Introduction to Point Set Topology, (Part I)
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Module - 46
Lecture - 46
Proof Alexander's Subbase Theorem

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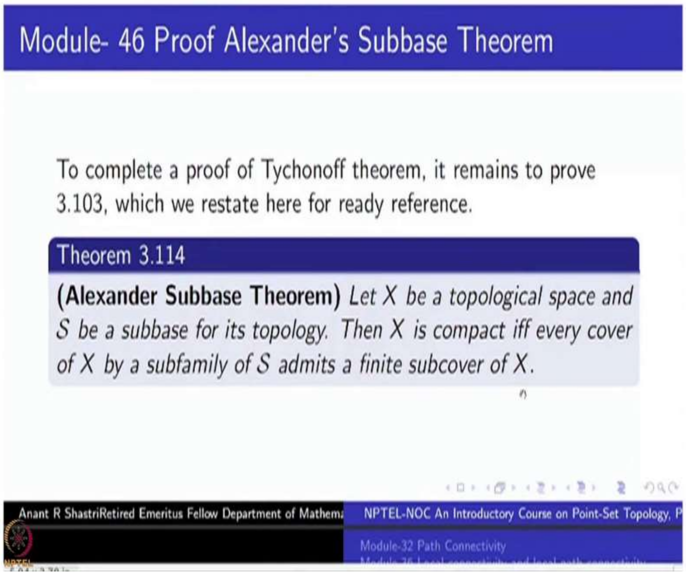

Welcome to module 46 of Point Set Topology Part-1. Last time we stated and proved Tychonoff's theorem. Of course, we assumed Alexander's Subbase Theorem. So, in order to complete the proof of Tychonoff's theorem, we should now prove Alexander's Subbase Theorem. We have already made some set theoretic preparation for that also last time.

So, let us start with the proof of Alexander's Subbase Theorem which I restate here, let X be a topological space and $\mathcal{S}$ be a subspace for its topology. X is compact if and only if every cover of X by a subfamily of $\mathcal{S}$, admits a finite subcover.

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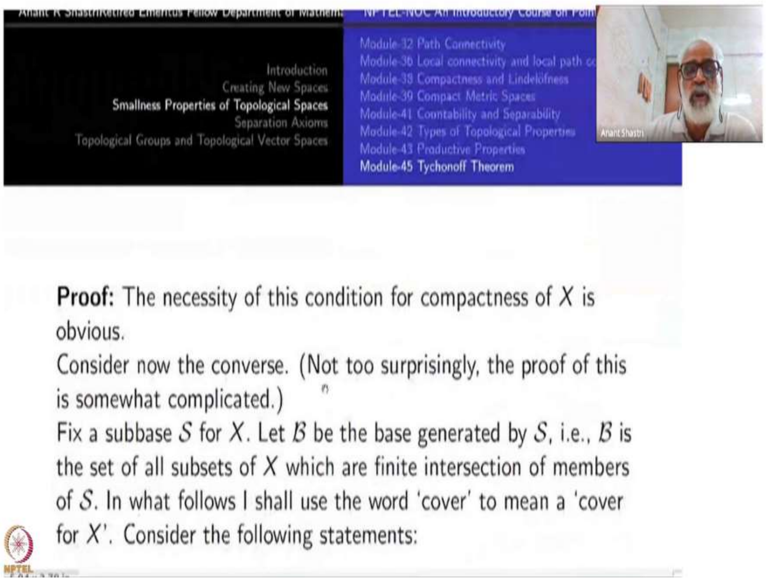

The necessity of this condition for compactness is obvious because, X is compact; every open cover has to admit a finite sub cover. After all members of $\mathcal{S}$ are open. So, some sub family of $\mathcal{S}$ covers means that is an open cover so; that means, that admits the finite suppose, that condition is necessary. The crux of the matter is that you have to prove the converse. So, we expected that this proof will be sufficiently complicated ok, so you have to be ready for that.

So, fix a subbase $\mathcal{S}$ for X ok? And let us have the notation that $\mathcal{B}$ is the base generated by $\mathcal{S}$, namely elements of $\mathcal{B}$ are those which are finite intersection of members of $\mathcal{S}$. So, in what follows I shall use the word cover to mean a cover for X ok. So, that much economy of notation what I mean, words ok little bit of small words that, what we have is let us let us just recall, what is the meaning of all whatever we had.

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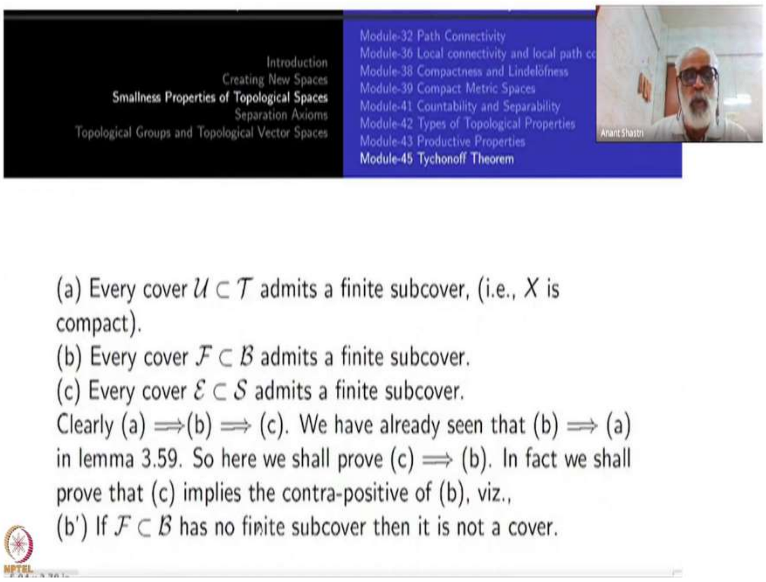

Every cover $\mathcal{U}$ of open subsets means contained in $\mathcal{T}$, means $\mathcal{U}$ is a sub family of $\mathcal{T}$ means they are open subsets admits a finite subcover, this is the compactness right. The second statement is every cover $\mathcal{F}$ contained inside $\mathcal{B}$; the base. that means what? Only members of this particular $\mathcal{B}$ are allowed, this is a smaller family than tau after all, that admits a finite sub cover.

Third one is even shorter, every cover $\mathcal{E}$ contained in the subbase admits a finite sub cover. Obviously, (a) implies (b) implies (c), ok? We have also seen that (b) implies (a), ok? Now, what we want to prove is (c) implies (b). So, this is the gist of Alexander's subbase theorem.

So, indeed what we shall do is (c) imply the contra positive of (b), namely if $\mathcal{F}$ is a subfamily of $\mathcal{B}$ which has no finite subcover, then it is not a cover which is the same thing as saying that if it is a cover then it has a finite subcover. So, it is in this form I am going to prove (c) implies (b), ok. So, these all just to clarify the ground situation.

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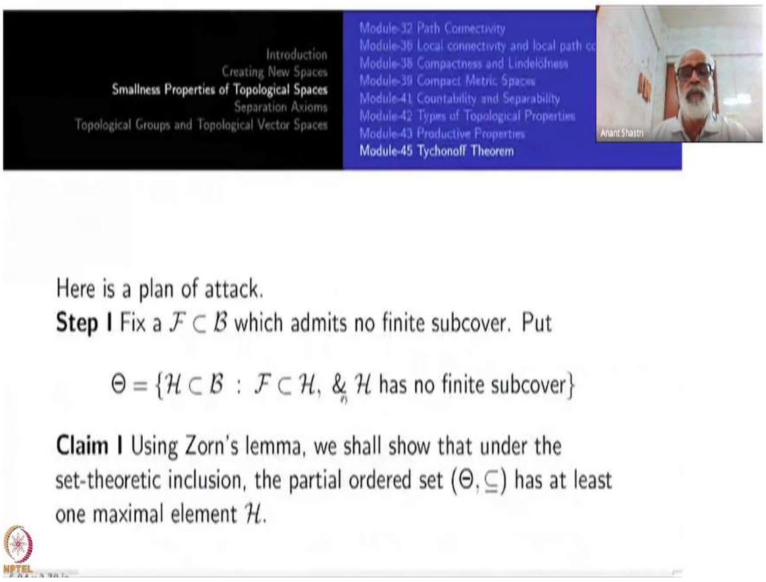

Now, the plan of attack is as follow; fix a family contained inside $\mathcal{B}$ which admits no finite sub cover. Construct another subfamily another one like this namely these are all now families of families ok, Θ is all $\mathcal{H}$ contained inside $\mathcal{B}$ such that, $\mathcal{F}$ is inside $\mathcal{H}$, this $\mathcal{F}$ is one one such family ok which admits no finite sub cover.

Finally, we want to show that it is not a cover for X ok. So, $\mathcal{F}$ contained inside $\mathcal{H}$ and $\mathcal{H}$ has no finite subcover ok. So, look at all those $\mathcal{H}$ which also have this property, no finite sub cover, but they are larger than $\mathcal{F}$. So, that is my family of sub families of $\mathcal{T}$ here. The first claim is that using Zorn's lemma, we shall show that under the set theoretic inclusion the partial ordered set Θ, (this is containment. So, it is a partial order), has at least one maximal element.

From one arbitrary $\mathcal{F}$, we want to have something which is maximal ok, with respect to this property, that it has no finite subcover ok. We have to assume there is one $\mathcal{F}$, then only this Θ will be non-empty I am sure. Then we have the maximal element for that we have to apply Zorn's lemma which means we have to prove something there ok, this is a plan of that diagram.

Second step will be take such a maximal element $\mathcal{H}$ in Θ.

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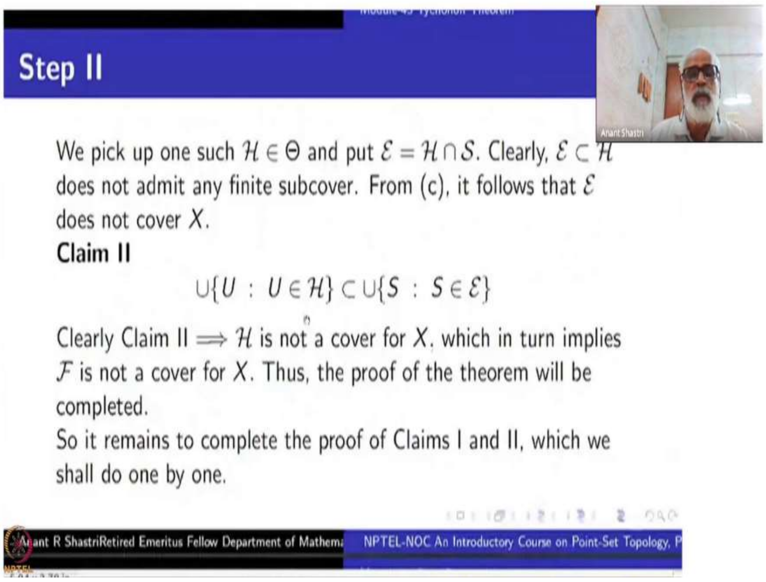

Now, put this $\mathcal{E}$ equal to $\mathcal{H} \cap \mathcal{S}$; that means, those members of $\mathcal{H}$ which are in the subbase, those members of $\mathcal{H}$ which are in the sub base, the sub base is fixed ok. So, that is a smaller family than $\mathcal{H}$, right? This does not admit any finite subcover, because $\mathcal{H}$ does not admit any finite subcover.

If this does not admit finite sub cover from the hypotheses in (c) it follows that this subfamily $\mathcal{E}$ is not a cover for X ok. So, up till here we have arrived by using Zorn's lemma and our assumption that there is a $\mathcal{F}$ such that which does not admits the finite subcovers ok.

Now, second claim is: look at the union of all all members of $\mathcal{H}$, take the entire set. We will show that this is contained in the union of members of this $\mathcal{E}$. So, what we have concluded for $\mathcal{E}$? $\mathcal{E}$ does not does not cover X. So, this will also does not cover X.

So, that will complete the proof. If this union of all elements in $\mathcal{H}$ is contained in the union of all elements in this $\mathcal{E}$ and $\mathcal{E}$ does not cover X, so $\mathcal{H}$ is also not a cover of X. If $\mathcal{H}$ is not a cover of X, remember $\mathcal{H}$ was having a some maximal property, in particular $\mathcal{F}$ is contained inside $\mathcal{H}$. So, if $\mathcal{H}$ does not cover X, $\mathcal{F}$ also does not cover ok? So, that will complete the proof.

So, we have to prove two steps here, in the first claim we have to prove the Zorn's lemma whatever hypotheses is needed that is the first part and then you have to prove this claim ok. Once we prove this claim, claim II also, the proof will be completed. So, it remains to prove claim I and claim II. So, let us do it one by one.

Look at claim I: let us just recall ok. This Θ has a maximal element is what you have to show, for that what is the ingredient that you have to put inside the Zorn's lemma?

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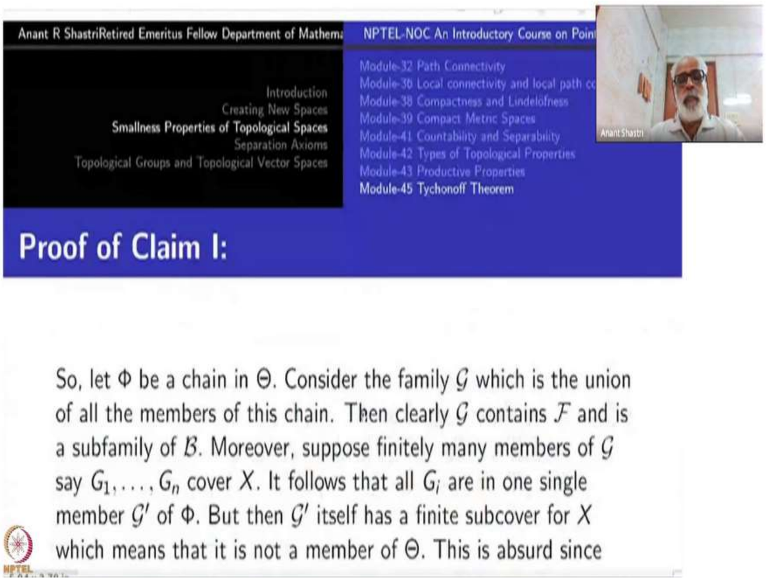

Take any chain inside Θ, we must show that that chain has an upper bound. This is what I have to show, this part is very easy as usual, or quite often.

Let Φ be a chain in Θ, recall what is the meaning of a chain. Chain is a totally ordered subset of Θ, under the usual inclusion here ok. Consider the family $\mathcal{G}$ which is the union of all members of this chain; obviously, under the inclusion that will be an upper bound for this one. But do not worry, that must be an element of Θ then only it will be justified. The larger set of all subsets it is always an upper bound fine ok.

So, clearly $\mathcal{G}$ contains $\mathcal{F}$ because each member of this chain they are members of Θ all the members of Θ contain $\mathcal{F}$, ok. So, it is a sub family of $\mathcal{B}$ also because each member of Θ is also sub family of $\mathcal{B}$. So, if you take unions of all these members you know, you take each member inside $\mathcal{B}$. So, that is not a problem.

Moreover suppose finitely many members of $\mathcal{G}$ say; $G_1, G_2, \ldots, G_n$ cover X, which is the last part which I have to show that it is inside Θ, that no finite family covers X that is what you have to show right? If not suppose there are $G_1, G_2, \ldots, G_n$ belonging to $\mathcal{G}$, which cover X remember what was $\mathcal{G}$; $\mathcal{G}$ is just union of members of this chain..

So, G_1 will be inside say some Λ_1, G_2 will be Λ_2, G_3 will be Λ_3 and so on. But these are all one contained in the other you take the maximum of these, then you have finitely many of them, you have the maximum. That will contain all the $G_1, G_2, \ldots, G_n$ ok, but that is a member of this Θ so it will not cover; so it is a contradiction ok.

So, I repeat, all these G_i's are in one single element $\mathcal{G}'$ of Φ, but this $\mathcal{G}'$ is a family which belongs to Θ by definition, it has no finite sub cover for X. So, this is observed because we assumed that $\mathcal{G}$ admits a finite subcover. So, $\mathcal{G}$ does not admit a finite sub cover that qualifies it to be a member of Θ.

Therefore every chain in Θ has an upper bound in Θ. Once you have guaranteed this property, Zorn's lemma tells you that Θ has a maximal element ok. So, first claim is done ok.

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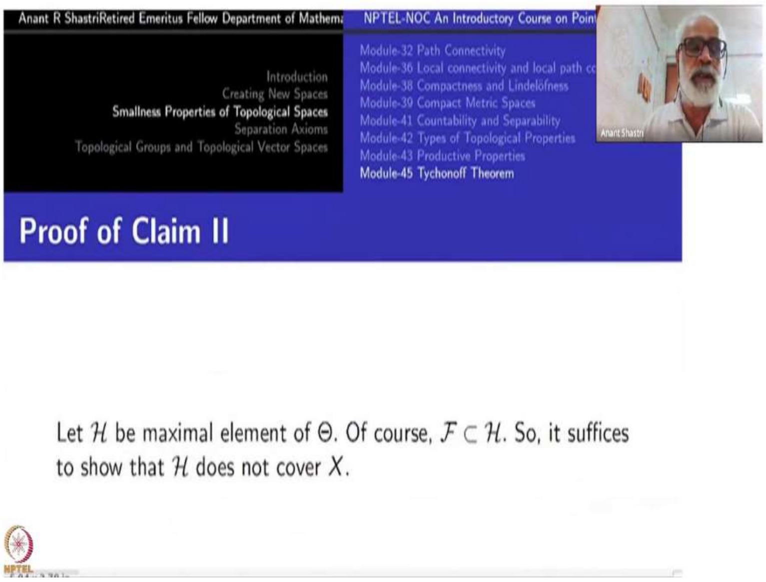
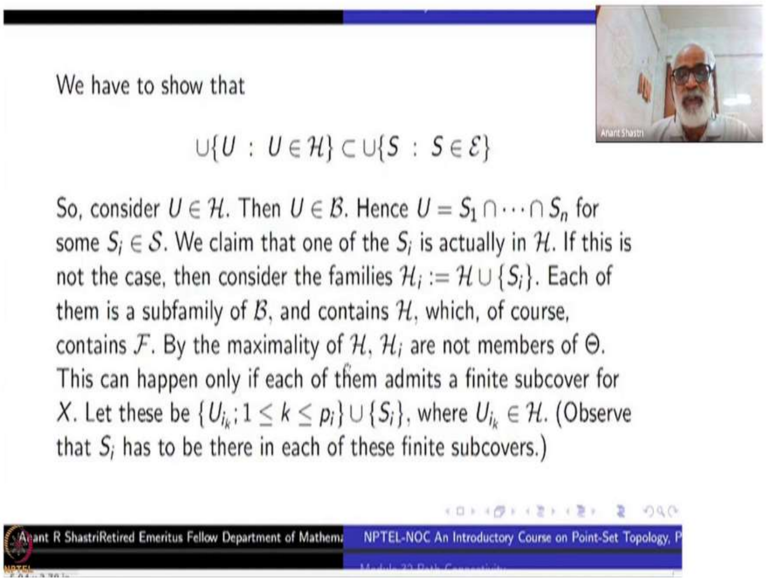


Second claim: take an element, which is a maximal element for Θ. Any maximal element, fix that; that you call $\mathcal{H}$, clearly $\mathcal{F}$ is contained inside $\mathcal{H}$.

So, our aim is to show that $\mathcal{F}$ does not cover X. So, we showed $\mathcal{H}$ does not cover X. In fact, what we will show is the following. Namely, the claim which says, (this claim is even stronger), than what we need. (Refer Slide Time: 14:23)



Namely, union of all members of $\mathcal{H}$, is actually contained inside union of all members of this $\mathcal{E}$, ok. Remember this $\mathcal{E}$ comes from the subbase $\mathcal{S}$, a subfamily of the subbase $\mathcal{S}$ ok? And we know that this does not cover. So this is what we have to prove ok, which is stronger than just showing that $\mathcal{F}$ does not cover ok.

Let us prove this one now. Take a member here U inside $\mathcal{H}$, by the very definition U is a member of the base. Member of the base means, it is intersection of finitely many elements $S_1, S_2, \ldots, S_n$ from the subbase $\mathcal{S}$.

We claim that one of the S_i's is actually inside $\mathcal{H}$ ok? See started with some U inside $\mathcal{B}$ ok we want to show that; that U is contained in the union of these things ok. So, now, what we end up with saying that one of this S_i is actually inside $\mathcal{H}$, ok.

If this is not the case, let us say none of them is inside $\mathcal{H}$. So, this is another subclaim you may say claim III, but claim II is not yet done--- it is part of claim II.

So, suppose none of these $S_1, S_2, \ldots, S_n$ is not inside $\mathcal{H}$. Then consider the family $\mathcal{H}_i$, which is $\mathcal{H} \cup \{S_i\}$, add this one more member, you get another family. Add S_1 to $\mathcal{H}$, you get one family $\mathcal{H}_1$. Add S_2, you get $\mathcal{H}_2$, add S_3 you get $\mathcal{H}_3$ and so on. So, you get $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ etc which are all larger than $\mathcal{H}$. So, I have put one extra element and that I am assuming is not inside $\mathcal{H}$, each of them is a sub family of $\mathcal{B}$, ok.

So, every element inside this $\mathcal{S}$ is inside $\mathcal{B}$, also and contains $\mathcal{H}$ which of course, contains $\mathcal{F}$, but by maximality of $\mathcal{H}$ these $\mathcal{H}$ are not members of Θ, you see they are larger than $\mathcal{H}$ because the each $\mathcal{H}$ is a maximal element. This can happen only if each of them, each of them admits a finite sub cover. There are members in this family such that union of them is a finite cover for X, ok.

So, what are those members? If you pick up only members from $\mathcal{H}$ that is not going to cover. So, each time you have to put S_i also, but just S_i will not cover, along with some members here finitely many, this S_i will cover that is the meaning of that ok. So, I get for each i say

$U_{i_1}, U_{i_2}, \ldots, U_{i_{p_i}}$, for $1 \leq k \leq p_i, U_{i_k}$; these are the members of $\mathcal{H}_i$, actually they are members of $\mathcal{H}$. All these U_{i_k}'s are members of $\mathcal{H}$.

And what are they? They are such that this union is whole of X along with S_i. S_i has to be there if S_i is not there that will be give you $\mathcal{H}$ is a finite cover that is the assumption that $\mathcal{H}$ is the $\mathcal{H}$ does not have a finite sub cover.

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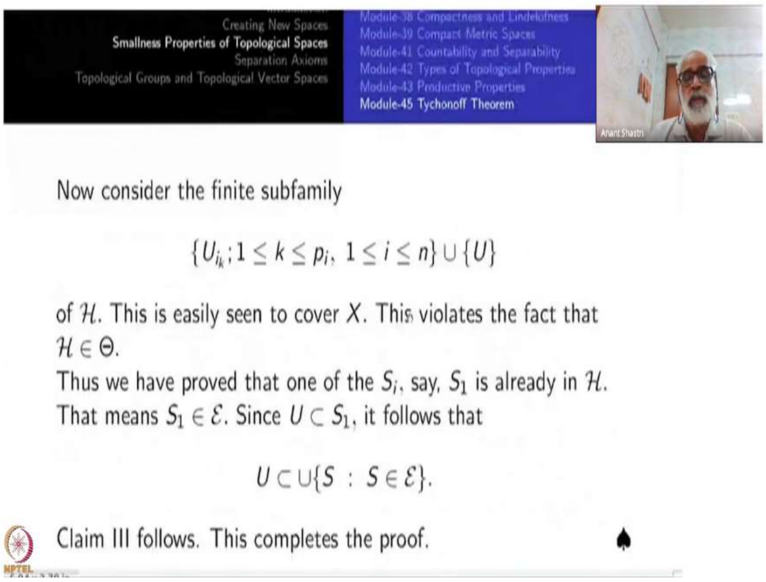

Now, look at all these U_{i_k}'s, from $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ and so on, along with instead of S_1, S_2, S_3 and so on ok. You just take one set U, viz, the intersection of S_i's, ok? And now the crux of the matter is that this will be a finite sub cover for X, clearly it is finite it is a finite sub cover for X, but these are all members of $\mathcal{H}$ only now. These are the members of $\mathcal{H}, U$ is starting with I remember it was $\mathcal{H}$. Why this is the cover for X? Take any point in X. If it is inside U_{i_k}'s for any i, that is fine.

Otherwise, the point will be in the corresponding S_i. Each time, if x is here that is ok, but if this is not here, i.e., none of these U_{i_k}'s contain x, then this point must be inside S_i. This happens for every i which is the same thing as saying that point is inside U therefore, this is a cover ok. So, this violate the fact that $\mathcal{H}$ is belongs to theta because there is a finite sub cover.

So, what we have proved is a sub statement here that one of the S_i's let us call it as S_1 is already in $\mathcal{H}$ ok. So, that just means that this S_1 is an element of this $\mathcal{E}$ right. See S_1 is already inside $\mathcal{H}$, S_1 where we started with $S_1, S_2, \ldots, S_n$ ok they are intersections of yeah, this each S_i's are members of $\mathcal{S}$ ok. This this is the base this is the basic elements here alright.

But, what is $\mathcal{E}$? $\mathcal{E}$ is $\mathcal{H} \cap \mathcal{S}$ right, if it is in $\mathcal{H}$ also and instead of just being a subbasic, instead of basic if it is a basic open set, that will be inside $\mathcal{E}$ by definition. Therefore, U is inside S_1, just means that U is contained in the union of all these. This is the second part that we wanted to show, the RHS here, RHS here, one of the members contains U because that element is inside $\mathcal{E}$, ok.

So, U is one of them, so started with this one. So, we have shown that this union is contained inside that one. So, that completes the proof of the claim 2 and therefore, Alexander subbase is proved therefore, we have completed proof of Tychonoff's theorem also ok.

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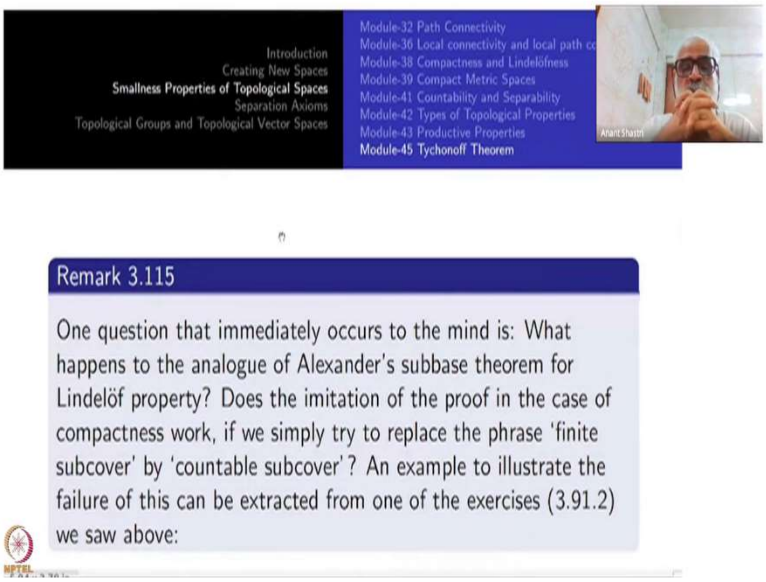

One question that immediately occurs to our mind is what happens to the analogue of Alexander's subbase theorem for Lindelof property?

Remember we have already seen that Lindelof is not even finite productive, namely we have seen that $(\mathbb{R}, \mathcal{L})$ where $\mathcal{L}$ is the semi interval topology has the property that the product is not Lindelof, whereas, $(\mathbb{R}, \mathcal{L})$ itself is Lindelof ok. But Alexander subbase theorem may still be true. Why that is not true? What is happening is it not true at all.

So, this is what we want to question. Does the imitation of the proof in the case of compactness work, if we simply try to replace the phrase, finite sub cover by countable sub cover. Wherever `finite' occurs, replace it with `countable', countable why, where do we go wrong?

You can figure it out where it is, but here is a concrete example which says that even in the Alexander sub base theorem you know the statement will not be true if you replace compactness by Lindelof property ok.

So, that makes the Alexander sub base more important in some way. See, it is just such a narrow thing and still he was able to figure it out. That is the whole idea ok. So, for this one, I will just quote this exercise 3.91.2 ok, but let me give you a little bit of this one what is it.

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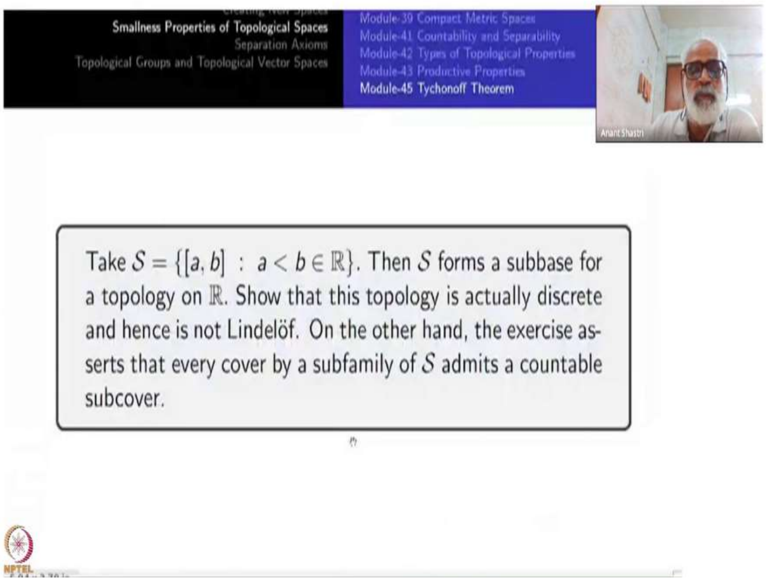

Look at the family $\mathcal{S}$ of closed intervals $[a,b]$, where $a < b$, ok. Then $\mathcal{S}$ forms a subbase for a topology on $\mathbb{R}$. Any family of subsets of a given set X will form a topology as a subbase ok. But this topology is nothing but the discrete topology ok. This topology is a discrete topology, why? Because given any x, I can take the interval $[x-1,x]$ and other one $[x,x+1]$, both closed intervals. Then the intersection will be just singleton x.

So, every singleton $\{x\}$ is open means, hence it is a discrete space. On the other hand so, once it is a discrete space by the way, an uncountable discrete space is not Lindelof ok, on the other hand, the exercise there asserts that every cover by a sub family of $\mathcal{S}$, admits a countable sub cover. This is for the usual topology of $\mathbb{R}$ ok, with the usual topology of $\mathbb{R}$ you show that $[a,b]$ are closed use the topology for that is a finite sub cover, we just show.

The finite sub cover part is just set theory, it covers $\mathbb{R}$ even this may be sorry this is not finite sub cover countable sub cover. This may be uncountable sub cover normally, but you can have a countable sub cover. So, this is the exercise. The point is if you take open intervals (a,b) then of course, you know it, because $\mathbb{R}$ is Lindelof, you have to use that also. But, now we have to show that even if you cover it by closed intervals it will have a countable sub cover ok.

So, granting that exercise what we have is the following, namely Alexander subbase theorem is not true for Lindelof's property.

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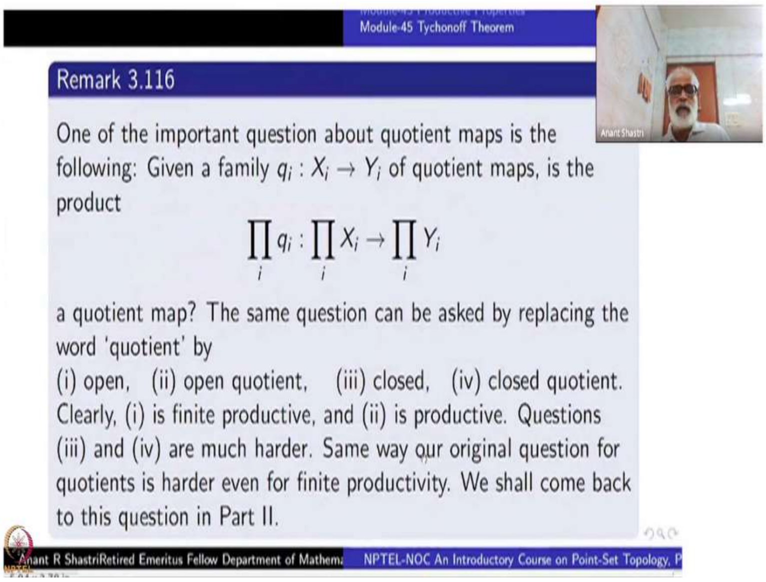

There is another remark which I would like to do, not with Alexander subbase theorem or quotient theorem and so on, but another property for whether something is productive. Namely quotient maps, take two quotient maps and take the product of this maps. So, X_1 to Y_1, X_2 to Y_2, then you have a product map from $X_1 \times X_2$ to $Y_1 \times Y_2$ ok.

More generally, you can ask for families X_i to Y_i or q is of quotient map then you can take the product map here from product of X_i to product of Y_i. Is it a quotient map? And then you can study the properties under quotient straightforward. This may fail, but suppose you take open maps. suppose you take open quotient maps or closed maps or closed quotient maps and so on. So, there are a number of such problems.

So, I will just sum it up we will not go into deeper study of these things, they are not too difficult or they are not too easy. In fact, some some of them will be taken in part 2 of this course. But right now you can observe that openness is finite productive, if you take an open map, two open maps, product will be an open map, that is very easy to see ok.

And if you have open quotient means open and surjective map that is an open quotient. Then it will be productive, no restrictions even you can take arbitrary products ok 3 and 4 are much harder, just if you take arbitrary quotients, just quotient maps even two of them, will not be a quotient map, need not be a quotient map ok, unless you assume some more hypotheses ok.

So, that is an interesting case here which is needed in many other places also so but that will be done in part 2. So, we will stop here with productive properties and so on whatever so far properties which we have studied. So, next time we shall start studying some more topological properties. They will be in general; they will be called what? What is the name? Largeness properties.

Thank you.