Introduction to Point Set Topology, (Part I)
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Lecture - 49
Examples and Applications

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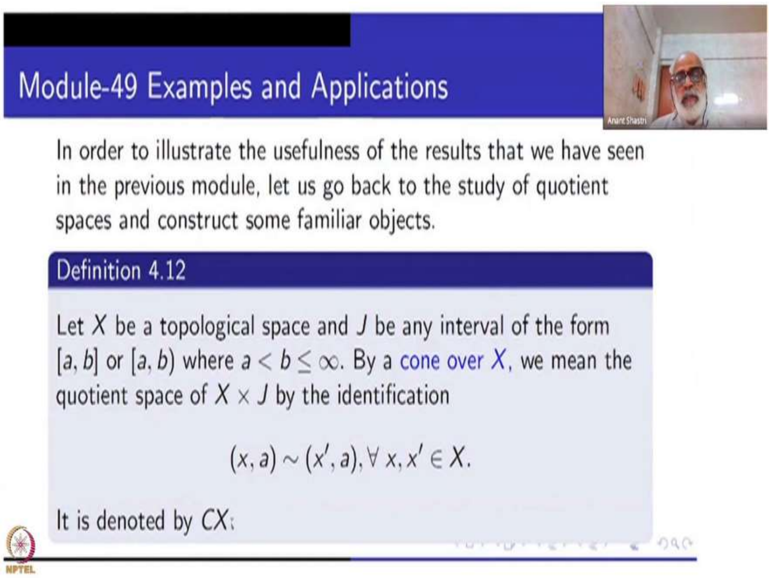

Welcome to module 49 of Point Set Topology course part I. Last time we introduced the notion of Hausdorff spaces and proved some very important theorems. One of them was a characterization of homeomorphisms. Characterization means only in some particular cases that is all. Namely if you have a bijection from a compact space to a Hausdorff space, then it is a homeomorphism iff it is continuous.

So, there are other cases also we have to prove. So, let us try to give some illustrations of usefulness of these concepts and theorems. So, we will go back to the study of quotient spaces now. Let X be a topological space and J be any interval of the form $(a, b]$ or $[a, b)$ and b could be infinity also, ok.

So, instead of writing $(a, b]$ or $[a, b)$, I will just write it as J it will represent any one of them. By a cone over J, we mean the quotient space of $X \times J$ by the identification: (x, a) is identified with (x', a), for every x and x' inside X and no other identifications. That is the meaning of when you define a relation by declaring some rule.

The rule is only this one. After that by reflexivity, transitivity, symmetry etc the relation is completed ok as an equivalence relation. So, in this case you know by reflectivity every point is related to itself. That you have as part of the definition, though it is not stated specifically. Symmetry is already there here and transitivity is also obvious. So, only points $X \times \{a\}$, they are identified to a single point, forming a single class here ok.

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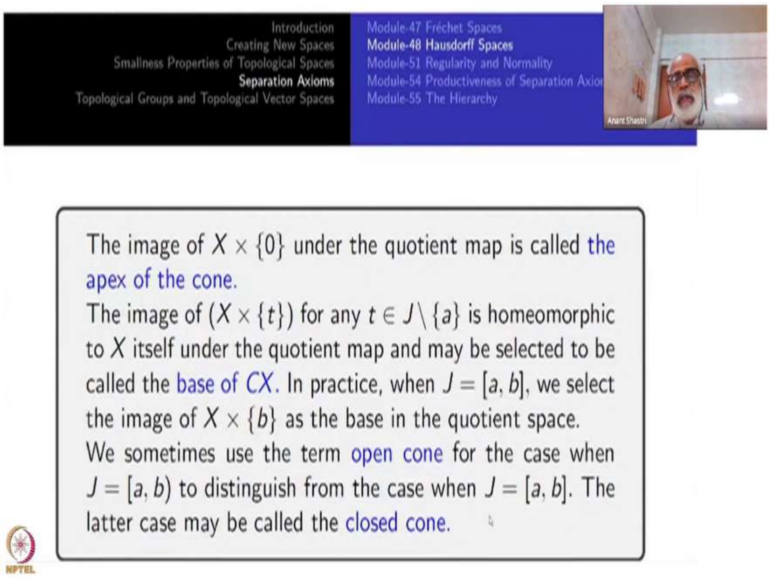

So, that is the meaning of this cone over X, the quotient space will be denoted by CX, ok. The image of $X \times \{a\}$ under the quotient map is called the apex of the cone, ok? A single point representing the entire $X \times \{a\}$. So, that will be called apex of the cone. The image of $X \times \{t\}$ for any $t \neq a$, see take a point t other than $a \in J$, they remain as they are: there is no identification. So, $X \times \{t\}$ to its image is a homeomorphic copy of X, ok? And it will be contained inside the cone under the quotient map. It is again a homomorphism and any one of them may be selected to be called the base of the cone CX.

Why I am saying `selected to be' because there is no definiteness you could have taken any t other than a. So, any one of them you can say is the base of the cone. In practice especially when J is a closed interval $[a, b]$, we take $t = b$ ok. We select $X \times \{b\}$ as the base. So, there is a definiteness ok. However, in the case of open cones namely when b open or b is infinity, then also it is open anyway there is no definiteness.

So, there is also these terminologies open cone and closed cone depending upon whether you have taken a closed interval here or a half-open interval $J = [a, b).J = [a, b)$ corresponds to open cone. An open cone does not have a unique base. You could have taken any t., they are all hoemeomorhic to each other. You know they all serve the same purpose, whereas, in the closed cone you can take the last point here b, then $X \times \{b\}$ will be the base.

The prototype of a cone is when X itself is a circle. You must have seen all the pictures of right circular cone and so on. In your twelfth standard you studied the cones also ok, conic sections and so on. So, the definition is generalized. Instead of X being a circle you can take any topological space and do this, ok?

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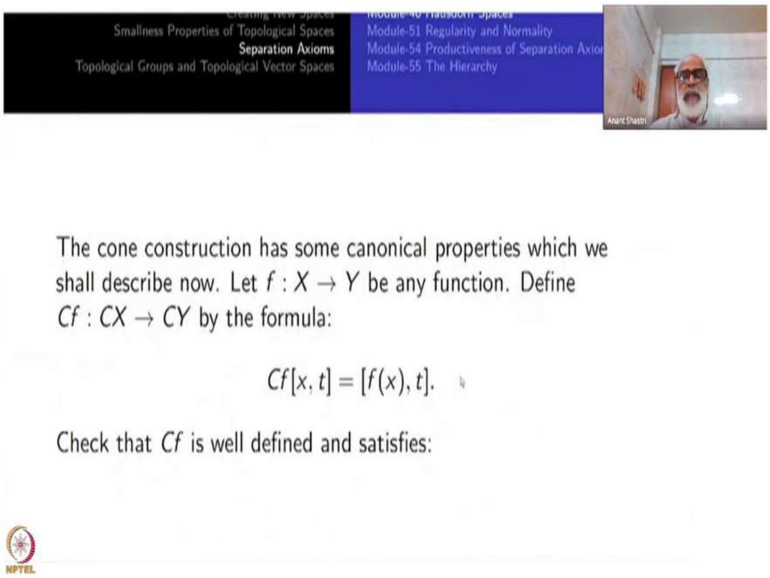

So, cone construction is one of the very important thing, especially in algebraic topology also in general topology ok. It has some canonical properties which we shall describe now. What is the meaning of this canonical property. Suppose you have a function from X to Y, any function. Then you can define Cf from CX to CY. So, you have defined a cone over a space, now I am defining the cone over a function here, cone of Cf.

So, what is Cf? Cf is a map from CX to CY. First, you define it from $X \times J$ to $Y \times J$ to be $Cf(x,t)$ equal to $(f(x),t)$ and then take the classes in both domain and codomain, ok? Here also you have take the class of (x,t) going to the class of $(f(x),t)$. Remember this class is the same point (x,t) unless t is the first point a and when it is first point a, all these (x,t) represent the same point, but here also they will represent same point. So, there is no problem here to in the definition of this function Cf, ok.

The point is if f is continuous, then f cross identify is continuous and hence the induced map Cf will be also continuous. Not only that, there are many other properties I want to list them, Cf has these properties.

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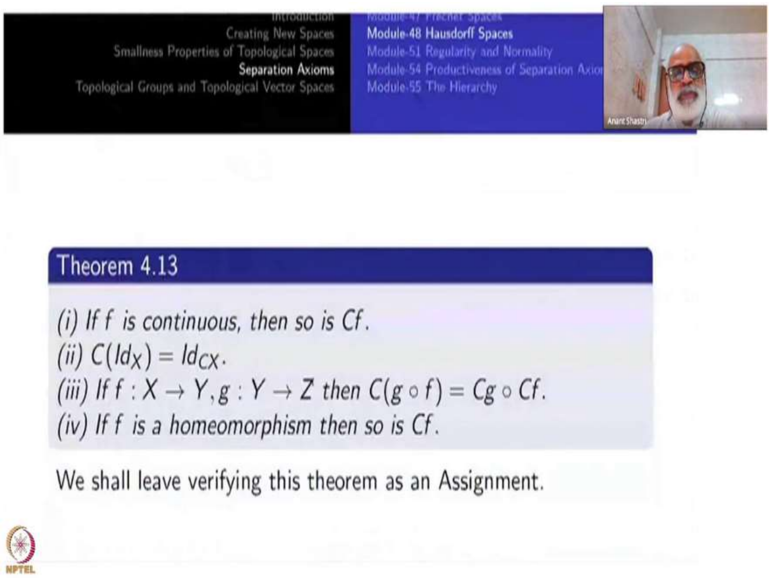

If f is continuous, then so is Cf. If you start with the identity map from X to Y, C of the identity is identity of CX. If you have f from X to Y and g from Y to Z, then the cone over $g \circ f$ is nothing but cone over g composite cone over f.

If f is a homeomorphism then so, is Cf ok.

So, (i), (ii), (iii) are all easy. (iv) follows from (iii) by taking f from X to Y and g equal to f inverse. If you put f inverse from Y to X, then $C(g \circ f)$ is identity. So, that is identity on this side also which means Cg is the inverse of Cf. Therefore, Cf will be a homeomorphism ok.

So, these things are easy to verify, there is no problem about them, ok. These are called canonical properties.

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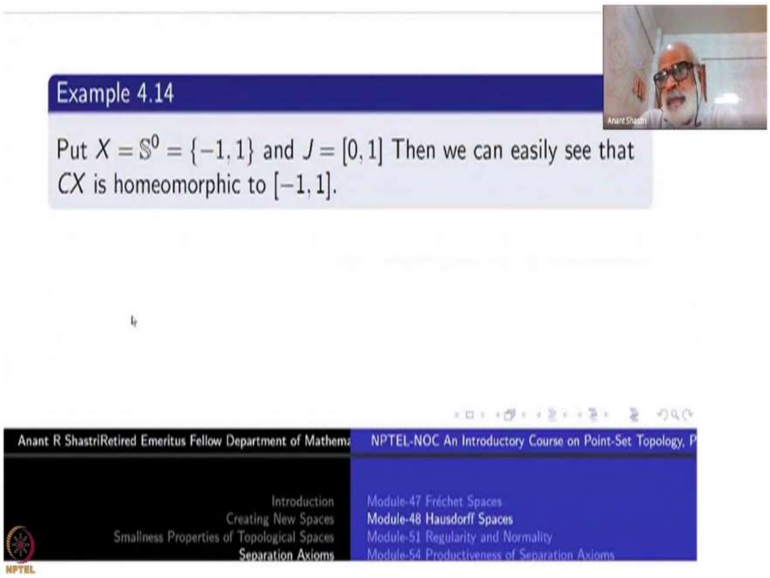

Now, I come to some specific examples. Suppose you start with $\mathbb{S}^0$ that is the 0-dimensional sphere which is just unit vectors inside $\mathbb{R}$ namely, $[-1, 1]$. The topology is discrete topology on that. What will be the cone over that? To be very specific now, I take J equal to the closed interval $[0, 1]$, ok. What will be the cone of $C(\mathbb{S}^0)$?

It is easy to see this. First you take $\mathbb{S}^0 \times J$. So, that will consist of two copies of the interval closed interval $[0, 1]$. $(\{-1\} \times [0, 1]) \cup (\{1\} \times [0, 1])$. They will be disjoint copies right, but when you carry out the identification, which one what is the identification? I have to take $a = 0$ here, right. So, $(-1, 0)$ will be identified with $(1, 0)$. Nothing else will be identified. That means what?

The bottom points of both these are $(0,-1)$ and $(0,1)$, when you hold them vertically, ok? So, -1 copy is there and 1 copy is there. Those -1 and plus 1 cross $0, 0$ they will come together. So, you will get a V shape right? But V shape is homeomorphic to the interval -1 to 1, open or closed interval as the case may be. So, this is the simplest case as such. I have already told you that if you take X as a cicrle, then the cone over that one will look actually an ice cream cone or a funnel and so on, ok?

If you flatten it out what you get is a disc, the 2-dimensional disc. So, that is the point. I will explain this a little more clearly, in more general generality, for all n.

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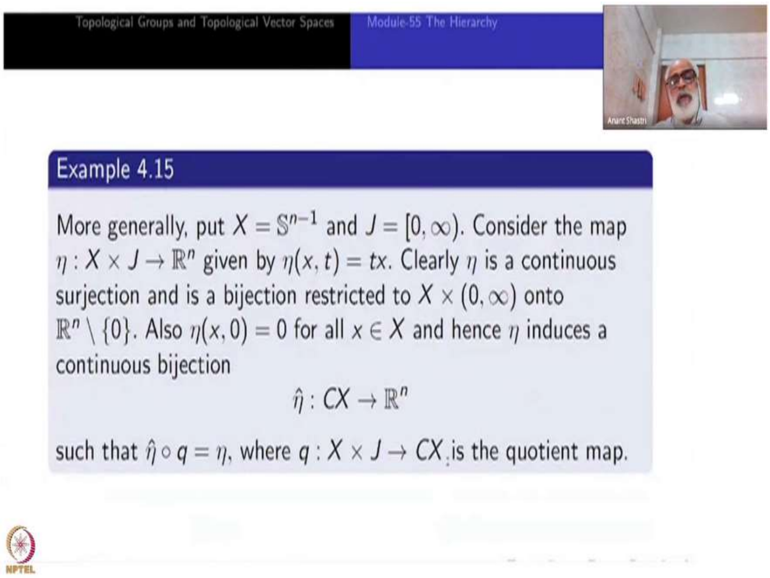

So, more generally, put X equal $\mathbb{S}^{n-1}$. So, $n = 2$. this will be just the circle, but I am now considering the general case, all of them together. $n = 1$ is over and $J = [0, \infty)$. Instead of the closed interval I take $[0, \infty)$. Consider the map η from $X \times J$ to $\mathbb{R}^n$ given by $\eta(x, t)$ goes to tx.

So, this is a unit vector I am multiplying by a scalar the scalar varies from $[0, \infty)$. Clearly this multiplication map is a continuous surjection. Every point inside $\mathbb{R}^n$ take a vector divided by its norm that will be unit vector that will be inside $\mathbb{S}^{n-1}$. What is the corresponding t? It is just the norm you multiply it by the norm you get back the vector V, ok?

So, this is surjective map ok, it is a bijection if you take nonzero vectors and do not take t equal to 0; that means, on the open interval $[0, \infty) \times X$ is a bijection and where does it go? What is the image? Image is $\mathbb{R}^n 0$ ok. So, this is precisely what you call as polar coordinates in the case of $n = 2$. At 0, you know 0 comma any vector v that will represent 0 into V which is just 0.

So, there is that 0 point is over represented ok. There are too many points which represent that point, but everywhere else there are unique representations. So, $\eta(x, 0) = 0$ for all $x \in X$ and hence η induces a continuous bijection when you pass on to the cone. See whatever η was taking? Several points, namely $(x, 0)$, they are all identified in CX to a single point.

Therefore, the induced map $\hat{\eta}$ from CX to $\mathbb{R}^n$, this is injective map also. It is already surjective this η is surjective. So, $\hat{\eta}$ is also surjective. So, this is a bijection ok. It is a continuous bijection by the very definition. How to check continuity on quotient spaces because η is continuous. So, if you write the quotient map q as $X \times J$ then $\eta = \hat{\eta} \circ q$.

So, it is continuous means $\hat{\eta}$ is continuous ok. So, $\eta = \hat{\eta} \circ q$, q is is the quotient map that is what I have got. So, why this $\hat{\eta}$ is a homeomorphism, right? If you had this domain were compact, the codomain is anyway Hausdorff, then you were done, right. In particular instead of taking infinity here suppose I take a closed interval $[0, 1]$, $[0, b]$ whatever b positive ok, then I would get a homeomorphism, but I will not get the whole of $\mathbb{R}^n$. What will I get?

Suppose, I take $b = 1$, then I get all vectors of length less than or equal to 1 which is nothing, but the disk $\mathbb{D}^n$. If I take r here, I get all vectors of length less than equal to r which is the closed ball $B_r(0)$, right. So, our theorem already says that all these balls are cones over what are the bases there? What is the last thing when when this then the second coordinate is equal to r that will be the sphere of different radius not $\mathbb{S}^{n-1}$.

$\mathbb{S}^{n-1}$ you get only when $r = 1$ ok, I want to say that even this infinite cone CX is homeomorphic to the entire $\mathbb{R}^n$, ok. So, there are different ways of seeing this one. The point is continuity of the inverse has to be established. This part ok. If you throw away the 0 from $\mathbb{R}^n$ and the apex from the cone, then we have the inverse of $\hat{\eta}$ given by a formula, because $\eta(x, t) = tx$ is unique.

So, how do you get the inverse of $\hat{\eta}$? You see it is just the first coordinate is a unit vector v divided by $\|v\|$, the second coordinate is just $\|v\|$. So, both of them are continuous there. I can divide by $\|v\|$ because v is a nonzero vector. But if this is the 0 vector you cannot do that. So, there is a doubt when you extend it to CX why this is continuous ok sorry this continuity is ok why the inverse is continuous. That needs to be justified.

So, let us prove that one rigorously once for all so that you would not have any doubt left in your mind.

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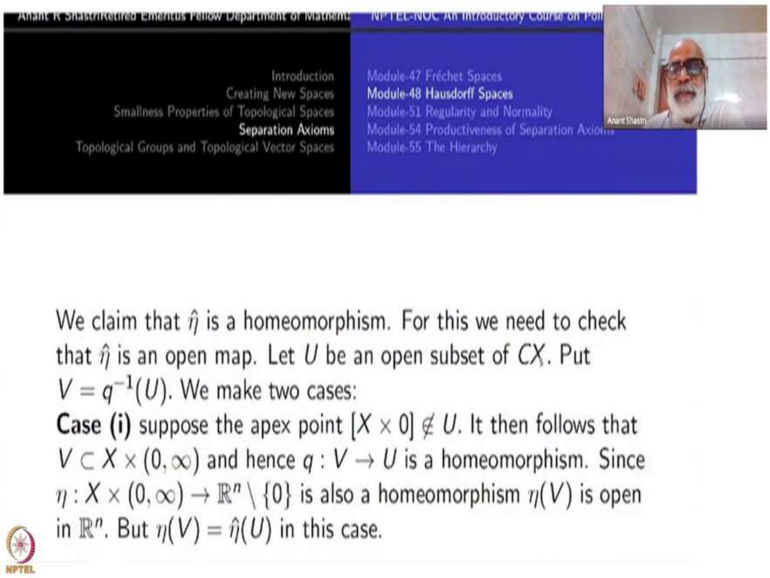

So, we claim that η is a homeomorphism. For this we need to check $\hat{\eta}$ is an open mapping (or a closed mapping) ok? Let U be an open subset of CX ok. Put V equal to $q^{-1}(U)$. See I start with a subset here, but then I go to $\mathbb{S}^{n-1} \times J$, via q inverse of that set. I am coming to $X \times (0, \infty)$ ok? $(0, \infty)$ here I am coming ok that is my V; V equal to $q^{-1}(U)$.

We make two cases. Suppose, the apex point $X \times \{0\}$ is not in U. That just means that V is completely contained inside $X \times (0, \infty)$ and q from V to U is a homeomorphism. This case is very easy. This we have already analyzed. Since $X \times (0, \infty)$ to $\mathbb{R}^n \setminus \{0\}$, η is a homeomorphism $\eta(V)$ is open ok and in this case $\eta(V)$ same thing as $\hat{\eta}(U)$ ok.

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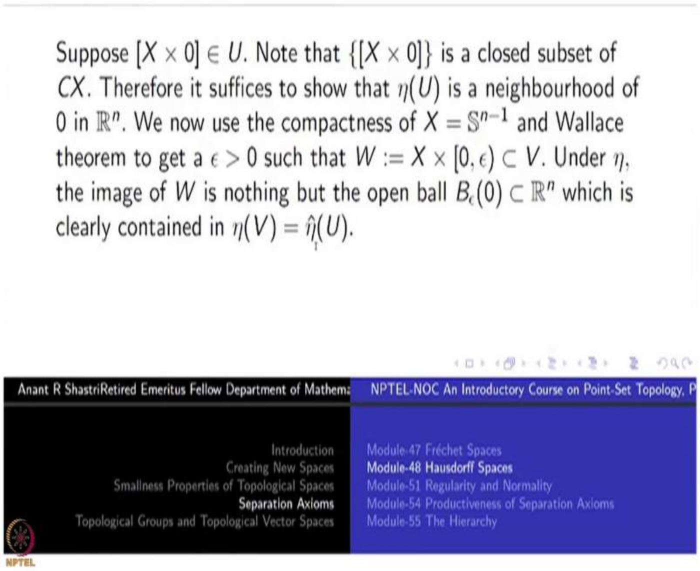

So, this part is ok. Second case is the important case; namely, suppose the apex point is inside U.

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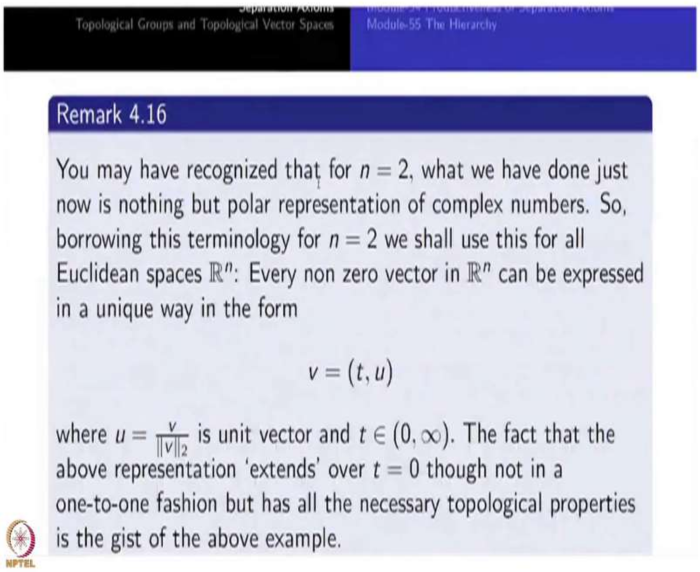

The singleton $X \times \{0\}$ is a closed subset of CX, ok. Why? Because the rest of them is open that is easy to see or inverse image of 0 under the projection map is just $X \times \{0\}$ the entire set; something is open something is closed. So, X cross singleton is closed in $X \times J$.

Now, it suffice to show that $\eta(U)$ is a neighborhood of 0 in $\mathbb{R}^n$. Under η the apex is mapped to 0, right? So, rest of them is no problem. So, why it is a neighborhood of 0 in $\mathbb{R}^n$? We now use the compactness of X which is equal to $\mathbb{S}^{n-1}$ and Wallace theorem to get an ϵ positive such that this W is equal to $X \times [0, \epsilon)$ is contained inside V.

See you have $\mathbb{S}^{n-1}$ which is compact cross $[0, \infty)$ you have ok. So, this 0 is your y and then use Wallace theorem. If you have an open subset cross some y, you have this neighborhood V of X cross 0 then you have a neighbourhood of 0 in [0, infinity) which is nothing but [0, epsilon) so that $X \times [0, \epsilon)$ is contained inside V. $X \times \{0\}$ contained inside V, for some > 0.

So, this is the Wallace theorem applied to $X = \mathbb{S}^{n-1}$, ok? Under η the image of W is nothing, but the open ball $B_\epsilon(0)$. This is a unit vector I am multiplying it by some number between 0 and ϵ. So, it will give you a vector of length less than or equal to less than ϵ.

All vectors less than ϵ are inside this one. So, this is an open ball, the image of W. This one is an open ball contained in $\eta(V)$, which is clearly equal to $\hat{\eta}(U)$ ok. So, once it is inside V, η of that one will contained inside V which is $\hat{\eta}(U)$. So, starting with an open subset U inside CX we have shown that $\hat{\eta}$ of that U is open ok. So, open bijective continuous map is a homeomorphism so.

Student: Sir.

I am making this remark which I have already told you. Yeah?

Student: Would you please explain it again how Wallace theorem is applied there.

Wallace theorem! You have to remember what is Wallace theorem. If you have a topological space X which is compact and any other space Y ok then you look at $X \times \{y\}$ inside $X \times Y$.

Suppose it is contained in some open subset V, $X \times \{y\}$ contained inside V, ok? Then there is a neighborhood of this y ok one single neighborhood of U this $y \in Y$ such that $X \times U$ is contained inside V.

Student: Yeah. So, here that neighborhood is 0 to x?

$[0, \epsilon)$. Because Y is $[0, \infty)$ here. This space Y here I am taking it to be $[0, \infty)$. So, how do you get a neighborhood whatever neighborhood of 0 you take, it will contain will contain $[0, \epsilon)$, ok.

Student: Yes, sir.

So, I have already told you that when $n = 2$, this is the popular representation of $\mathbb{R}^2$ or complex numbers in polar coordinates: take any unit vector in the complex numbers, you can write it as $e^{2\pi it}$. But if you are working in $\mathbb{R}^3, \mathbb{R}^4$ and so on you cannot write that way you just write your unit vector.

So, then this is also polar coordinates inside $\mathbb{R}^n$. Every nonzero vector is uniquely written as what? $v = tu$, where u is a unit vector what is this u? This u is nothing, but $v/\|v\|$. What is t? It is norm of v. as soon as the vector $v = 0$, you have to take $t = 0$, but u could be anything. So, that much of ambiguity is there in polar coordinates, but if you think of this as a cone there is no ambiguity.

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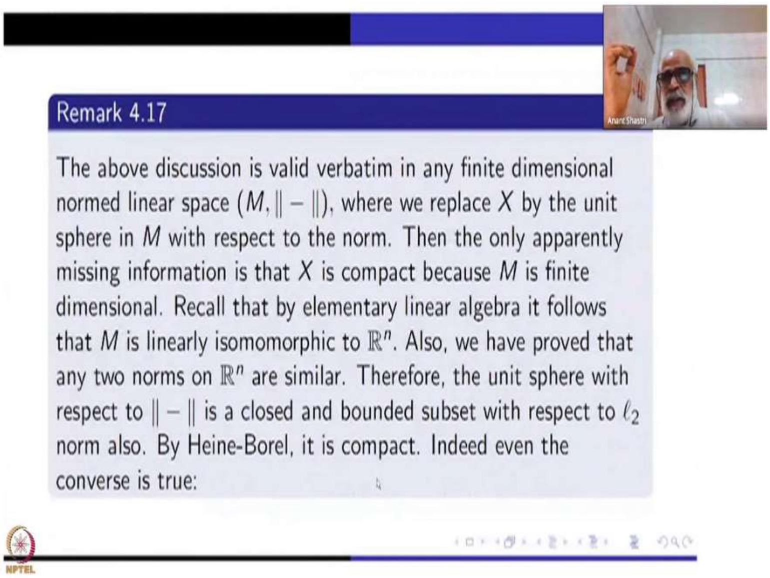

So, now, I want to do one more serious thing here. The above discussion is valid verbatim in any finite dimensional normed linear space $(M, \| - \|)$. So, we are doing it in $\mathbb{R}^n$, but the same thing holds for every normed linear space which is finite dimensional, where we replace X by the unit sphere in M with respect to the norm.

If the norm is ℓ_2 norm, you will get the standard sphere. If it is ℓ_1 what you get? You get a diamond shape? If it is ℓ_∞, you get a square and so on, right. So, all these things we have seen. Even if it is any arbitray norm, not necessarily, one of these ℓ_p's, this statement is true is what I am claiming. What is the missing part? The only missing part is that perhaps compactness of X; X is what? The unit sphere. Why X is compact? Because M is finite dimensional we want to say that X is compact.

So, you might have already seen this one, but I will complete this one this argument. First of all recall that by elementary linear algebra it follows that any finite dimensional linear space is isomorphic to some $\mathbb{R}^n$, where n is the dimension. Also we have proved that any two norms on $\mathbb{R}^n$ are similar ok? Similarity preserves buondedness and closedness.

Therefore, this norm whatever norm, I do not know its form, the unit sphere with respect to this norm is closed and bounded subset with the ℓ_2 norm also, alright. In the ℓ_2 norm, a closed and bounded subset, by Heine Borel theorem, is compact. So, there is nothing missing there. An apparently missing information, but on a second thought everything is available.

Therefore, what we have is that whatever analysis we did, namely C of X is homeomorphic to the whole of $\mathbb{R}^n$ that is valid inside any finite dimensional normed linear space.

Now, we come to the converse of this one, namely, if the sphere is compact in a normed linear space M ok, then what? Then M itself is finite dimensional.

So, that is the next theorem that we are going to prove. This is a standard result in function analysis ok.

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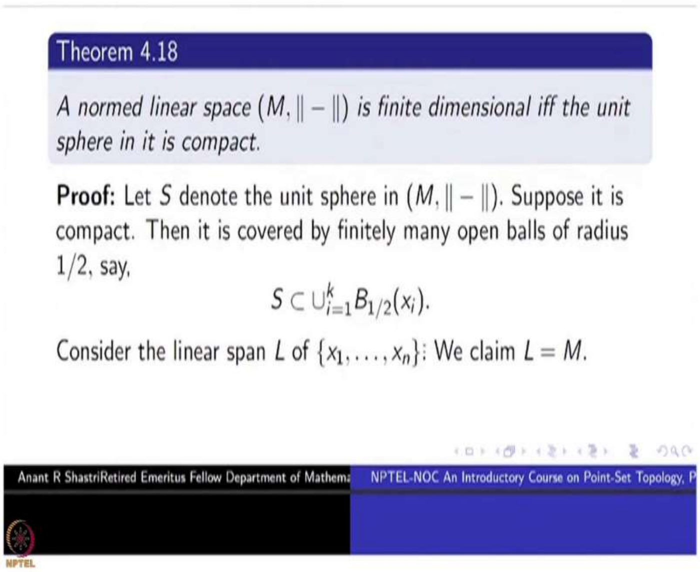

So, let us prove this one as an application of whatever we have done so far. A normed linear space $(M, \| - \|)$ is finite dimensional if and only if the unit sphere in it is compact. So, let us denote by S the unit sphere. Suppose, it is compact, then you can take balls of radius half around each point that will be an open cover. So, that should admit a finite cover.

So, there will be finitely many points $x_1, x_2, \ldots, x_n$ inside S such that S is contained inside union of all the balls of radius half centered at x_i where i ranging from 1 to k. I have got some points inside a vector space M. So, I can take the linear span of them, right. Let L be the linear span of $x_1, x_2, \ldots, x_n$. So, this is a linear subspace of M. We want to claim that L is equal to M.

If we prove this one, then it follows that M is finite dimensional dimension may be equal to n or less because these may not be independent. So, why L is equal to M? So, the proof here is a very important the step, I would not have much time to spend on that one, but in functional analysis they do much more elaborately about first order quadratic approximations and so on.

So, all those things will be easy for you once you learn what is going on here.

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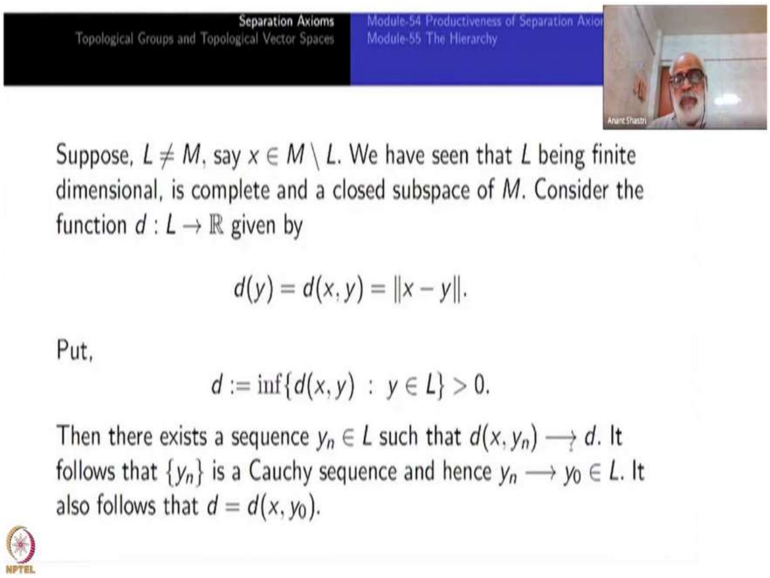

Namely, suppose you have a proper subspace, (this part itself is an independent result now). Suppose L is not equal to M that is L is a proper subspace ok? Say x is in the complement of L. I want to find the nearest point to x inside L. So, that is the approximation, nearest point ok. So, I do not want to elaborate that one.

So, what we want to do is L being a finite dimensional subspace ok? It is a complete and a closed subspace of M. This part we have seen ok? Every finite dimensional normed linear space is complete and once it is complete it will be automatically closed subspace of M, ok.

So, consider the distance function given by the same norm nothing else, d from L to $\mathbb{R}$ given by $d(y) = d(x, y)$. Here x is fixed. So, this is a continuous function it is just equal to norm of $x–y$, ok? Take d to be the infimum of all these $d(x, y)$'s where y ranges over L, ok. So, why infimum is makes sense? First of all see the distance to some point is already finite. So, that is fine right. So, all they are all bounded fine so, this non-empty and so on. Also, the distance function is always bounded below by 0. So, infimum makes sense.

Why infimum is positive? If the distance is 0, ok then we know that x will be inside L because L is closed. So, distance must be positive ok. So, this d which is infimum of $d(x, y)$ is actually the distance of x from L ok? I have just recalled this one here.

So, that is positive because L is closed and x is outside L. So, then there exists a sequence $\{y_n\}$ inside L ok such that distance between x and y_n converges to d. What is d? d is infimum. by the definition of infimum you must have points here converging to that point, right. So, those points are nothing, but $d(x, y_n)$ they are real numbers $d(x, y_n)$ converging to d ok.

It follows that $\{y_n\}$ is a Cauchy sequence. See $d(x, y_n)$ converges these are real numbers, but this implies $\{y_n\}$ is a Cauchy sequence ok, but $\{y_n\}$'s are inside L which is complete. So, $\{y_n\}$ converges to some y_0 belonging to L. This y_0 is the point which realizes this distance.

It follows that d is equal to $d(x, y_0)$. The limit will be $d = d(x, y_0)$ ok. When you take the limit $\{y_n\}$ converges to y_0.

So, infimum is actually minimum and it is attained, alright? So, in fact, now you see that this y_0 as such may not be unique in general, but in this case because L is a linear space it happens to be unique also, but we are not interested in that part right now.

So, we have got a point such that d equal to $d(x, y_0)$, which is the infimum.

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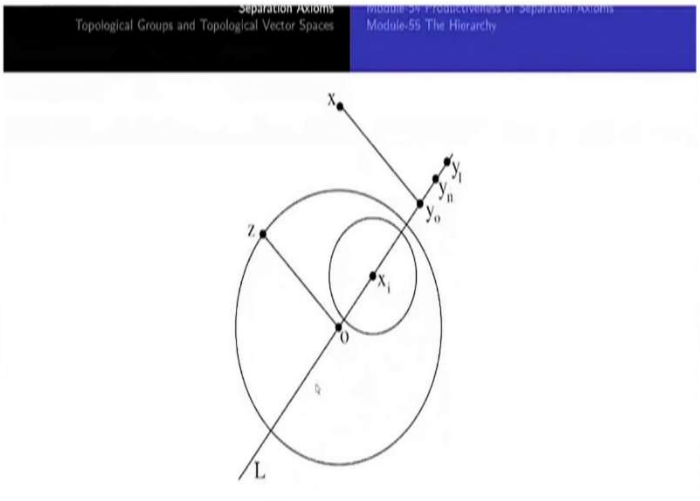

Figure 17: M is finite dimensional iff the unit ball is compact

So, here the picture. So, what I have taken? This is L, I am assuming that it is smaller than the whole space M. It is not the whole space, this is the origin ok. This is some ball of maybe positive radius whatever, alright. This is my x and on L, I have located y_0.

So, these are $y_1, y_2, \ldots, y_n, \ldots$ converging to y_0 ok, it turns out to be if you know what is the meaning of perpendicular and so on it happens to be like that, but this happens only in an inner product space. If you have just a normed linear space you do not have the notion of angle. So, there is nothing like perpendicular and so on.

So, you do not have to draw these pictures perpendicularly and so on ok to be very precise, but what you can do is you look at $x - y_0$ that is a nonzero vector, right? You divide by it is norm that will be a point here on the unit sphere ok. So, this z is nothing, but $x - y_0$ divided by its norm ok? So, that will be a point of unit sphere. So, this is what I am doing here ok in the next slide.

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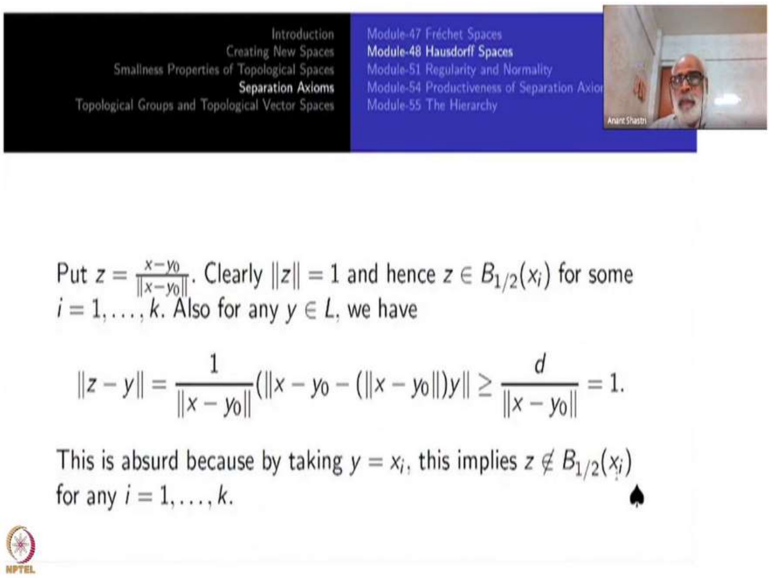

Put z equal to $x - y_0$ divided by its norm, then norm of z is 1. Therefore, z is one in one of the $B_{1/2}(x_i)$'s right? it must be in one of the $B_{1/2}(x)$, because the entire of the sphere S is covered by these balls for some i equal to 1 to up to k. Also for any $y \in L$, we have norm of $z - y$ you can rewrite it as z is $x - y_0$ divided by norm of $x - y_0$, right?

So, here is $x - y_0$, but I am pulling out the denominator outside ok. Then I have to multiply by that number $x - y_0$ times y, but now y minus this one is a linear combination of y_0 and y_0 as well as y are inside L. So, this whole thing is inside L right? $x - y_0$ plus this thing. So, that is an element of L. Therefore, this norm minus this one must be bigger than or equal to d.

And, then divide by this, this denominator is there, d divided by norm of $x - y_0$, ok. So, that is equal to 1 because d is nothing but the norm of this is equal to norm of $x - y_0$(distance between x and y_0), ok. So, what we have shown is that for every point $y \in L$, the distance between z and y is bigger than equal to 1. Now, that is the absurd because all these x_i's are inside L, right?

We have chosen those things inside L. So, that is the problem. So, that is why this picture is funny because it is absurd picture. So, this one of these balls should contain z, but they are all of half radius, but this is this z is away from all of them ok. So, this is the contradiction. So, that contradiction because L, it is not the whole space.

So, once again this method of minimizing this norm etcetera is important elsewhere, but we have used it to prove that compactness of the unit sphere implies that the normed linear space is finite dimensional, ok? So, next time we will continue with applications of this nature and more and more examples we will study.

Thank you.