Introduction to Point Set Topology, (Part I)
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Lecture - 48
Hausdorff Spaces

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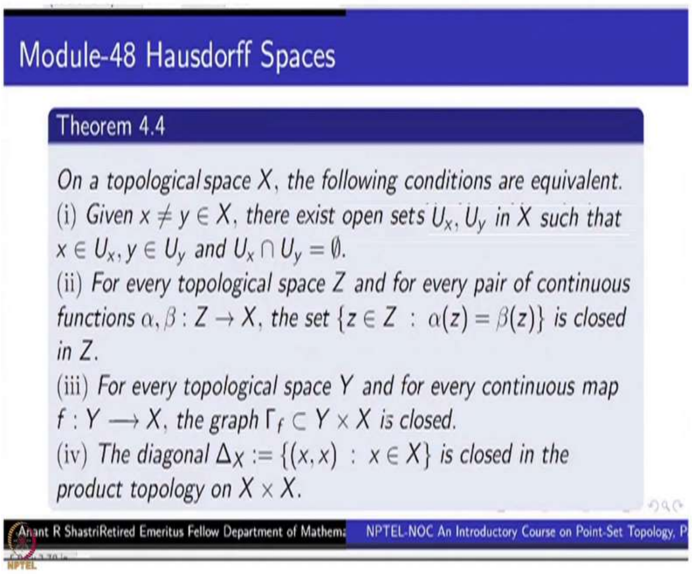


Welcome to module 48 of Point Set Topology course, Part 1. So, today we will continue the study of largeness properties, one property we have studied, namely Frechet spaces. Now today it will be Hausdorff Spaces. Once again similar to what we did for Frechet spaces, but this time only four equivalent conditions. So, I will make a statement on a topological space X, the following conditions are equivalent.

Given two distinct points x and y there exist open sets U_x, U_y in X such that x is in U_x, y is in U_y and $U_x \cap U_y$ is empty. So, this is also stated like every pair of distinct points can be separated by open sets. 2nd condition is: for every topological space Z and for every pair of continuous functions α, β from Z to X, the set z belonging to Z such that $\alpha(z) = \beta(z)$, is a closed subset of Z.

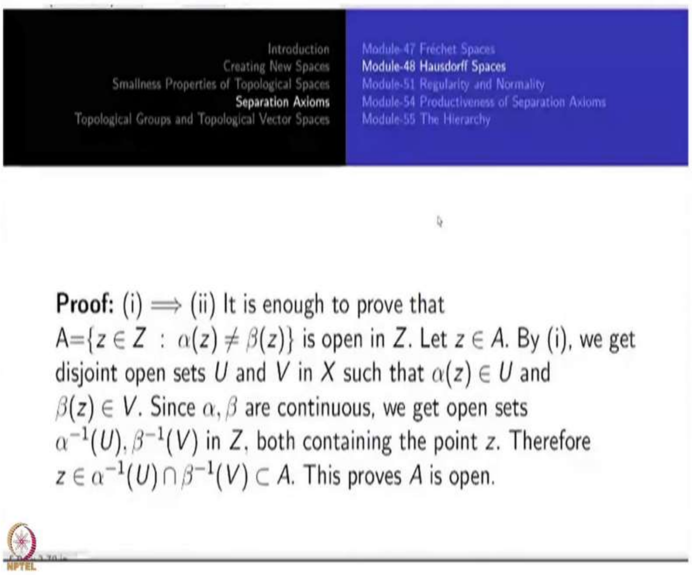
3rd condition: for every topological space Y and for every continuous map from Y to X, the graph of f, which we denote by Γ_f is closed in $Y \times X$. Remember graph of f is nothing but the set of points $(y, f(y))$.

The 4th condition is a very simple condition: the diagonal $\{(x,x) : x \in X\}$ inside $X \times X$ with product topology is closed, that is a closed subset. So, let us go through the equivalence of these four statements first.

So, starting with the condition that two distinct points can be separated by open sets, I want to show that set of all points z such that $\alpha(z) = \beta(z)$ is closed, whenever α and β are continuous functions from Z to X. That is the same thing as saying that set of points where in $\alpha(z) \neq \beta(z)$ is an open set alright?

$\alpha(z)$ and $\beta(z)$ are points of X, $\alpha(z) \neq \beta(z)$ implies you can apply the previous condition 1, you will find U containing $\alpha(z)$ and V containing $\beta(z)$, open subsets such that their intersection is empty.

If the intersection is empty their inverse images will be also open subsets which are empty, intersection will be empty, they will be also disjoint. That just means that for any point inside $\alpha^{-1}(U)$ and any point inside $\beta^{-1}(V)$ will never be equal. (Refer Slide Time: 04:33)



So, that is how you get that the complement is open ok. So, that is I am repeating it here. put A equal to all z such that $\alpha(z) \neq \beta(z)$, we want to show that it is open. Take a point z inside A, by condition 1 we get open subsets U and V such that $\alpha(z)$ is in U and $\beta(z)$ is in V and U, V are disjoint.

α and β are continuous. So, we get open subset $\alpha^{-1}(U)$ and $\beta^{-1}(V)$ in Z, both containing the point z, because $\alpha(z)$ belongs to U and $\beta(z)$ belongs to V. Therefore, when you take the intersection, z belong to the intersection also. Then take any point in the intersection, take α of that it will be inside U, β of that will be inside V. So, they are not equal. So, that is a subset of A ok. These sets may not intersect then what happens? Because I am not taking inverse image under same map, different maps are there right? But I begin with a point in $z \in A$, that is a common point for both the inverse sets. That is a neighborhood now. That neighborhood is contained inside A, that is the whole idea.

Second statement implies third statement: For every topological space $Y$and for every continuous function f from Y to X, the graph Γ_f contained in $Y \times X$ is closed ok.

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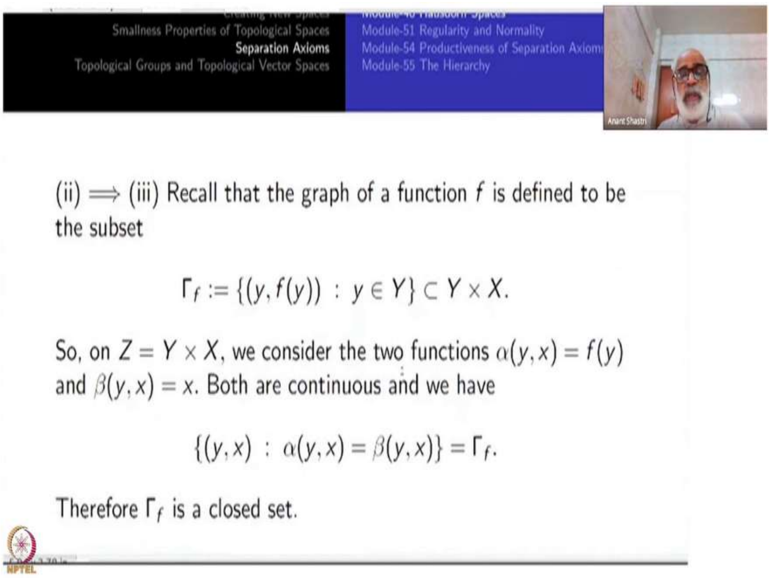

So, this is what we want show. So, recall that a graph is nothing but points $(y, f(y)), y$ belong to Y, this is a subset of $Y \times X$. So, let us take Z as $Y \times X$ itself ok? And consider two functions $\alpha(y,x) = f(y)$, (which happens to be the second projection here on Γ) and $\beta(y,x) = x$ ok. $\alpha(y,x) = f(y)$ from Z to X and $\beta(y,x) = x$ is again from Z to X ok. α is nothing but the second projection followed by f.

So, these are my α and β now and Z is $Y \times X$, both α and β are continuous and we have put (y,x) such that $\alpha(y,x)$ is equal to $\beta(y,x)$. Then what is it? First coordinate is Y, the second coordinate the (y,x) is x this is β, this is second coordinate β here, that is equal to $f(y)$. So, this coordinate is $f(y)$ means it is in Γ_f ok? Points wherein $\alpha(y) = \beta(y)$. So, that is a closed subset of $Y \times X$, ok.

See you can directly prove (i) implies (iii), but I want to prove (ii) implies (iii), that is why I have to do this. Take a special case. (ii) is true for all Z and take a special Z equal to $Y \times X$ and α and β defined this way.

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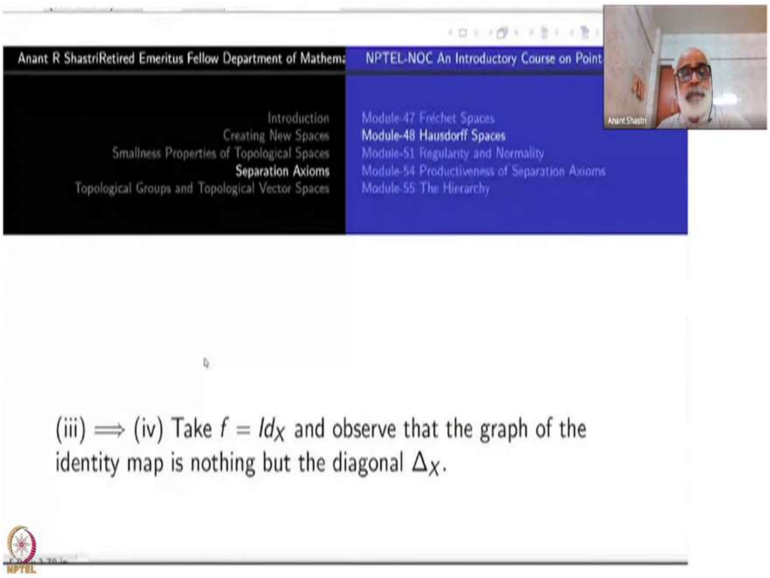

Now, (iii) implies (iv) I have to show. Δ_X is the diagonal, diagonal is nothing but graph of the identity map (x,x). So, apply (iii) to the case where you know Y is equal to X and f is identity the diagonal is closed ok.

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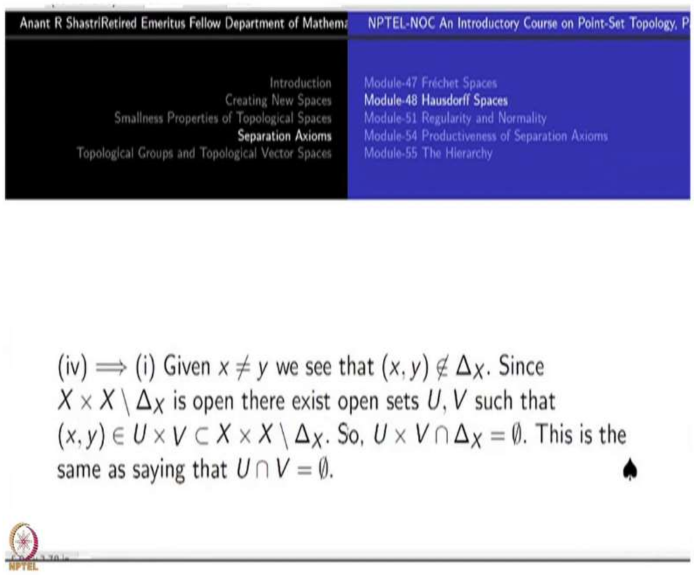

Now, (iv) implies (i), that is also easy. If x is not equal to y, we see that (x, y) is not on the diagonal ok. The diagonal is closed means the complement is open. So, (x, y) is in the open subset, in the product topology right? So, there must be basic open sets U and V such that (x, y) belongs to $U \times V$ contained inside the complement of Δ_X. That will mean that $(U \times V) \cap \Delta_X$ is empty.

This is the same thing as saying that $U \cap V$ is empty ok? What is the meaning of $(U \times V) \cap \Delta_X$ is empty? Take a point here, the same point here you have to take to be inside Δ_X, ok. Then only you will get non empty set, something non-empty. If you cannot do that; that means, any point here cannot be taken as point here, if they are different. And it is this if and only if actually.

This set-theoretic observation can be used in many other places also. This is purely set theory, but it is quite useful here ok.

So, that completes proof of the equivalence of these four conditions.

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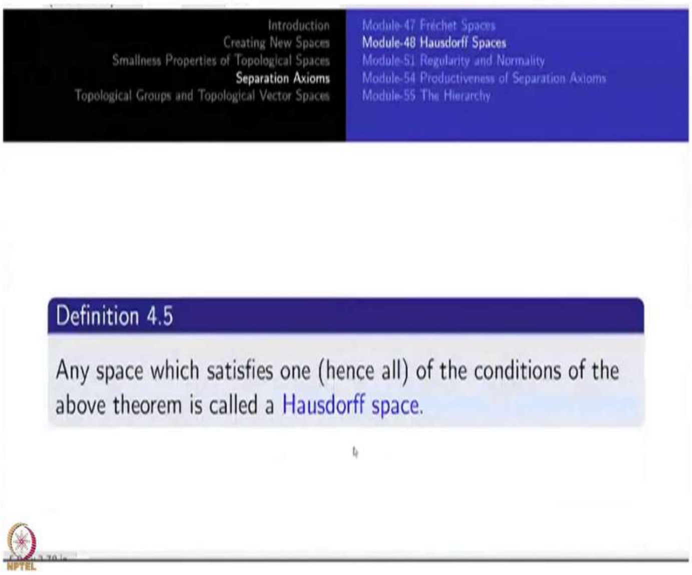

And we make a definition. A space is called Hausdorff, if it satisfies any one of the conditions and hence all of the four conditions ok.

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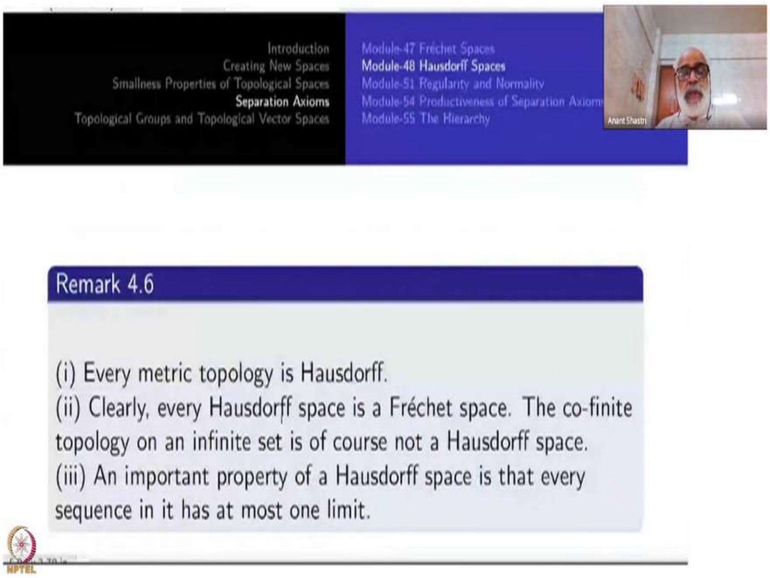

So, let us make some immediate remarks here. The very first thing is you know that every metric space is Hausdorff. The topology coming from a metric is always Hausdorff. Because given two distinct points you can take the distance between them and take half of that and take the two open ball and they will be disjoint.

The second example is the co-finite topology on an infinite set is not Hausdorff. Indeed the co-finite topology has a fantastic property that any two non-empty open sets intersect. Of course, I have to take infinite set to begin with. On a finite set, cofinite topology is not very interesting, it is a discrete space that will be Hausdorff of course, ok. As soon as X is an infinite, a non-empty open set means its complement is finite, and therefore, two non-empty open sets cannot be disjoint.

An important property of Hausdorff space is that every sequence in it has at most one limit point. A sequence may not converge that you know, but if it converges the limit is unique. So, this was one of the properties which you have been all the time using and you are familiar with from real analysis, from metric spaces, and also real life ok? So, that is one of the motivations to keep this Hausdorffness property of metric spaces and make it an axiom for some topological spaces. If not the metric let us at least keep this property. That is the motivation.

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Now, we start mixing up some of these properties. The first thing is: every compact subset of a Hausdorff space is closed ok. This is a very very good reuslt. Later on we will keep on mixing compactness and Hausdorffness often. So, this is only a starting point.

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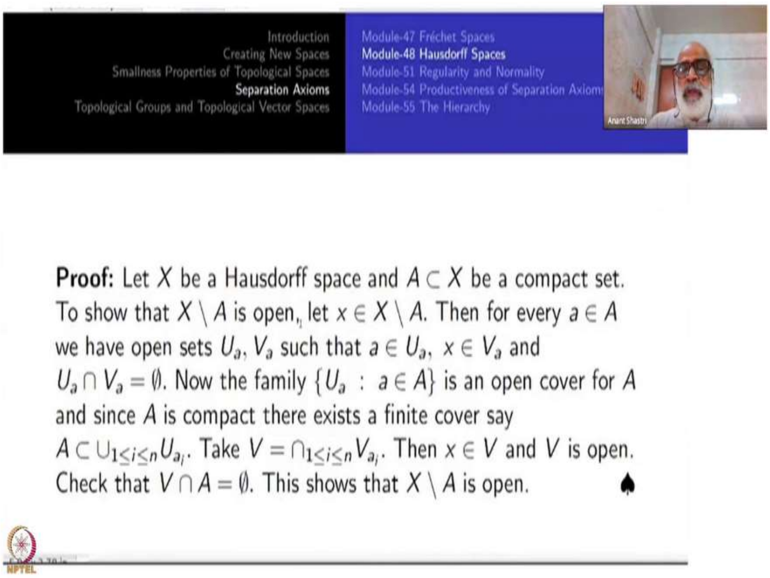

So, how does one prove that? You take a compact subset A of a Hausdorff space X, look at $X \setminus A$. I want to show that, that is open. So, take a point x in $X \setminus A$. to each point inside $a \in A$, ok, we can get an open subset U_a and another open set V_a, U_a is a neighborhood of a and V_a is a neighborhood of x, $U_a \cap V_a$ is empty.

So, this I do for all $a \in A$ then I get an open cover for A. This open cover will have a finite sub cover because A is compact. So, let us assume that A is contained inside $U_{a_1} \cup U_{a_2} \cup \cdots \cup U_{a_n}$ union, correspondingly, you take $V_{a_1} \cap V_{a_2} \cap \cdots \cap V_{a_n}$ and intersect them. Take $V = V_{a_1} \cap V_{a_2} \cap \cdots \cap V_{a_n}$. You should watch this game carefully. Here I get a finite cover there I take a finite intersection of corresponding pen sets. So, this technique will be used again and again ok.

So, what is this V good for now? V is open and x belongs to V, ok? Why x belongs to V? Because x is inside each of V_{a_i}, you know x is all the time here ok. The point is now V intersection A is empty why? Because take a point inside A, it will be in one of the U_{a_i}'s.

Then U_{a_i} intersection corresponding V_{a_i} is empty, but this is even smaller, this is V is contained in the V_{a_i}. So, U_{a_i} intersection V is empty ok. So, that is why $V \cap A$ is empty; that means, this V is contained inside $X \setminus A$. So, this we have done for each point of $X \setminus A$. Therefore, $X \setminus A$ is open, alright.

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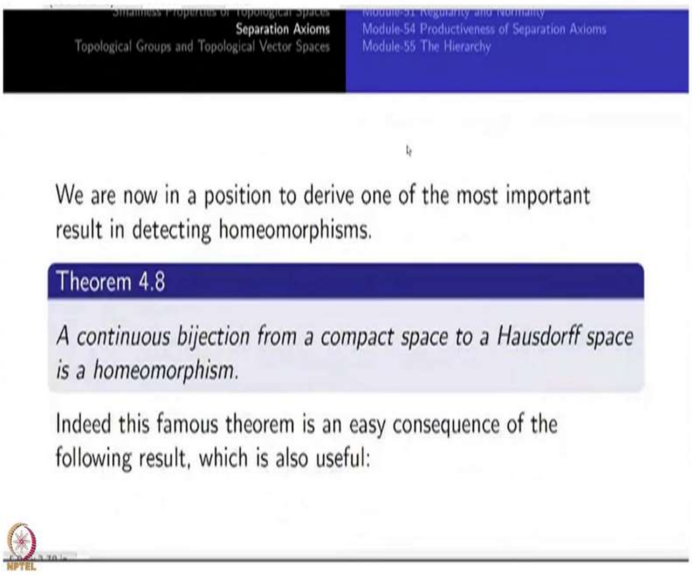

So, every compact subset in a Hausdorff space is closed. We knew this in metric spaces right? The proof is exactly same except that we should not use open balls; there we are all the time using metric and open ball and so on. There also you can use this same proof, because this proof works in general ok.

Now, we are in a position to derive one of the most important results in detecting homeomorphisms. It is quite an application oriented result yeah.

A continuous bijection from a compact space to a Hausdorff space is a homeomorphism. The domain must be compact, the co-domain must be Hausdorff. A continuous bijection is a homeomorphism. What is missing? Either you should prove that this map is open map or this map is closed map ok. So, what we shall prove here is that this map is a closed map.

Since there is a more general useful result, but not so popular as this theorem 4.8, and theorem 4.8 is a mere consequence of that one, I will state and prove that one ok? You will see that while proving that you will get a proof of this one also.

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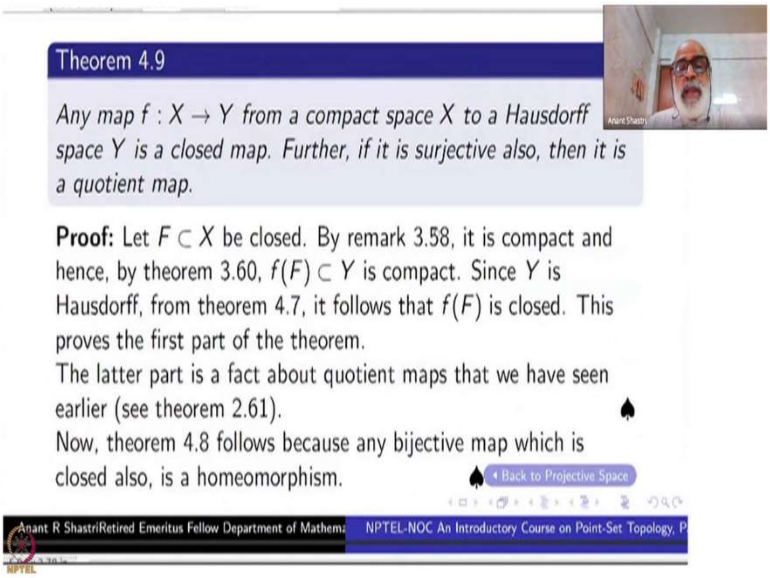

So, this is 4.9. Any map f from X to Y from a compact space to a Hausdorff space is a closed map. You see I am not assuming injection, surjection etc. nothing ok? So, this is a more general result. What does it say? You take any continuous function from compact space to Hausdorff space; it is also a closed map. Further if it is surjective then it is a quotient map. This part we have already seen. Every surjective closed map is a quotient map; surjective open map is a quotient map. All that we have seen ok? The second part we have seen.

So, this one we want to show why this f is a closed map. Start with a closed subset in X, say F be a closed subset of a compact space X. Use remark 3.58 or whatever, F is compact right; we have proved that one while studying compact spaces. By just proved theorem here, the partial converse also holds: compact subset of a Hausdorff space is closed ok.

So, what I am going to do? I am going to apply one more theorem, namely conitnuous image of a compact set is compact. So, $f(F)$ is what? Is a compact subset of a Hausdorff space. So, that is end now, by that theorem which you have proved just now, it shows $f(F)$ is closed ok?

So, I repeat. By a result about compact subset, being a closed subset of a compact set X, F is compact. By the theorem that continuous functions preserve compactness, $f(F)$ is a compact subset of Y, ok. But from theorem which we have proved just now, since Y is Hausdorff, $f(F)$ itself is closed.

So, closed set goes to closed set; f is a closed map ok. So, I repeat now, if it is surjective a closed surjective map is a quotient map.

If it is bijective, it is a homeomphism. Beacuse then the inverse of f will be continuous. So, the proofs of both the theorems are over now ok.

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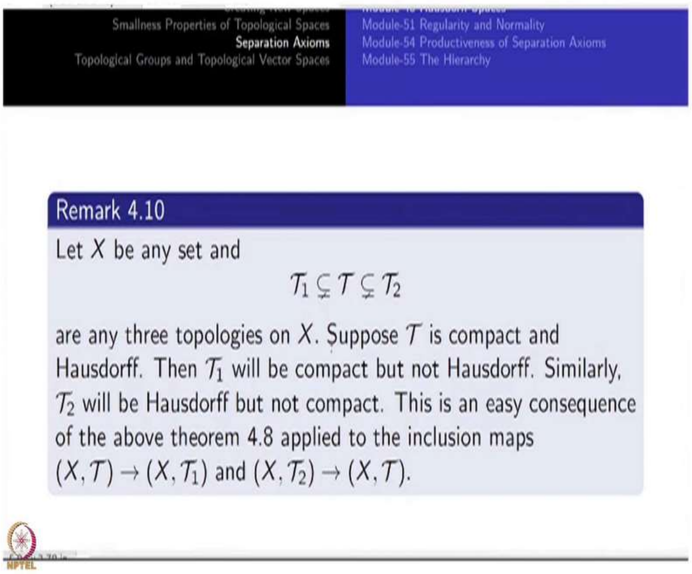

Here is a remark about mixing compactness and Hausdorffness. Suppose you have some set and three different topologies on it, one contained in the other, but not equal: $\mathcal{T}_1$ contained inside $\mathcal{T}$, $\mathcal{T}$ contained inside $\mathcal{T}_2$, ok. So, $\mathcal{T}$ is trapped between $\mathcal{T}_1$ and $\mathcal{T}_2$. And equality does not hold, that is what we have to assume ok? They are distinct. Suppose these are topologies on X and the middle one is compact as well as Hausdorff ok? You have mixed up two somewhat opposite natured properties--compactness is a smallness property and Hausdorffness is a largest property. So, you mix them. Something wonderful happens which justifies the names--the largeness and smallness properties.

Then $\mathcal{T}_1$ be compact but not Hausdorff. Why it is compact? Because it is smaller than $\mathcal{T}$ which is compact. But it will not be Hausdorff, that is the claim. Similarly, $\mathcal{T}_2$ will be Hausdorff because it is larger than $\mathcal{T}$, but will not be compact ok. So, how to see these things? All that you have to do is start with $(X, \mathcal{T})$ and take the identity map ok or inclusion map whatever you say.

This is the identity because X to X is same set here ok, but topologies are different. Since $\mathcal{T}$ is larger than $\mathcal{T}_1$, the identity map will be continuous. $(X, \mathcal{T})$ is compact ok therefore $(X, \mathcal{T}_1)$ is compact. If this were Hausdorff also, then this will be a homeomorphism which means $\mathcal{T}$ is equal to $\mathcal{T}_1$, but $\mathcal{T}$ is not equal $\mathcal{T}_1$ by assumption.

Similarly, $(X, \mathcal{T}_2)$ to $(X, \mathcal{T})$, you take again the identity map ok, which is continuous. We know that $(X, \mathcal{T}_2)$ is Hausdorff. If it is compact also then by the above theorem the identity map will be a homeomorphism. But $\mathcal{T}_2$ is not equal to $\mathcal{T}$. So, this is not a homeomorphism ok. So, $\mathcal{T}_2$ fails to be compact. You go above $\mathcal{T}$ which is compact Hausdorff, everything above fails to be compact, everything below fails to be Hausdorff. So, this is like optimizing both of them, whether such things exist always or not that is a completely different question on a given set such things may not exist, one does not know.

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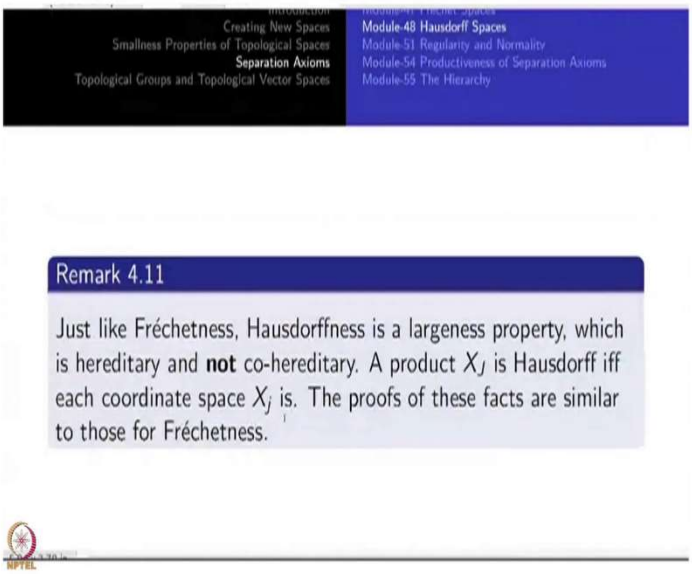

Just like Frechetness, Hausdorffness is a largeness property. So, that is what we have just seen and a nice illustration here. It is hereditary, but not co hereditary ok, very easy to produce examples. All through we will have such examples a product X_j is Hausdorff if and only if each coordinate space is.

So, in this sense it is a productive property ok. All these proofs are exactly similar to what we have done for Frechetness and they are straightforward, there is no catch there. Only thing is you may not be able to see immediately why it is not co-hereditary, exactly similar example. Same example we will do of collapsing in open interval say $(0, 1)$ inside $\mathbb{R}$ ok. The real line is Hausdorff, the quotient need not be Hausdorff. So, do not make the mistake that quotients are Hausdorff ok? Quotients are very fussy.

Now there are many things to do about Hausdorffness as such. We are now going to mix it other properties just like compactness. Also we may go back to some linear spaces, metric spaces again and so on. So, at this point let us break. So, that is enough for today.

Thank you.