Introduction to Point Set Topology, (Part I)
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Module - 44
Lecture - 44
Productive Properties - Continued

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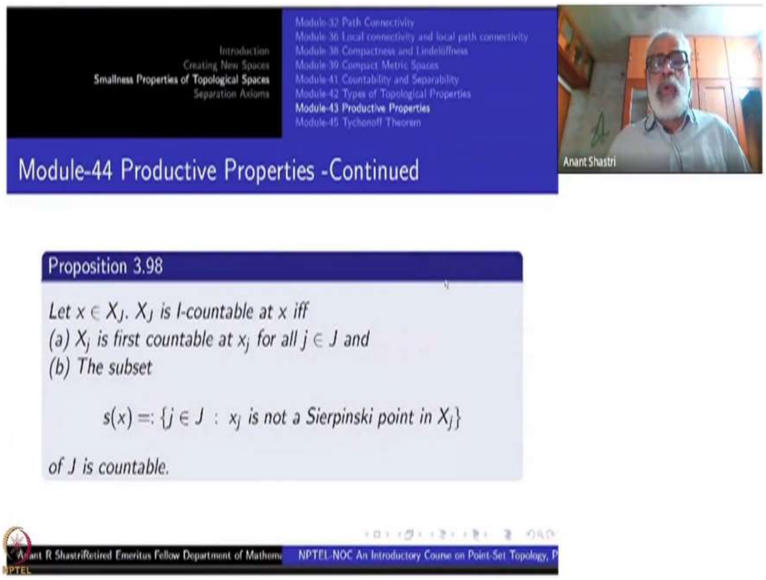

Welcome to module 44 on Point Set Topology course. So, today, we will continue our study of properties which are preserved under products, Productive Properties. So, let us begin with this proposition.

You start with an arbitrary product of topological spaces X_j, pick up a point in it. So, X_J will be first countable at that point if and only if each factor X_j is first countable at x_j for all $j \in J$. So, all coordinate points are having a countable base that is the first condition.

Second condition is that the subset $s(x)$ of the indexing set all j belong to J such that this x_j is not a Sierpinskiski point in X_j. Look at all such indices. That set $s(x)$ must be countable. Then the other part is if and only if. So, then the converse is also true. This is what the proposition says. These are somewhat not very straightforward so, let us go through the proof carefully ok.

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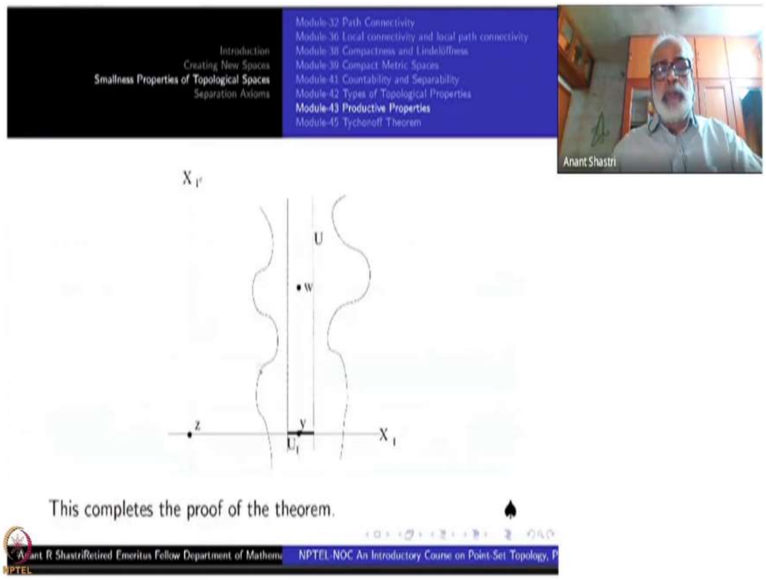

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So, first, suppose X_J is I-countable, the product space, at the point x belonging to X_J. That means that we have a countable local base for X_J at the point x. That is the meaning of it is first countable at x. Then, (a) follows from the fact that all the coordinate projections are open and surjective.

So, if you take $p_i(B)$ where B ranges over the countable local base, that will give you a countable base for X_i at the point x_i ok? So, this we have seen that first countability is weakly hereditary in the sense that under open surjective maps, it is preserved. So, part (a) is proved, the second part is something peculiar ok?

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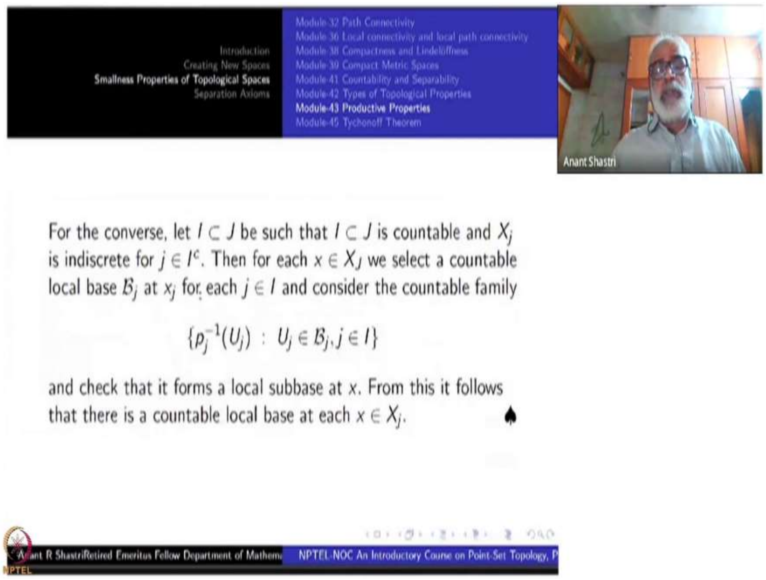

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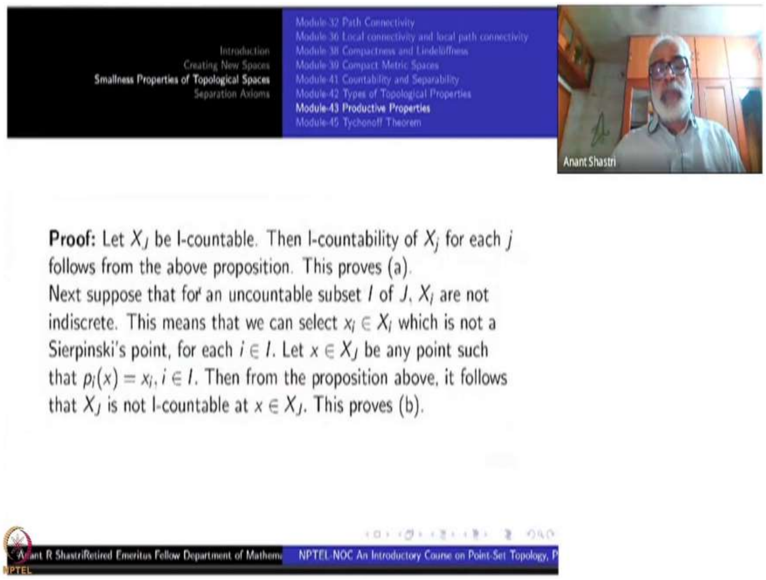

Suppose that for some uncountable subset I of J, X_i is not a Sierpinski's point ok? ....

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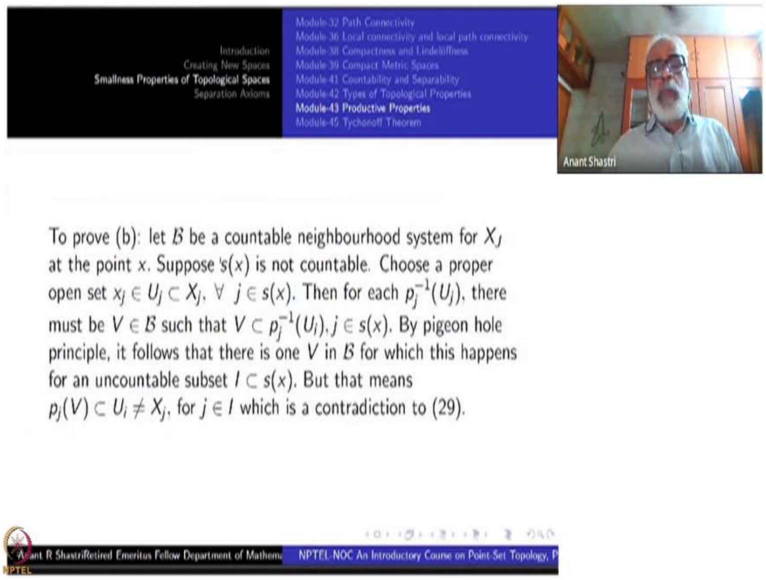

What I am doing here yeah to prove (b), suppose $\mathcal{B}$ is a countable neighbourhood system of X_j at each point and $s(x)$ is not countable. You want to prove that $s(x)$ is not countable; that means, what? There are uncountably many j such that x_j is a not Sierpinski's point means that they have proper open subsets as neighbourhoods, Sieripinski point is defined to be such a point wherein the only open set containing that point is the whole space right?

So, not a Sierpinski's point means x_j belongs to U_j; U_j open and U_j is a proper subset of X_j, for every j inside $s(x)$ with $s(x)$ is uncountable. So, now I am assuming that this is not countable. Now, what happens? This is countable, that is not countable, so something happens.

Then, for each $p_j^{-1}(U_j), p_j^{-1}(U_j)$ is a subbasic open set right. There must be a V belonging to $\mathcal{B}$ that is $\mathcal{B}$ is the countable base at the point x for X right? So, this V must be inside $p_j^{-1}(U_j)$ for j in $s(x)$, ok?

By pigeon hole principle, while for each V there is such a thing I mean for each, but number of members in $\mathcal{B}$ this is only countable, but these j's are coming from an uncountable set $s(x)$. So, it follows that one of the V inside $\mathcal{B}$ for which this happens for an uncountable subset I of $s(x)$, right?

If all of them are countable, countable union of countable sets will be countable so $s(x)$ will be countable. So, that means, that $p_j(V)$ when you project j^{th} coordinate that is contained inside U_i because V is contained in the $p_j^{-1}(U_i)$ and this is happening and U_i's are not the whole space. So, $p_j(V)$ are proper open subsets, containing proper open subset for every j that is a contradiction to this basic fact that we have observed earlier ok.

This must happen at most for finitely many right? at all other coordinates it must be equal to whole of X_j, but now we have got uncountably many. That is a contradiction. So, that proves what? One way. So, I have to prove the converse. Suppose (a) and (b) are true. Then I have to show that X_J has a countable base at the point x. So, that is easier actually.

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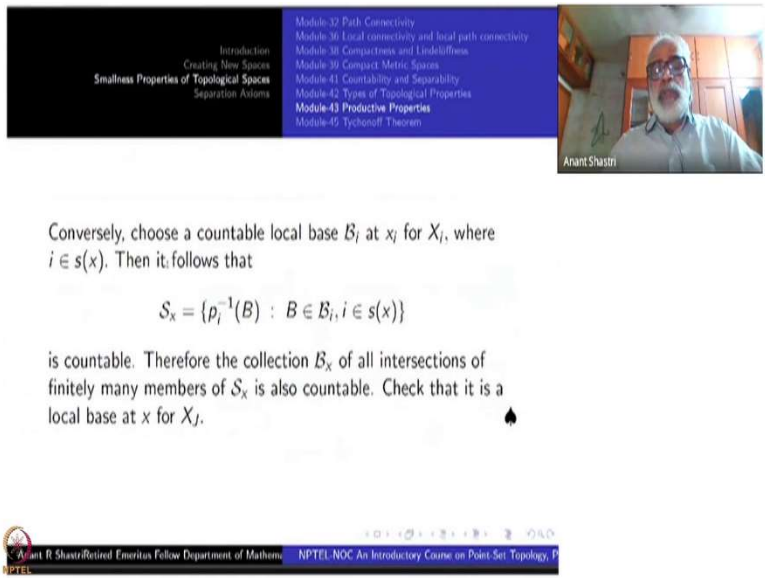

Choose a countable local base $\mathcal{B}_i$ at x_i for every X_i, where i is inside $s(x)$. Remember what $s(x)$ is? $s(x)$ is set of points wherein these x_j's are not Sierpinski's point. For those indices I am choosing countable base. For others what should I choose? It does not matter because the only open set containing x_j will be the whole of X_j, it does not matter. you can choose singleton X_j, that itself is the countable base. So, you can ignore them, that is the meaning ok.

But now, I have chosen for each j, I have chosen a countable base at X_j and $s(x)$ is countable. So, this $\mathcal{S}_x$ is the set of all $p_i^{-1}(B)$, where B is inside this countable family and i runs over all of $s(x)$, this is countable. So, these are countably many open subsets So, this is a countable family ok.

Therefore, if you take finite intersections of members of this family, that will be also countable ok, but once you take all finite intersections that becomes a local base at x_J for the product space X_J.

The Sierpinski points where you see all those indices, they do not trouble you at all ok because, as soon as you take some open subset around corresponding X_j there around x_j, it will be the entire space X_j. ok? So, for those things you do not have to take finite intersections so only on this family you have to take. So, that will become a countable base at x.

Now, why I have proved this one so carefully is that this is a pointwise statement. Now, the same proof we will go through if you want to do globally ok, proof will be the same, but statement will be slightly different. What is that? Because now, you are taking for all points this is happening ok, you want to do that; that is for first countability at all the points of X_J. This was first countability at a single point right? First countability at all the points now. So, that is our next theorem.

X_J is I-countable if and only if each X_j is I-countable that is the first part, this is pointwise; we saw this first part. But the second part says that X_j is indiscrete space for all, but a countable number of $j \in J$.

If all the points are Sierpinski's points in a space right that is an indiscrete space, that is the difference between condition (b) here and condition (b) in the previous proposition. For a countable subset of J, anything can happen. Other than that, all of them must be indiscrete spaces ok? Yeah, product with indiscrete space, does not disturb the rest of the things that is what the theme is here ok.

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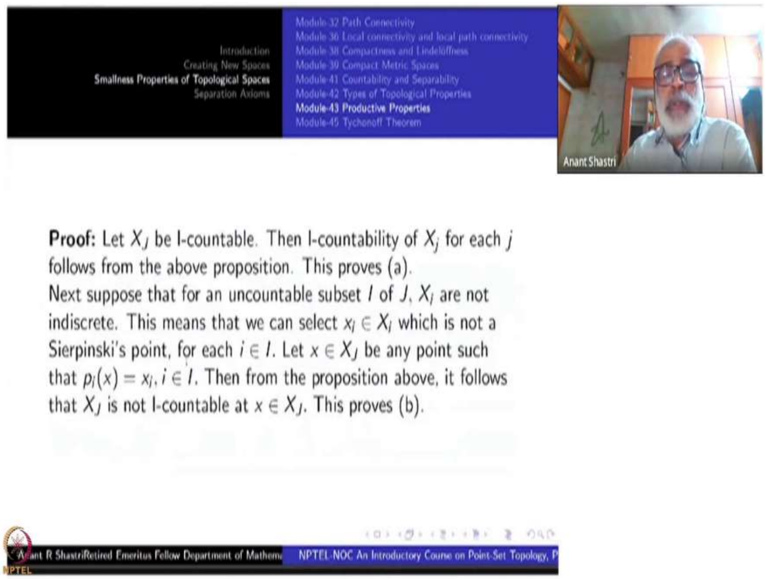

So, let us go through this one, but this is more or less like the previous proposition. Let X_J be I-countable. Then I-countability of each X_j for j follows from the above proposition ok? Same argument like open surjective maps preserve the I-countability. This proves (a).

Next suppose for an uncountable set I of J, X_i's are not indiscrete ok. This means that we can select x_i belonging to X_i which is not a Sierpinski point for each $i \in I$ and this is uncountable set ok. So, having chosen x_i's, you take a point x which has these as the coordinates i^{th} coordinate is equal to x_i, you choose an x belong to X_J such that $p_i(x)$ is equal to x_i on this uncountable set, other things can be anything.

Then from the proposition above it follows that X_J is not I-countable at that point x because condition (b) also proposition is not satisfied. So, by contradiction, you have proved property (b), ok?

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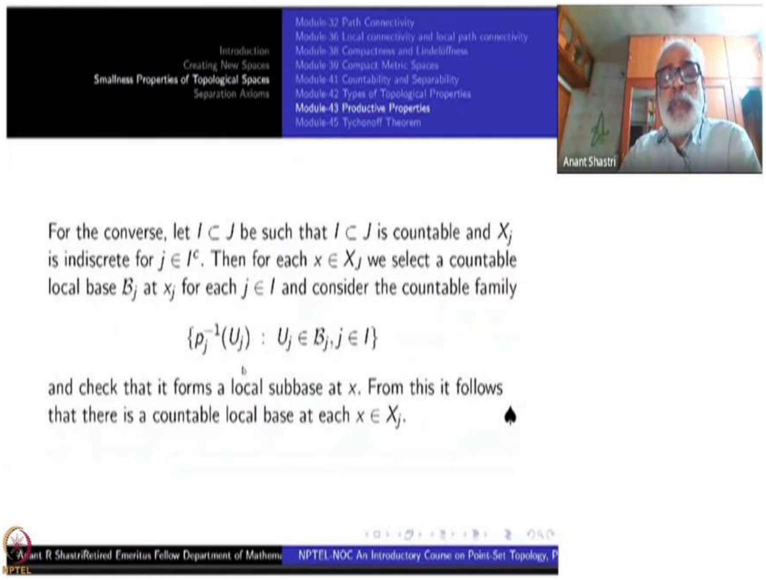

The converse is also easy here. Suppose I contained inside J is countable and X_j is indiscrete for j in the complement. This is the condition we want, away from a countable set it all of them X_j's should be indiscrete. Then for each $x \in X_J$, we select a countable local base $\mathcal{B}_j$ at x_j for each $j \in I$.

See again you have to do it by pointwise construction ok? The indiscreetness gives you for all point the Sieripinski condition (b) is satisfied. Therefore, the same conclusion as in the propesition will work for case also, ok? $p_j^{-1}(U_j)$, where U_j's are inside $\mathcal{B}_j$ and $j \in I$, this I is countable, you check that this forms a local base ok.

This is repetition of the previous part, the only thing is having an indiscrete space away from a countable set will give you the Sierpinski points ok just naturally the condition $s(x)$ is countable will be satisfied.

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Now, we just make a statement, and its proof is straightforward. Let X_J is countable product of an arbitrary family of topological spaces. It will be II-countable if and only if each factor X_j is II-countable and X_j's are indiscrete except for a countable number of $j \in J$. So exactly same thing as I-countability. So, this time you do not have to worry about pointwise, you take a base here, go back come back and so on, the same proof will work.

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Let us come to now separability in our list. Separability is a tricky business ok? One has to be a bit careful here, but whenever things happen, it happens easily also. It is easy to check that if each X_j is separable and J is countable, then X_J is separable ok? All that you have to observe is that product of A_i closure is equal to the closure of the product of A_i's ok? First take the product and then, take the closure, that is the same thing as first take all the closures and then, take the product.

So, if A_i's are dense in X_i, $\bar{A}_i$ will be the whole of X_i ok? Then, you take the countable product, then take the closure that will the whole space. The only thing is (this is always true of course), that a product of countable set is countable. That has to be used. If you take uncountable product, it need not be countable alright?

So, countability has to be preserve so, you have to take J to be countable that is all. However, there is a curious phenomenon here, namely, even if the cardinality of J is the first uncountable, which is denoted by the letter c (this is the cardinality of the reals, For example, ok?) I-continuum ok, then X_J turns out to be separable though the argument I have given does not work and it is not an easy argument here.

So, we have no time to do that one and you know it is not used by us anyway anywhere. So, I have given you a reference you can look into that namely Willards's book. However, if cardinality of J is bigger than c, then even this will fail. Each of X_J may be separable, but the product may not be separable ok?

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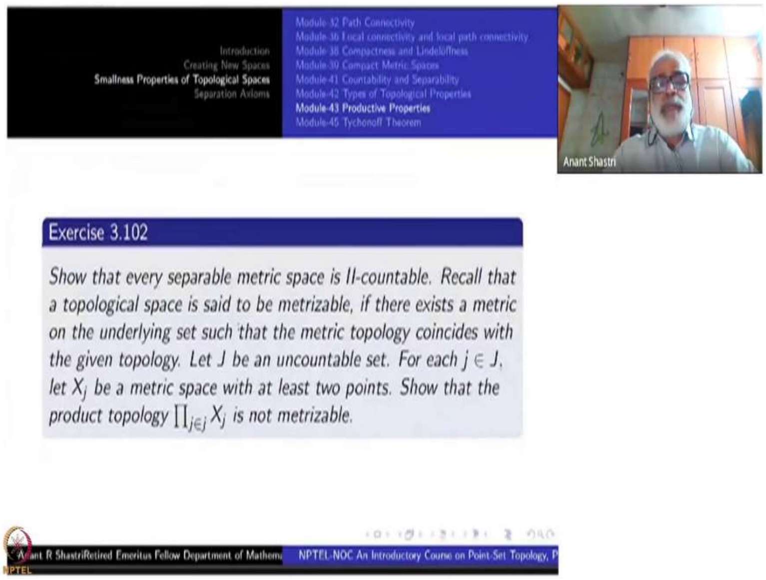

So, let us stop here for today, just summing up what we have done so far. So, after hereditary property and co-hereditary properties checking for various things namely connectivity, path-connectivity, compactness, lindelofness, I-countability, II-countability and separability. Then, yesterday and today, we checked about I-countability and II-countability and separability ok.

So, this is the list of the various properties of topological properties that we have studied so far alright. So, we have still more thing to do with product properties namely compactness and lindeloftness, we have to worry about that. So, that is another topic. So, that will be taken next time ok.

So, thank you.