ, there exist a δ; that is the condition right, for all t should happen. What is the negation of that? There is some ϵ, is that no matter what δ you take, somewhere, something goes wrong. That point you can choose freely here. That is the point, because there is no for every; given x, x is not a starting point or anything, ok.

So, you can take any number, 1 for example, and then show that there is no such δ which will give you the continuity for all the points. So, that is what you have to show. So, having told that much, I will leave it to you. The same thing with $t \to e^t$ is not uniformly continuous, ok. On the other hand, closely related to them are the so called trigonometry function sine and cosine, they are uniformly continuous, ok?

I do not want to reveal it. There are tricks to see, why they are uniformly continuous. I do not want to reveal that to you right now, you think about them, alright. If you have problems with this one, you should not hesitate to contact us on the platform namely, the discussion forum there ok. So, there you can ask, I tried this way, I tried this way, but I am not getting it, please explain. So, our TAs will explain that.

So, let us look at another example here, namely tangent and cotangent functions, ok. They are also not uniformly continuous. So, this time, the domains are of finite domains here, but, but the co-domains are the whole of minus and plus infinity the entire of ℝ, ok. So, entire of ℝ. If these are actually homeomorphisms, some of the mappings here, but they are not uniformly continuous, ok.

So, uniform continuity is something, somewhat funny thing it is not a topological quote and quote topological property at all. So, in topology you would not much see it at all. In general topology, there is no uniform continuity concept, to bring it you have to do something like a metric space, not exactly something like metric space. What are they called? They are called uniform spaces, ok. Because, with them we can talk about uniformly continuous functions.

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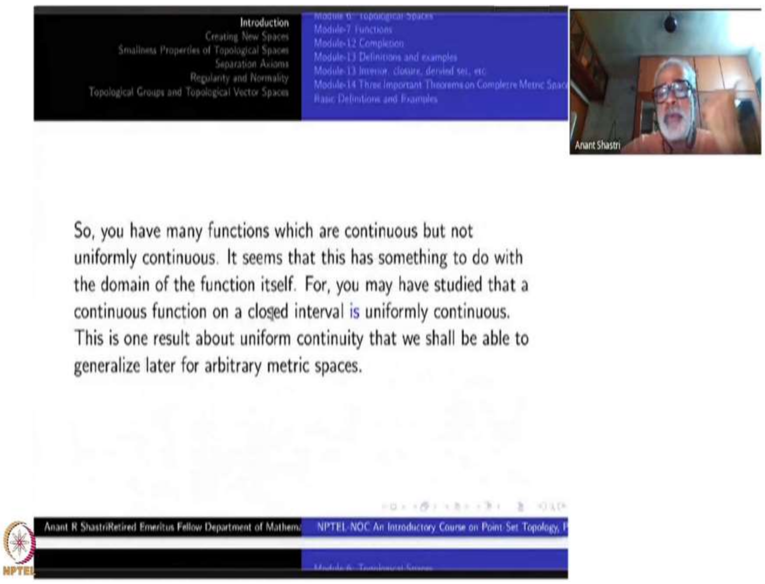

So, you may have many functions which are continuous, but not uniformly continuous, But, look at the thing here, it seems to have something to do with the domain and co-domain ok. So, indeed you have also studied in your real analysis course, namely every continuous function on a closed interval is always uniformly continuous, ok.

So, though I have given you these examples of, not uniformly continuous functions, if you restrict them to any closed interval they will be uniformly continuous. The emphasis here is this is not uniformly continuous on the whole of $\mathbb{R}$. Similarly, $t \to e^t$ is not continuous on the whole of $\mathbb{R}$, that is the point. If you restrict it to closed intervals, they will be uniformly continuous that is one of the theorem, that theorem namely a continuous function on a closed interval is uniformly continuous, that one, we can extend it in some sense and that is, comes back in topology also, definitely in metric spaces later on we will do that.

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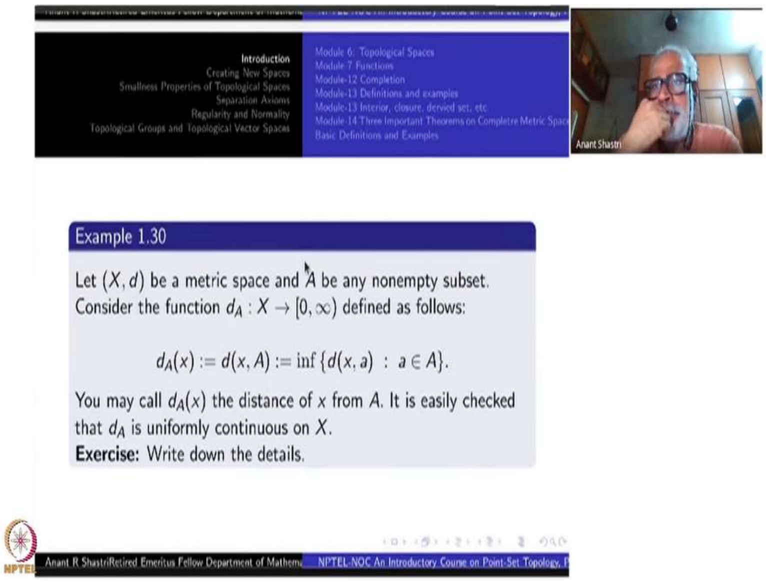

So, here is one example which you might not have seen in your analysis course at all, because this is about metric spaces. Take a metric space, take any non-empty subset, ok. So, the layman's language of distance, I am going to use that. So, I am going to take arbitrary points and then talk about the distance of that point to the set, ok. So, that is what $d_A(x)$ or $d(x, A)$, whatever notation you want to use you can use.

So, this is defined as, infimum of all the distances between a and x, where x is fixed, a varies over A, look at all these, these are all non-negative real numbers, ok. Take the infimum. Why does this infimum make sense, because this is bounded by 0 below. So, infimum may, at worst be 0 or it may be some positive number, it makes sense.

This infimum is called the distance of x from A, ok? So that is the notation $d_A(x)$ or $d(x, A)$. This function, you can easily check that it is uniformly continuous, you do not have to hunt for a δ, it is there already. I have given you a hint. So, write down the details yourself, ok. Any doubts? We will stop here, until next time.

Thank you.