Introduction to Point Set Topology, (Part I)
Prof. Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay

Module - 45
Lecture - 45
Tychonoff Theorem

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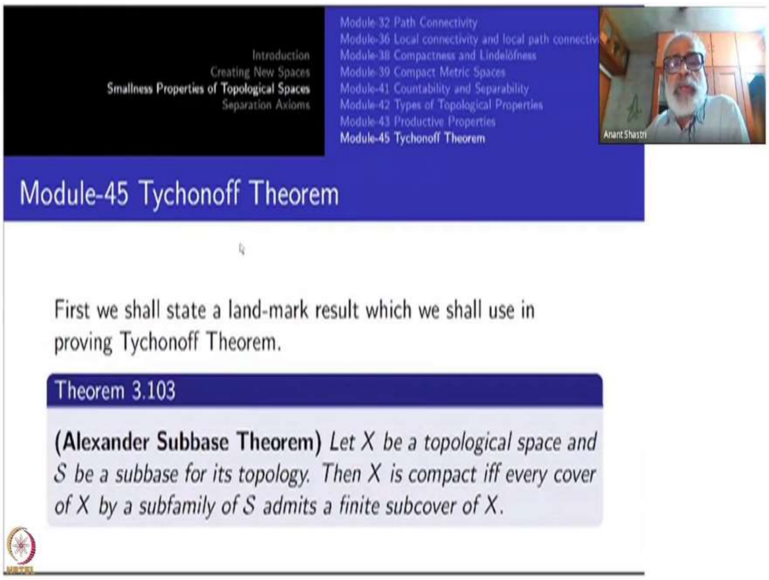

Welcome to module 45 Tychonoff Theorem. So, Tychonoff theorem is a landmark result in the development of point set topology. The key result that we are going to use as a step for Tychonoff theorem is called Alexander's Subbase Theorem. Let X be a topological space and fix a subbase $\mathcal{S}$ for this topology. Then X is compact if and only if every cover of X by a subfamily of $\mathcal{S}$ admits a finite sub cover for X.

We have seen that if X has a base such that open covers from this base, admit a finite subcover, then X is compact. That much we have seen already. But Alexander's subbase theorem goes one step ahead. It says instead of base you can just use a subbase. The proof of Alexander's subbase theorem itself is not very straightforward. So, we should postpone the proof of this one, but use this one to prove Tychonoff's theorem ok?

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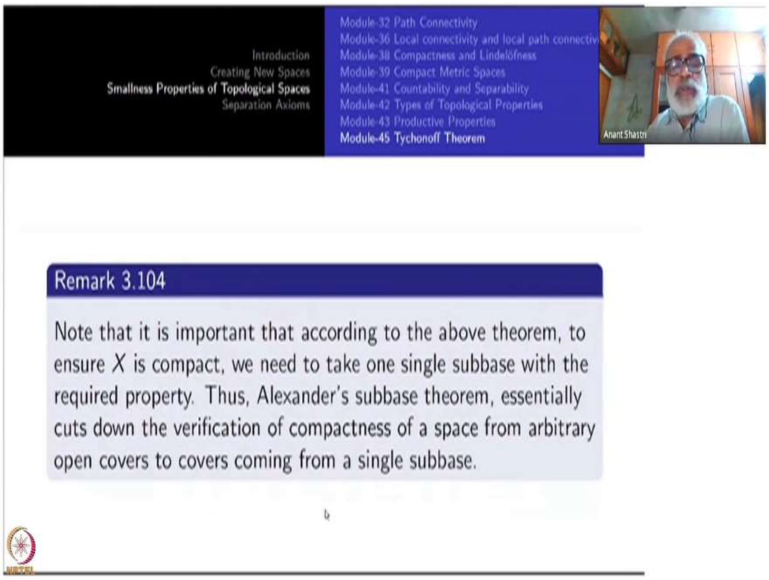

So, once again I repeat that Alexander's subbase theorem essentially cuts down the verification of compactness of a space, from arbitrary open covers to open covers coming from a single subbase. So, that is the whole idea.

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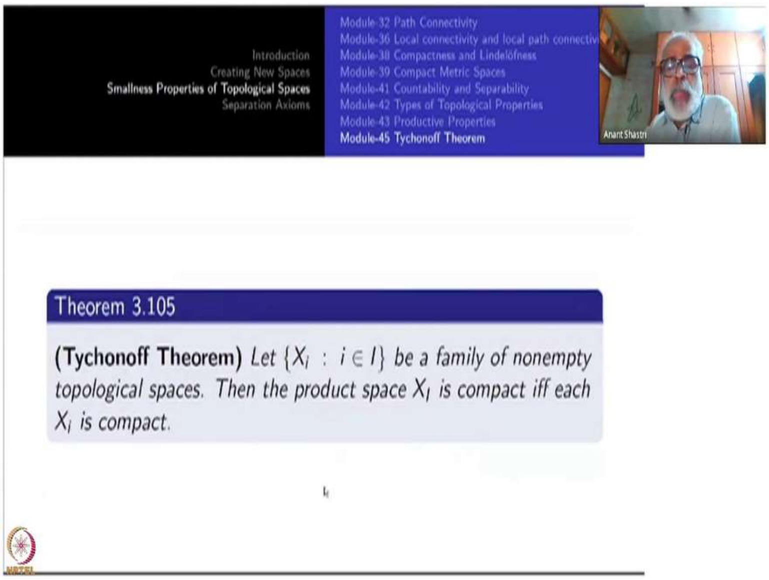

Tychonoff's theorem can be stated as follows, start with a family of topological spaces each of them non empty. Then the product space X_I is compact if and only if each X_i is compact ok? What we have seen is that, if the product space is compact each factor space is compact of course. For this you have to use the that X_i's are non empty. Therefore, the projection maps are surjective. A surjective continuous function takes compact sets to compact sets. That is a very easy theorem that you have proved. Using that it will follow that if product space is compact then each X_i is compact.

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Now, we have to prove the converse. Assume that each X_i is compact. Let $\mathcal{S}$ be the standard subbase for the product topology which we have been using, namely, consisting of all $p_i^{-1}(U_i)$ for all open sets U_i inside X_i and for all $i \in I$. By Alexander's subbase theorem if we show that an open cover $\mathcal{S}'$ from $\mathcal{S}$, you know members of $\mathcal{S}$, admits a sub finite sub cover that is enough ok? Any arbitrary open cover, but members are from $\mathcal{S}$, if that has a finite subcover then X should be compact.

So, that is what we want to show now. Starting with an arbitrary open cover with members of $\mathcal{S}$ we will show that there is a finite subcover.

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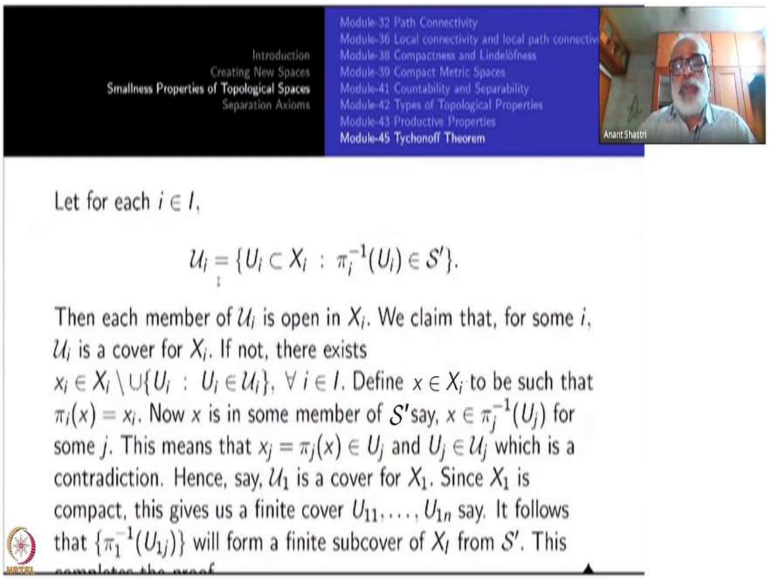

So, for each $i \in I$ put U_i equal to those U_i inside X_i such that this $p_i^{-1}(U_i)$ is in $\mathcal{S}'$. This $\mathcal{S}'$ is a cover that you have been given, members of $\mathcal{S}'$ cover X. From this we want to extract a finite sub cover. So, first I define $\mathcal{U}_i$, this is a subfamily of open subsets in X_i, consisting of those U in X_i such that $p_i^{-1}(U)$ belongs to this $\mathcal{S}'$.

Then each member of $\mathcal{U}_i$, by the very definition is open in X_i, ok? We claim that for at least one i, $\mathcal{U}_i$ is a cover for X_i, ok? We have not claimed that all the $\mathcal{U}_i$'s will cover the corresponding X_i here. At least for one of the indices this must happen is what we want to say. If not, what happens? there exists some little x_i belonging to capital X_i minus the union of all the U in $\mathcal{U}_i$ because this is not a cover for every i this will happen ok?

So, pick up one point x_i in the complement of this. So, this gives you one element x belonging to X_I such that $\pi_i(x) = x_i$. Now x is in some member of $\mathcal{S}'$ say because this this is my this $\mathcal{S}'$ here ok say x belongs to $\pi_j^{-1}(U_j)$ for some $\pi_j^{-1}(U_j)$ belonging to some $\mathcal{S}'$, that is what for some j. This means that, if x belongs to this one means its j^{th} projection x_j which we have chosen must be inside U_j and with U_j belonging to some $\mathcal{U}_j$. That will be a contradiction, because we have chosen this one to be such that they are in the complement of all this ok.

So, therefore, one of the $\mathcal{U}_i$ will cover the whole of X_i. Say $\mathcal{U}_1$ just for definiteness sake, $\mathcal{U}_1$ is a cover for X_1. Since X_1 is compact this gives you a finite sub cover which you call $U_{11}, \ldots, U_{1n}$ ok. It follows that $\pi_1^{-1}(U_{1j})$'s you take π_1 inverse of these things, they will form a finite sub cover for X, from $\mathcal{S}'$; they are members of $\mathcal{S}'$, that is how we have chosen.

But this is now cover because X_1 is the union of all these things. So, p_1 inverse of for all these things will be the whole inverse of this and will cover for X_I and these are members of this. So, that completes the proof ok.

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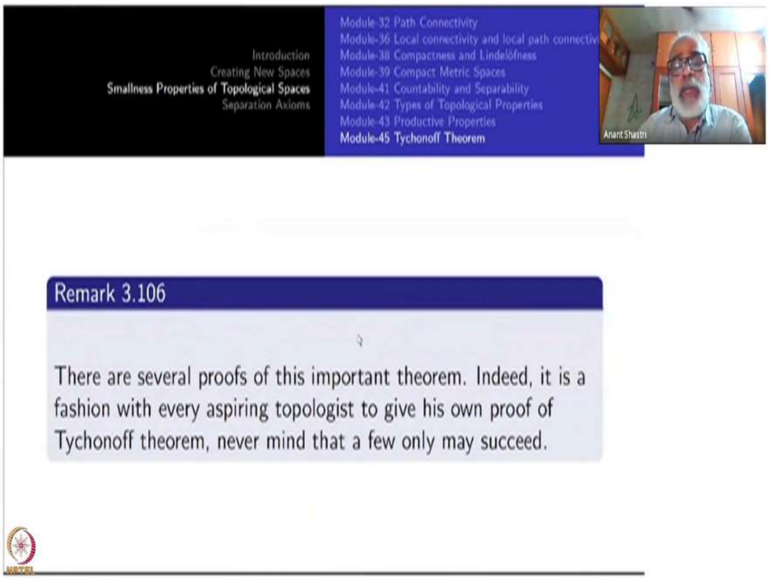

That completes the proof of Tychonoff's theorem. There are several proofs of this important theorem. Indeed it is a fashion with every aspiring topologist to give his own proof of Tychonoff's theorem, never mind that only a few of them may succeed. Nevertheless there are quite a few proofs of this theorem ok.

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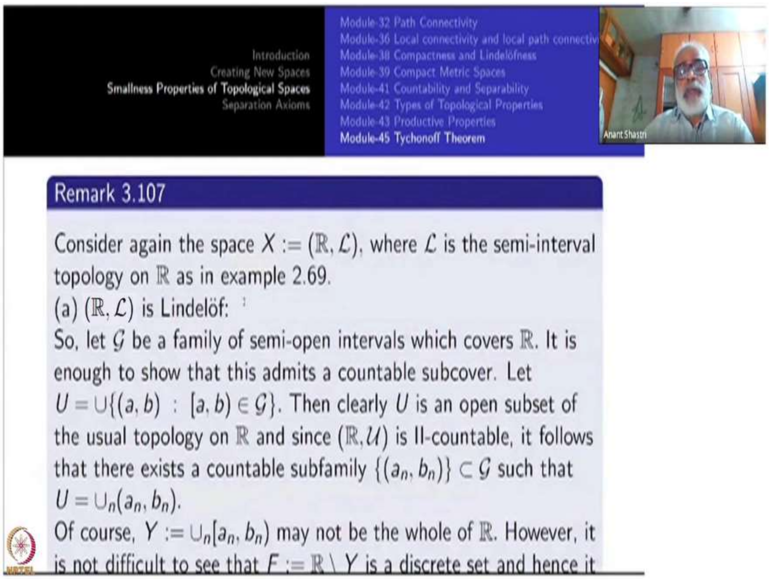

Now, let me give you an example here, namely, our favourite example $\mathbb{R}$ with semi interval topology lower limit topology and so on there are various names or this. This we have seen earlier several times ok. This space is a Lindelof space ok.

So, let $\mathcal{G}$ be a family of semi open intervals which cover $\mathbb{R}$ ok. So, I am trying to prove why it is a Lindelof space it is not very easy to see that ok?

It is enough to show that this admits a countable sub cover ok. So, I am using that this $\mathcal{S}$ is actually a base here open subsets of the form $[a, b)$ that is the definition of this topology $\mathcal{L}$, ok? Take a cover by these open sets there is no need to take unions of these things and so on this is the base.

So, you can take this one and then show that it admits a countable sub cover ok. So, what do we do? We will use the property of $\mathbb{R}$ in the usual topology and then compare it with this one. So, put U equal to the union of open internals (a, b) where this $[a, b)$, a closed here, these are members of this $\mathcal{G}$ ok? So, we started with a family $\mathcal{G}$ of semi open intervals which covers $\mathbb{R}$.

Now you drop out the the first point here in each interval ok? and take only open intervals (ab). Then this union will be clearly an open subset of the usual topology in $\mathbb{R}$, but now the usual topology is second countable.

Therefore, it follows that there exists a countable sub family $[a_n, b_n)$ contained inside $\mathcal{G}$ such that this U is union of countably many open intervals (a_n, b_n). So, what I am using is every open subset of a second countable space is second countable. Therefore, it is a Lindelof ok. If I just use directly $(\mathbb{R}, \mathcal{U})$ is Lindelof, I do not know how to conclude this one is Lindelof because it is an open subset not a closed subset ok. But if you say second countable, then every subspace is second countable and a second countable space is Lindelof.

So, you get a countable sub cover for this subspace U which is an open subset. Now I put back all the points a_n's but only taking from this countable sub cover, put Y equal to union of $[a_n, b_n)$ put back these points. This may not be the whole of $\mathbb{R}$. If that were $\mathbb{R}$, you are fine.

See we dropped out all these initial point of the interval is right? Now you put back some of them, but that may not be the whole of $\mathbb{R}$; however, it is not difficult to see that whatever is left out namely, put $F = \mathbb{R} \setminus Y$. What are they? They are all the starting points of intervals $[a, b)$ coming from $\mathcal{G}$, some of them some of them are already here in this countable family, some of them are left out.

That space $F = \mathbb{R} \setminus Y$ is definitely a closed subspace. It is a discrete set. No open interval will be contained inside that one, the open parts have been taken care by this (a_n, b_n) that is easy to see that. And hence it follows that F is countable. Once F is countable you pick up open subsets which cover F, ok, for each one of them, one open subset from $\mathcal{G}$. Along with that you put all these $[a_n, b_n)$'s also which you have got already. Together you get a countable family that will cover the whole of $\mathbb{R}$, ok?

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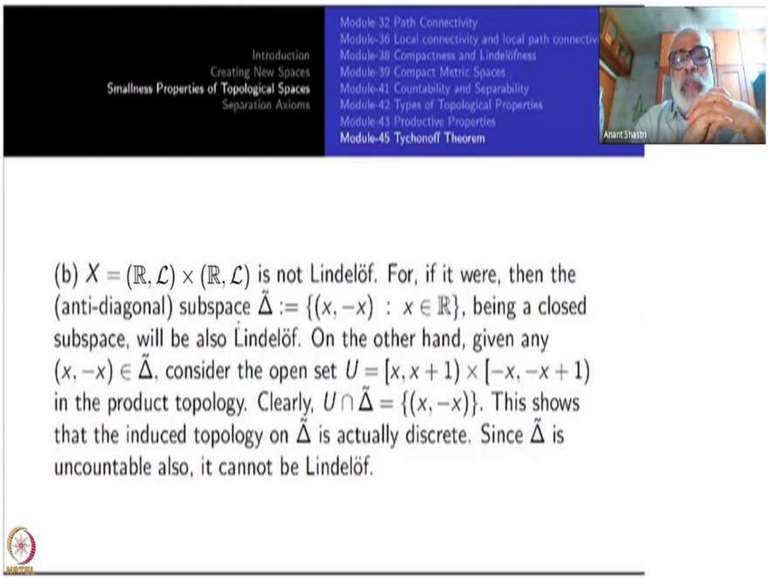

So, that shows that the semi open interval topology is Lindelof. But now I am going to show that the product is not Lindelof, see that was my idea. You know we showed products, even infinite products and so on of compact space is compact right? And we are all the time telling that Lindelof property keeps tagging along. So, this is one place where it does not, even product of two of them need not be Lindelof ok.

So, take X equal to $(\mathbb{R}, \mathcal{L}) \times (\mathbb{R}, \mathcal{L})$, I should be take this should be taken $\mathcal{L}$ because I have used this notation $\mathcal{L}$ here ok. So, look at that, it is not Lindelof. For if it were then the anti-diagonal $(x, -x), x \in \mathbb{R}$, we have used this one earlier ok? The anti diagonal, being a closed subspace ok, will be also Lindelof because closed space of a Lindelof space is Lindelof.

On the other hand given any point $(x, -x) \in \tilde{\Delta}$. Consider the open subset, $[x, x+1)$ or $x + r$ whatever, cross $[-x, -x+s)$ some positive number $-x+s$ and not necessarily 1, you take these open subsets.

These are open subsets in $(\mathbb{R}, \mathcal{L}) \times (\mathbb{R}, \mathcal{L})$, ok, in the product topology alright? What is the intersection of this one with the anti diagonal? It would be just the first point $(-x, x)$ ok. So, intersection with the subspace $\tilde{\Delta}$ is just the singleton; that means, all the singletons are open in $\tilde{\Delta}$ in the subspace topology. It just means that $\tilde{\Delta}$ is a discrete space.

However its cardinality is the cardinality of $\mathbb{R}$, its uncountable ok? For each $x \in \mathbb{R}$ there is $(x, -x)$. So, an uncountable discrete space cannot be Lindelof ok.

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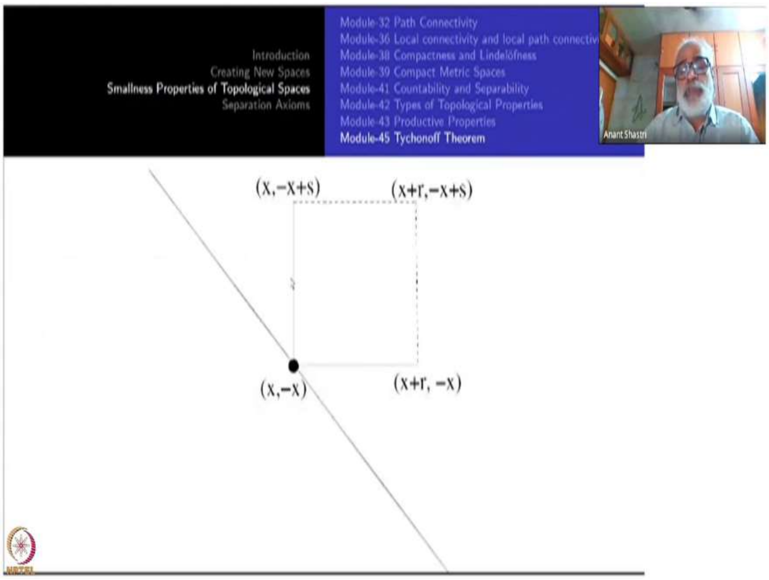

So, that is a picture for showing that the anti diagonal is what? It is a discrete subspace. So, we have given an example of a Lindelof space, the product with itslef is not Lindelof.

We have shown that the product is compact if each factor is compact, but the Lindelof of property is not even finite productive ok.

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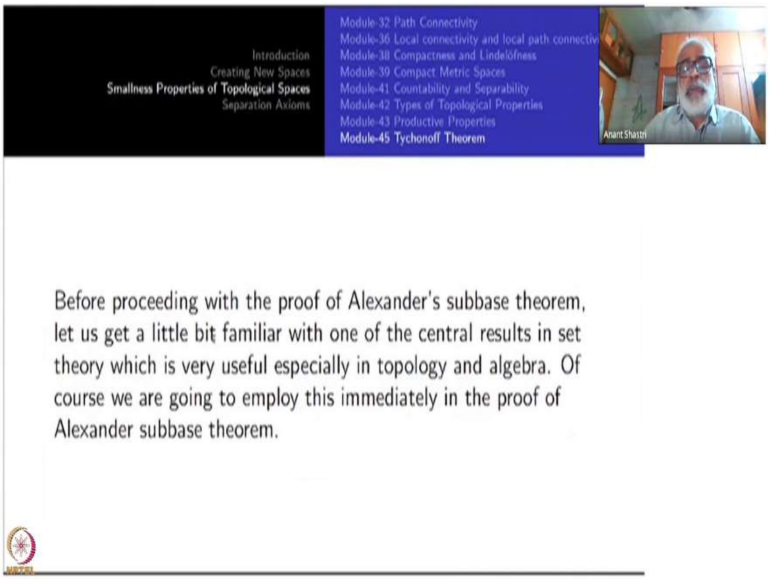

Coming back to this Alexander subbase theorem, let us get a little bit familiar with one of the central results in the set theory which is very useful especially in topology and algebra. Of course, we are going to employ this immediately in the proof of Alexander subbase theorem ok. So, a little more point set topology here today, and then we will wind up today. Tomorrow we will again proceed with Alexander's subbase theorem.

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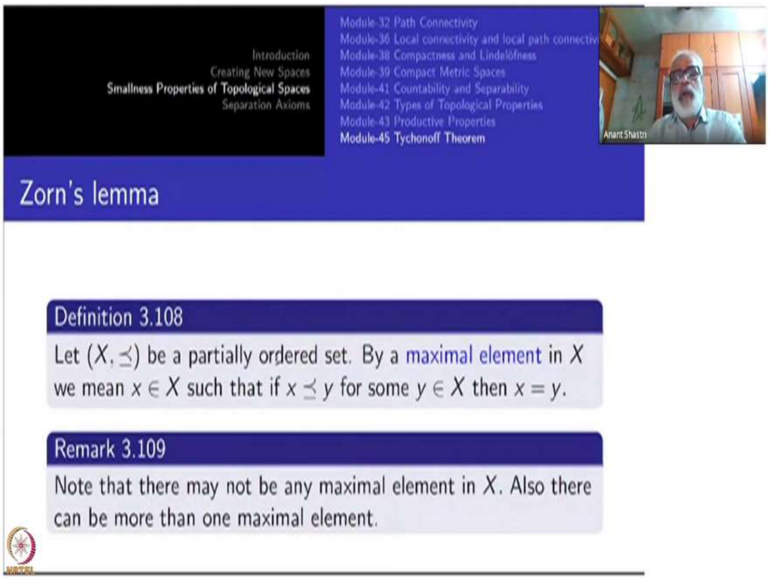

So, this is about partial order and so on. Start with a set which is partially ordered. Preferably nonempty ok? Do not take empty set and all. By a maximal element in X, we mean x belonging to X such that x is less than equal to y for some y will imply $x = y$.

There is nothing sitting over x. So, that is the meaning of maximal element alright? Note that there may not be any maximal element inside X. Like if you take $\mathbb{R}$ with the usual order it has no maximal elements. That means, it is not bounded above. So, there is no maximal element.

Also there can be more than one maximal element ok. So, you can think about that. You know if you take some subsets of any set, then put inclusion relation, it can have bigger ones one bigger ones another bigger here, those two are not comparable and so on. There are lots of such examples alright. So, maximal elements may not exist and also there may be plenty of them also either of them can happen.

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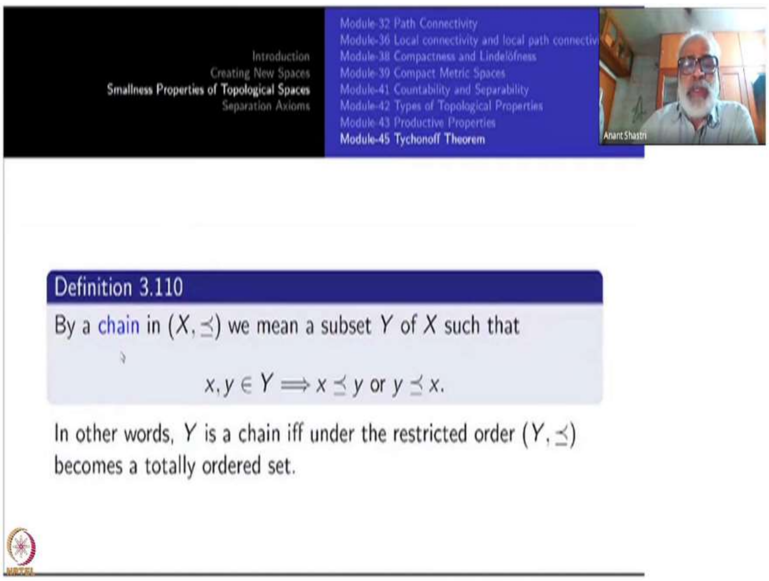

Now, there is another simple definition. By a chain in a partial order set we mean a subset of the form whenever x and y are inside this subset either x must be less than or equal to y or y must be less than or equal to x ok. Of course, if both of them happen that is also allowed, but then x will be equal to y, that is by definition. In other words, Y is a chain if and only if under the restricted order it becomes a totally ordered set. That is another name for it, totally ordered sets always have this property ok.

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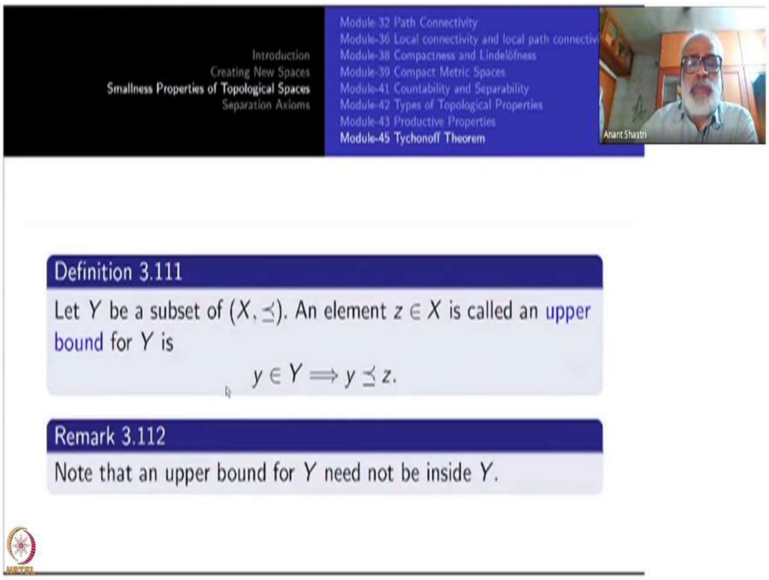

A totally ordered subset of a partially ordered set would be called a chain. We have just introduced another word here which is popular in set theory. Let Y be a subset of some partially ordered set. An element $z \in X$ is called an upper bound for Y if y belongs to Y implies y is less than equal to z. So, this is an upper bound.

So, all these things are very straightforward definitions, but these definitions are now made in arbitrary partially order set that is what you have to be careful not inside $\mathbb{R}$ or $\mathbb{Q}$ or integers and so on ok.

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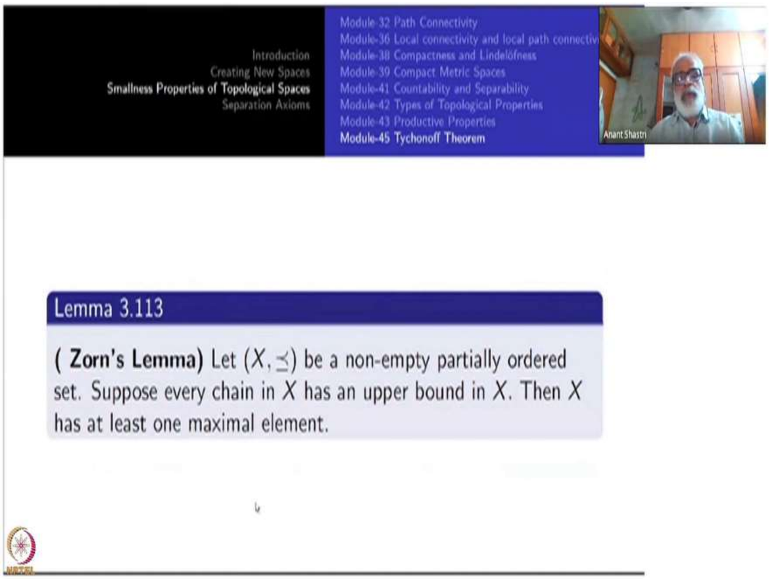

They will of course apply to $\mathbb{R}, \mathbb{Q}$ etc, all those things also. But this is a general partially ordered set alright. Now, here is what is called the Zorn's lemma which is almost like one of the axioms of Set theory, as such, equivalent to axioms of choice. Zorn's lemma says that start with any nonempty partially ordered set. Suppose every chain in this X has an upper bound.

Then X has at least one maximal element. It does not say anything about uniqueness of course, there may be plenty of them. It just assures you there is a maximal element ok, that is Zorn's lemma alright.

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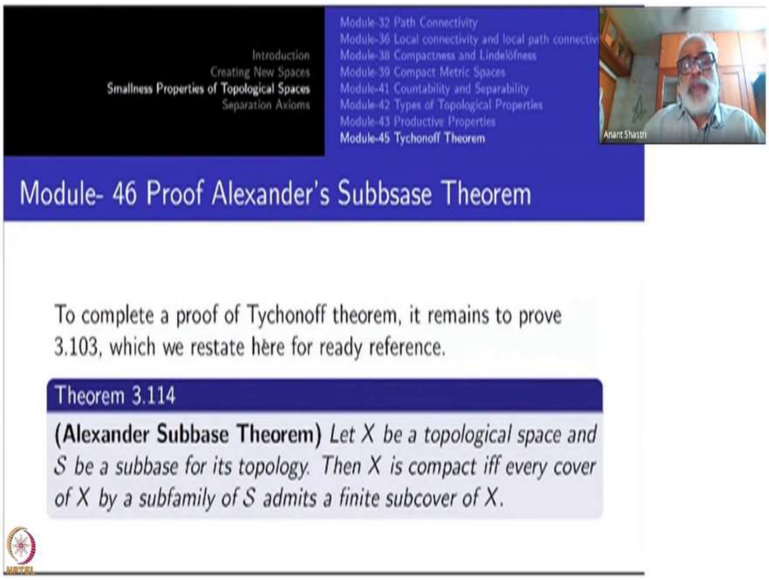

So, next time we shall use this one very effectively to prove Alexander subbase theorem.

Thank you.