Introduction to Point Set Topology, (Part I)
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Lecture - 47
Frechet Spaces

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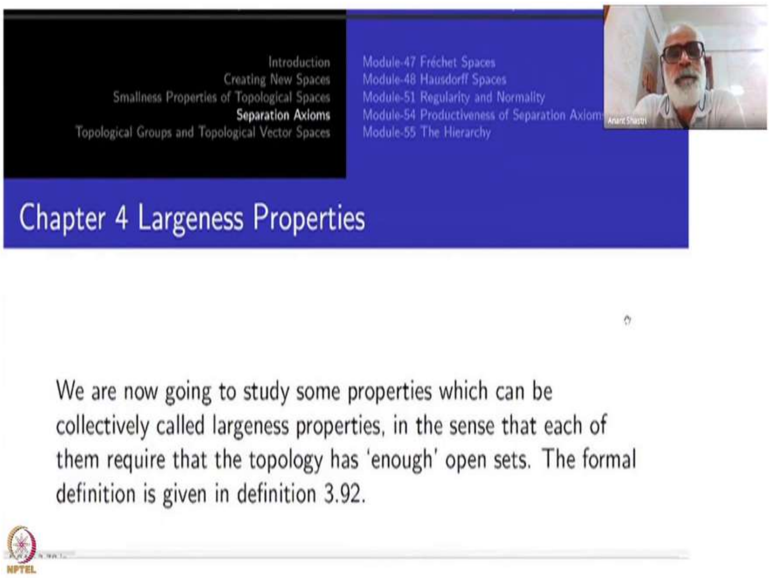

Welcome to chapter 4 of Point Set Topology course Part 1. So, this I have named as largeness properties. Essentially we are going to study properties of topological spaces, such that whenever a topology has it something which is larger than that will also have it. So, that is the rough definition, formal definition we had already given.

But there are some more here, which we feel that just like the case of I-countability among countability conditions. They are the opposite here, something called regularity and normality, they do not exactly fit into this definition.

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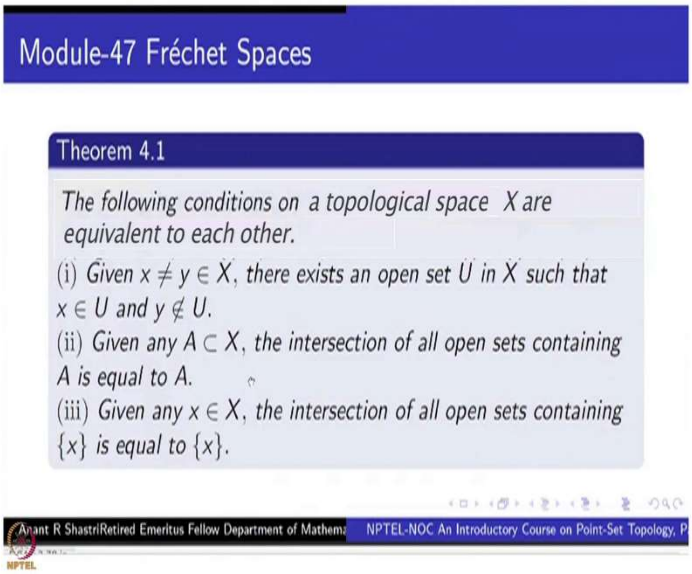

So, to begin with you will take the celebrated Frechet Spaces. So, that is module 47. So, we begin with a Bourbaki style presentation here. Starting with a topological space X several statements are made which are equivalent to each other; that is the statement of this theorem. Any one of them can be taken as the definition of Frechet space finally. So, that is the whole idea ok. So, x, y are different points, there exists an open set U in X, such that x belongs to U and y does not belong to U.

So, read it carefully you take two distinct points, then the statements is that there is an open subset containing x, but not containing y. So, this statement also implies that there is another open set V, which contains y and not containing x. So, that is the logical conclusion of the statement. You have to understand that.

Because x and y are two different points ok. Just because I have written first x here second y, this is not an ordered pair ok? So, take the statement: x belongs to U and y does not belong to U. There is such a U. You can interchange x and y also. So, that is the 1st statement.

The 2nd statement is any subset of X is the intersection of all open sets containing A ok. Take intersection of all open sets containing A, that will be equal to A; any subset ok. If you take empty set what happens, you can take empty set also as an open set the intersection will be empty.

Given any x belonging to X, the intersection of all open sets containing this singleton x is equal to singleton x. So, this is the 3rd statement; from arbitrary subset arbitrary set you have come to singleton set.

So, obviously, 2 implies 3 is clear here right. I have some more statements there is only three of them, there are more statements here; the statement of the theorem continues on the next slide.

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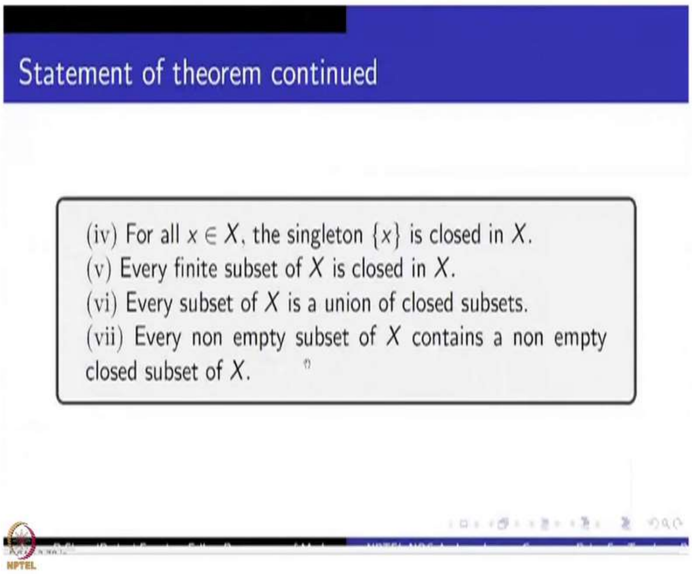

The 4th statement says that for all x belong to X, singleton x is closed in X. So, this one statement is very easy to remember. All the singleton sets are closed ok?

The 5th one says that every finite subsets of X is closed. So, that is an easy consequence of (4) because, finite union of closed sets is closed.

The 6th statement is that every subset of X is a union of closed subsets.

So, that is also easy from (4) directly. In any case it also follows from (5), because every subset is the union of finite subsets. The 7th statement is every non empty subset of X, contains a non empty closed subset of X. So, this does not seem to be immediate, but we will see all these things are very easily implying each other. They are all equivalent to each other that is the statement ok. So, let us go through the proof once again, because I have not indicated all the proofs.

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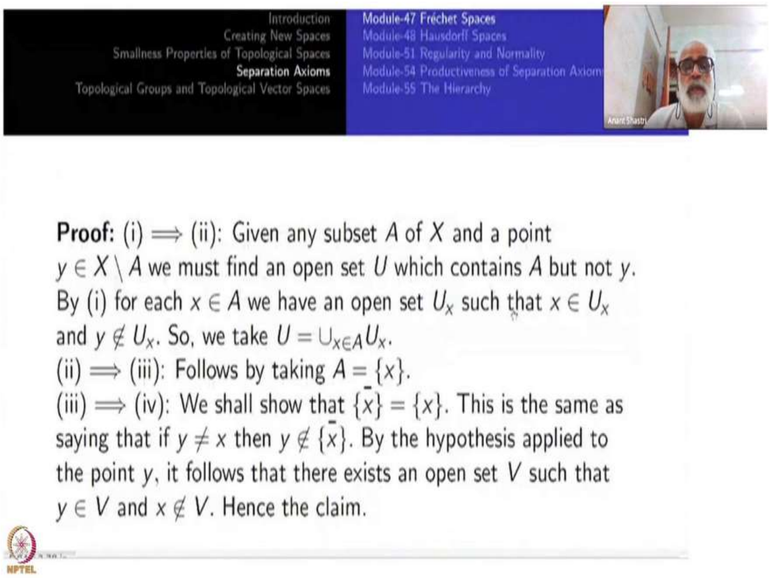

(i) implies (ii): given two distinct points ok, there is an open set containing one and not containing the other is the condition (i) ok. Take any subset A of X and a point y which is not in A; then we must find an open set U which contains A, but not y ok. So, let us look at statement (ii) here. Any subset A is the intersection of all open sets containing A ok?

So, if you want to show that it is equal to A you must produce some open set containing A, but not containing the point y, whenever y is a not a point of A. So, that is what we have to prove right? So, that is what we have to prove here, (i) implies (ii) ok; By (i) for each x inside A, we have an open subset U_x, such that U_x, I have written because, it depends upon x and x belongs to U_x and y is not in U_x. So, you vary the point x, but keep y as it is. Take U equal to union of all U_x is x varies over A. So, this will contain A, since none of them contain y, the union also will not contain y. So, when you take intersections of all open subsets containing

A, this y will be left out. So, since every point away from A will be left out, the intersection is exactly equal to A; So, proves (ii).

Now, (ii) implies (iii): By merely taking $A = \{x\}$. This I have already indicated. So, what is the statement (iii)? Take the collection of all open subsets containing x, the intersection is just $\{x\}$ ok, that is the statement (iii).

So, (iii) implies (iv): We want to show that $\{x\}$ are closed; that means, take $\{x\}$ take the closure, I should show that it is $\{x\}$ itself ok. This is the same thing as saying that, if y is not equal to x, then y does not belong to $\overline{\{x\}}$ right? $\overline{\{x\}}$ is $\{x\}$ means that. That is what you have to show.

But by the hypothesis applied to the point y now, you see for every point intersection of all open sets is just that point ok. It follows that there exists an open set V such that y is inside V, x is not in V; because x is not equal to y ok. So, I am not using number (i) here, I am using number (iii) here to conclude this one ok. So, if this open set does not intersect the $\{x\}$ that means y is not in the $\overline{\{x\}}$ that is all. So, this is the definition of the closure. So, (iii) implies (iv) is over all the singletons are closed alright.

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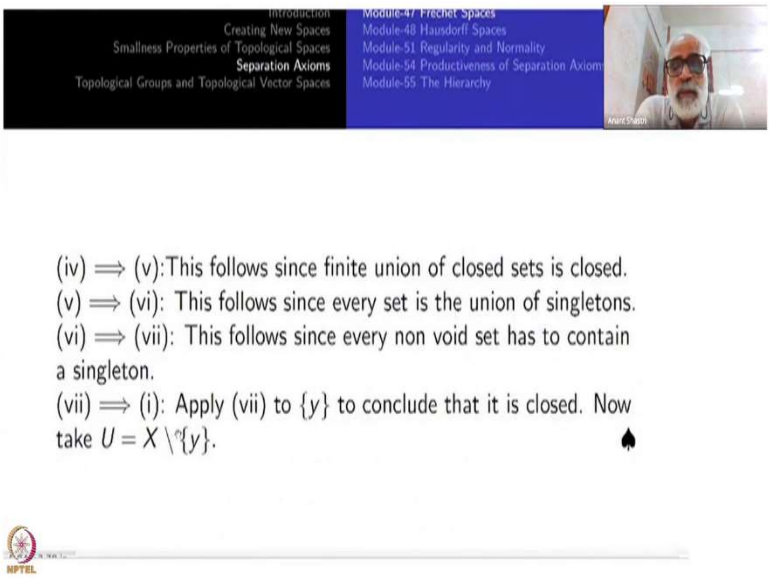

Now, (iv) implies (v). Since every finite union of closed sets is closed, every finite subset will be also closed ok?

Now, (v) implies (vi) is again obvious, because every set is the union of finite sets. Every set is the union of singletons is one thing, next one is every set is a union of finite sets.

Now, (vi) implies (vii) follows since every non void set has to contain a singleton right? See let us go back and see what is (vii). Every non empty set of X contains a non empty closed set; singleton is closed set because it is a finite set. Every finite set is closed set we have this hypothesis in (vi). So, a set is non empty implies what? there is a point that point singleton point is closed. So, this implies (vii) ok. You see singleton set from (vi) every subset of X is union of closed sets. $\{x\}$ how can it be union of anything, it is union of $\{x\}$ itself right. So, that must be closed. So, it is already implies that gives (vi) implies already (iv) ok. So, anyway you can jump from (iv) to (vii). Now, (vii) implies (i): I will complete this finally, that will complete the whole chain of implications. (vii) implies (i) is what we have to show.

So, apply (vii) to the $\{y\}$ to conclude that it is closed. $\{y\}$ should contain a non empty closed subset, right? Therefore, $\{y\}$ is closed. That means what? the complement of the $\{y\}$ is open and that will contain the point x. So, you can take U to be $X \setminus \{y\}$, ok.

So, all these statements are equivalent. Proofs of one implies the other is very easy. So, we approved it in a single chain ok.

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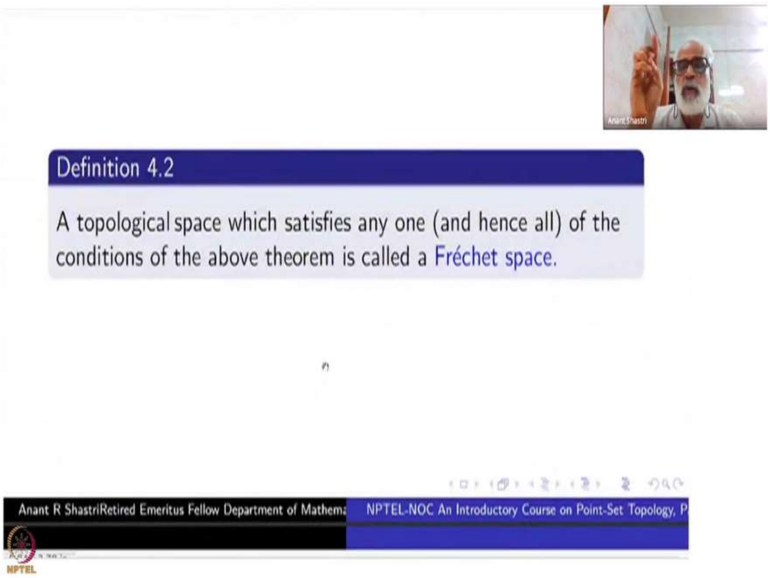

Now, the definition is the following. A topological space which satisfies any one and hence all of all the above conditions, namely in the above theorem, such a space is called a Frechet space. Incidentally you know Frechet is one of the founders of this point set topology. He had his own definition of topological spaces, in which he put this extra condition for a topology just like Hausdorff had put his his own extra condition. So, the present day definition of topoogy, whatever has been adopted, is called a general topology. General than Frechet's as well as Hausdorff's definitions, which does not have these extra conditions.

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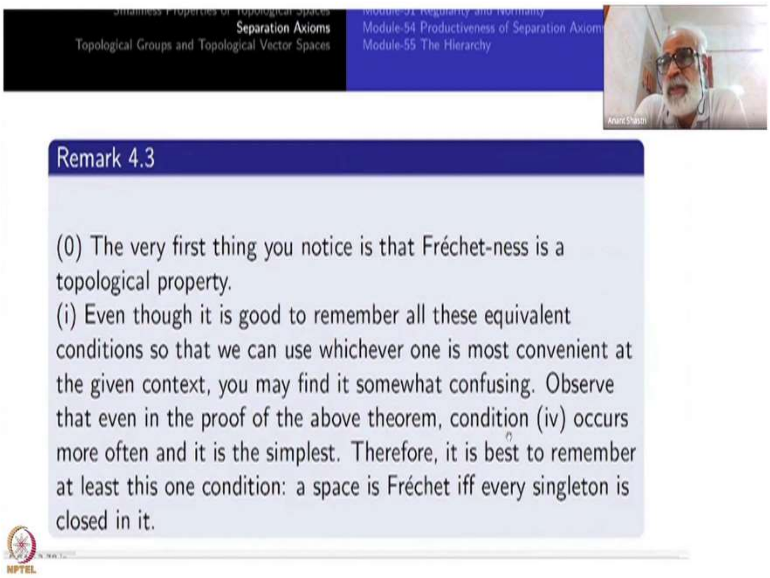

Student: Sir. So, all the metric spaces will be Frechet spaces right.

Obviously, they are Frechet spaces you know they will satisfy many more properties. In this chapter now, you watch out for them. Metric space is still our motivation after all, but we will see that lots of Frechet spaces are there which are not metric spaces. Very first thing you notice is that Frechet-ness is a topological property.

If X is homeomorphic to Y, X is Frechet implies Y is Frechet ok. You can take any one of the them for example, I will take singleton sets are closed under the homeomorphism closed set will be go to closed set. So, singleton sets in the other space will also be closed. That is all.

Even though it is good to remember, all these equivalent condition so that we can use whichever one is most convenient to us ok, you may find it somewhat confusing to remember all of them, whereas, condition (iv) that just now I used, namely, singleton sets are closed is the easiest thing to a to remember. For me at least ok?

Therefore I remember Frechet space by this condition. Other things I can derive whenever I want ok. This question which one to remember. So, Frechet space is, for me, that space in which every singleton is closed. That is it.

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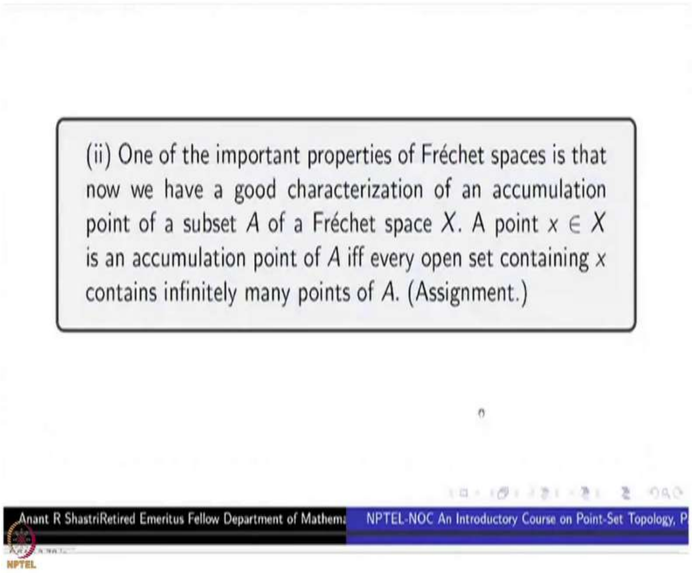

One of the important properties of Frechet space is that now we have a good characterization of an accumulation point of a subset inside a Frechet space. Equivalent to what we do in metric spaces. A point x belonging to X is an accumulation point of a subset A, if and only if every open set containing x contains infinitely many points of A. For a limit point or a closure point for a closure point, all that you needed is that intersection is non empty, one point is enough. But if it is an accumulation point in particular, inside a Frechet space just like in a metric space ok? Every point which is accumulation point of A should have the property that every open set containing x intersection with A must be infinite ok.

Not very difficult to see. You have to carbon copy the argument in a metric space, but do not use the metric, use the property that given x and y different there is an open subset here blah blah blah or compliments are singleton sets are open or singleton sets are closed etc. that I will need an assignment.

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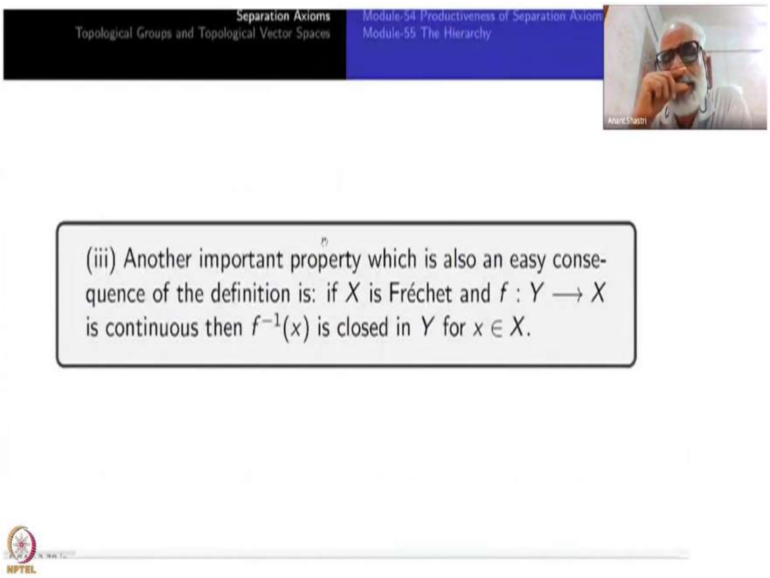

Another important property is also an easy consequence of the definition, that is the following. If f from Y to X is continuous, remember Y to X, not X to Y, and X is Frechet here. Then the inverse image $f^{-1}(x)$, that is, the fibres of f, they are all closed in Y. Why? Because singleton sets are closed in X, ok?

So, this is very useful thing. So, that is why I have included it here. So, for example, we have been using this inverse image, continuous functions, continuous real valued functions, inverse image of a point. Set of all x such that $f(x) = 0$, it is a closed set. So, this is because $\mathbb{R}$ is Frechet, ok? So, that is why it was working. So, now you can see that inverse image of singleton set is closed, $\{x\}$ is closed. So, inverse set is also closed ok.

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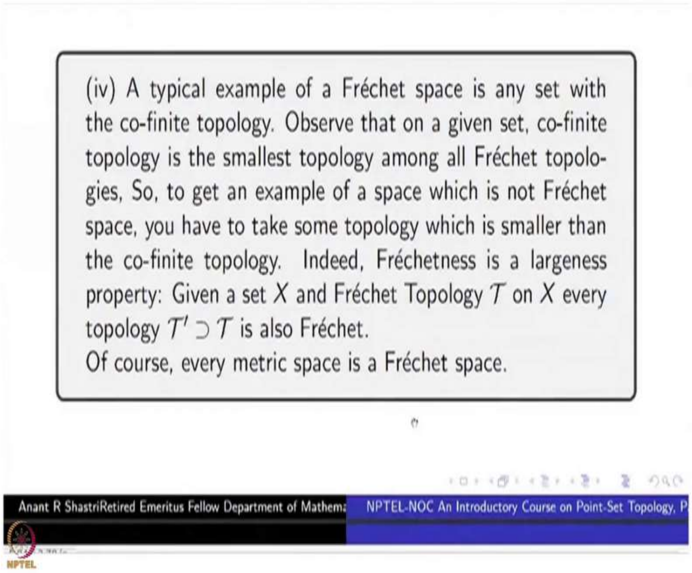

A typical example of Frechet space is any set with co-finite topology. See I do not take metric space as a typical example, metric space is too good. Take any set with co-finite topology. Especially take an infinite set, if you take a finite set and co-finite topology, then it is just the discrete space ok. Discrete spaces are obviously, Frechet spaces.

But co-finite topology on an infinite set is a typical example of Frechet space, it is not metrizable ok. It is does not come from a metric ok. So, that is why this is nice. Observe that on a given set co-finite topology is the smallest topology among all Frechet topologies.

So, that is why I take this as a typical example. So, to get an example of a space which is not Frechet, you have to take some topology which is smaller than co-finite topology. Indeed Frechet-ness is a largeness property; once some topology tau is Frechet, if $\mathcal{T}'$ is a larger topology than $\mathcal{T}$, that will be also Frechet.

It is very easy because singleton sets are closed here. So, they will be here also right here means what? If something is closed its complement is open here, that will be in the larget topology and its complement which is the singleton is closed there. Every metric space is a Frechet space.

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Frechetness is clearly hereditary. See singleton sets are closed in X. Take A a subspace of X, and take $a \in A$, $\{a\}$ is closed in X itself right? So its intersection with A is closed in A. So, all singletons inside A are also closed. So, every subspace of a Frechet space is Frechet very easy. That is hereditary property. However, it is not co-hereditary ok. So, here is an example. You can give lots of examples later on.

Let us begin with some example take X to be the quotient space of $\mathbb{R}$, where all the points of an open interval, any open interval, let us say $(-1, 1)$ is identified to a single point. Collapse the open interval to a single point. -1 will remain separately and 1 will remain separately. All the points outside $(-1, 1)$ will remain separately. No identifications there.

Only whenever x and y are both lie in $(-1, 1)$ they will be all be equivalent to each other. So, that is one single class. So, let us look at that point that class $*$, this is just a notation. So, in the quotient topology $*$ and the image of every real number are there. Image of everything between $(-1, 1)$ is denoted by $*$ now, one single element right.

So, in the quotient topology this star and image of 1 will be distinct points ok and every neighbourhood of 1 will contain this point $*$.

So, you cannot separate them in that way ok. Every neighbourhood of 1 will contain the image of $*$; Why because in $\mathbb{R}$, every neighbourhood of 1 intersects the interval $(-1,1)$. So, there will be some point here. So, that point when you go down it is just the $*$, right. So, this is not a Frechet space.

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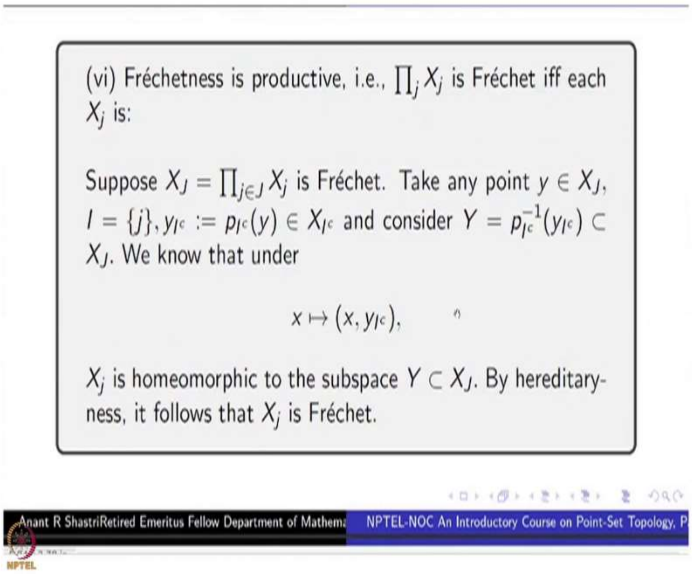

So, here is another property of Frechet spaces: It is productive. Take a family of topological spaces X_j, the product is Frechet space, if and only if each X_j is a Frechet space ok?

The first thing is suppose the product is a Frechet space ok. Suppose first that we are dealing two factors only, $X \times Y$. Then I can take X as a subspace of $X \times Y$ by choosing some point $y \in Y$ and looking at $X \times \{y\}$ sitting inside $X \times Y$, right? That is a subspace. Since I am assuming $X \times Y$ is Frechet, this subspace will be also Frechet, but $X \times \{y\}$ is homeomorphic to X. It is a copy of X under the map x going to (x, y). So, X is Frechet. Similarly, Y is Frechet. Similar argument can be used in the case of infinte products also.

But you have to write down it carefully that is all. Every factor X_j can be thought of as a `coordinate' subspace of the product. Just like x going to (x, y) ok. So, that is what I have done here, pick up any point $y = y_J$ in X_J, then forget about the j^{th} coordinate of that point. So, that is take y_{I^c}, where I is equal to singleton j.

So, I^c is complement $J \setminus I$. So, all the other things are there that is in a point of X_{I^c} product taken over this complement only j is missing here ok then if you go take $x \in X_j$ and x going to (x, y_{I^c}), this will give you an embedding of X_j inside product of X_J.

So, each X_j is a subspace of X_J in that sense, ok? By hereditariness each X_j is Frechet ok. We have to prove the converse. Suppose each X_j is a Frechet then why the product is Frechet? That is much easier.

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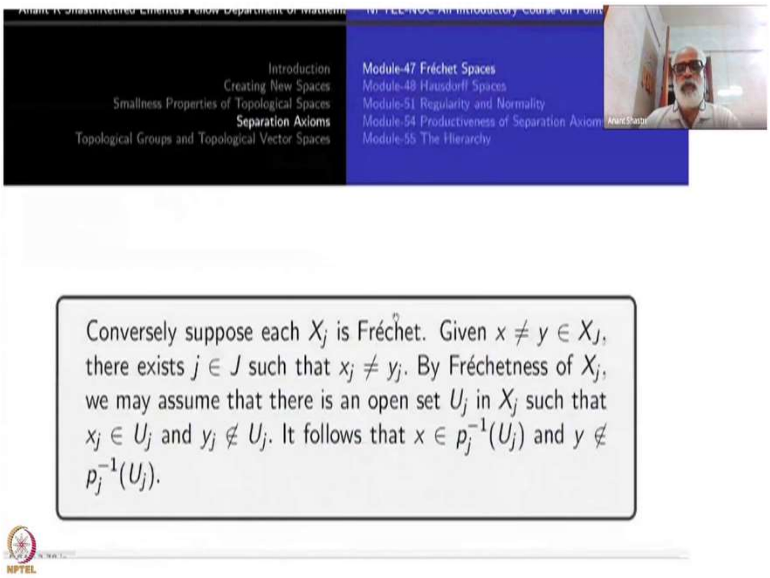

Conversely suppose, each X_j is Frechet take any two distinct points inside X_J. What is the meaning of that the two points are distinct? At least one of the coordinate is different right? If all the coordinates are the same then the points are the same. So, there exist $j \in J$ such that $x_j \neq y_j$.

Now, use the fact that this space X_j is a Frechet space, you will get an open set U_j inside X j such that x_j is inside it and y_j is not there. Now look at $p_j^{-1}(U_j)$. That will contain x and y will not be there, y is there precisely if the y_j coordinate of y is inside U_j, but y_j is not inside U_j. Over ok?

So, you havef used Frechetness of just one of the coordinates! But but to conclude that X_J is Frechet you have to say all the coordinates, because starting with $x \neq y$, you do not know which coordinate is different right. So, that is the whole idea.

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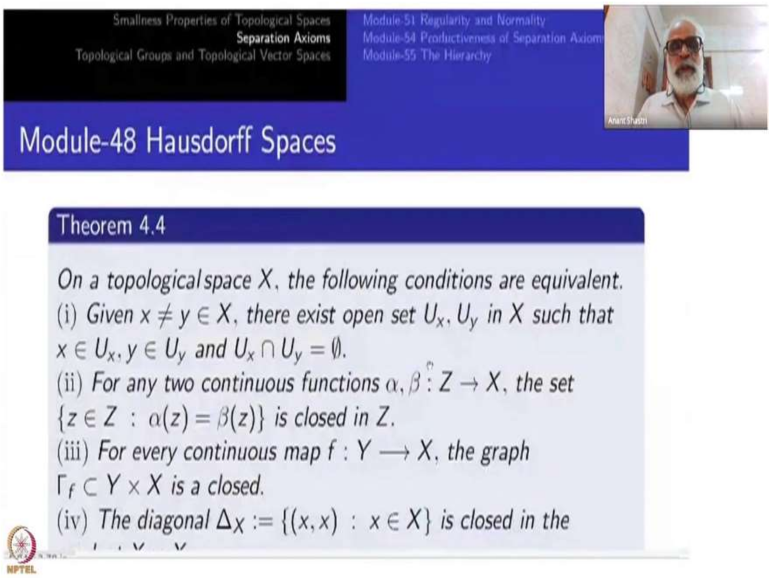

I think we will stop here. We have done a good work with Frechet spaces. Next time we will study Hausdorff spaces ok? Somehow the Hausdorffness has taken the limelight, it is more important than T_1 spaces ok. So, next time.