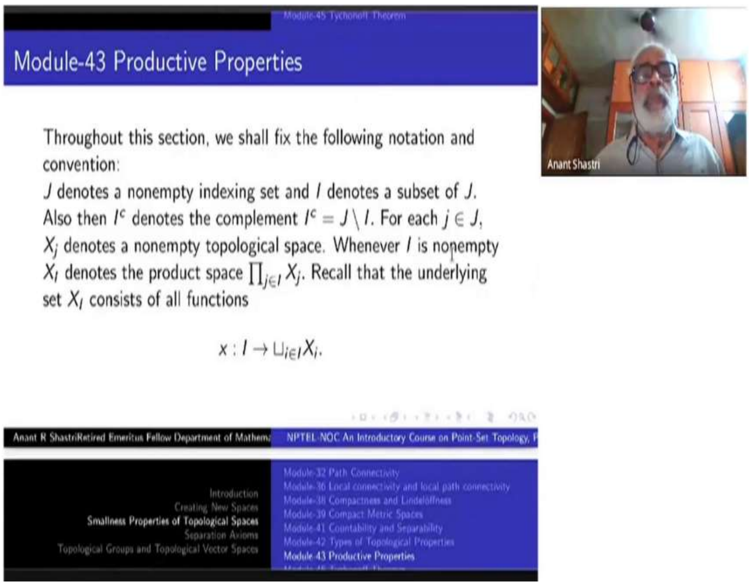
Introduction to Point Set Topology, (Part I)
Prof. Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay

Module - 43
Lecture - 43
Productive Properties

(Refer Slide Time: 00:16)



Welcome to module 43 of Point Set Topology Part-I. As promised last time, today we shall study Productive Properties or whatever topological properties we have made, we will verify which one is productive which one is not. That is the whole idea of today's talk ok.

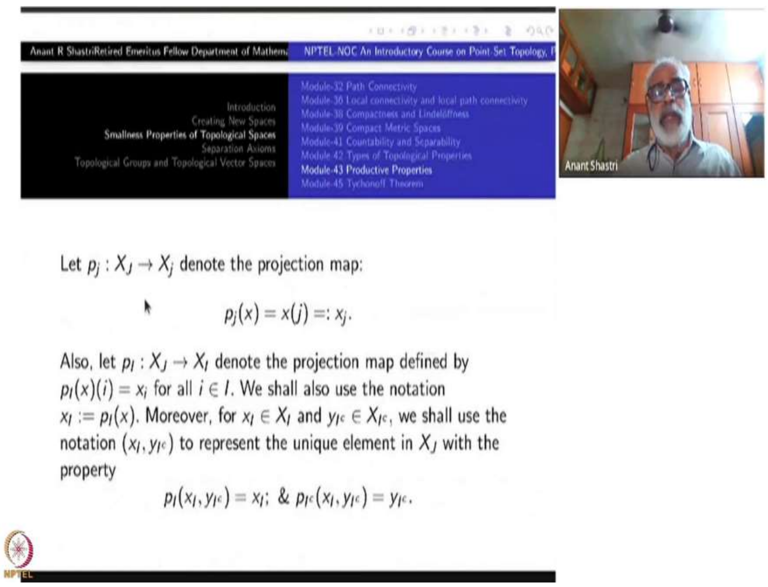
So, when you are studying products, I would like to recall a few conventions and notation that we have made earlier and also introduce some more convenient notation. The fundamental idea here is that when you have finite products, it is ok to use the ordered tuple notation, the Cartesian coordinate space notation. But when you go to infinite products, it is totally inconvenient and whatever you use, it will not be exactly rigorous, you will have to keep on using identifications ok.

So, we will try to reduce that kind of notation as far as possible and use a logically consistent notation and that is possible only if you think of the product space as a function space ok? So, having said that let me start telling you what are these notation.

J will denote a nonempty indexing set. So, this is all for the entire of this course you can say, but definitely in this chapter. I denotes a subset of J and I^c will denote the complement of I inside J ok? For each $j \in J, X_j$ denotes a nonempty topological space. So, I have used non-emptiness twice here, pay attention to that. Whenever I is also non-empty, X_I denotes the product space over this indexing set I, This suffix I is the indexing set ok?

Remember I used X^I, superscript I only when all the X_i's are equal to X. So, that is a different notation, I am not going to use that one here at all ok? The underlying set of X_J, I want to recall, namely, it consists of some functions. I will call them element of the product, but they are function ok? From where? from the indexing set J to the disjoint union of all these sets X_i's ok. So, any function like this, will be an element of this product. So, that is the underlying set.

(Refer Slide Time: 03:46)

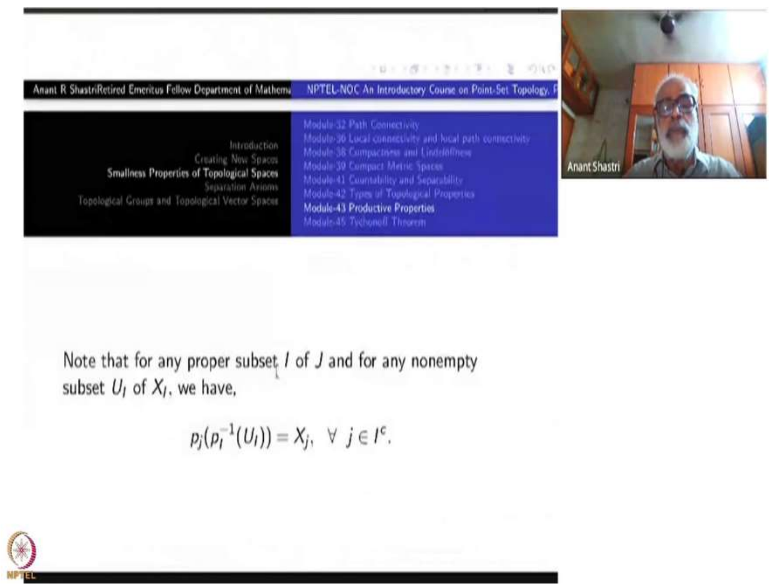


Then, we have these projection maps p_j from X_J to X_j, What are these? $p_j(x), x$ is a function remember, you evaluate x at the j^{th} element here, j is an element of J, so $x(j)$, this is another notation x_j, this is our usual coordinate notation. So, I do not want to give up that one completely, but I have given up the ordered n-tuple notation, ordered pairs and so on ok.

So, let p_I now, I is capital, this also denotes the projection map from X_J, the entire product space to X_I, which is the partial product because I is a subset of J, this denotes the projection map defined by the same principle namely $p_I(x)$ is a point in X_I. So, it is a map on I to the disjoint union of X_i's, on the point i of I, it is x_i; other coordinates are forgotten that is the meaning of this projection map p_I. We also use the notation x_I to be image of x under p_I, to be consistent with this one, whether it is a singleton or a whole set of bunch of indices, you can use that notation.

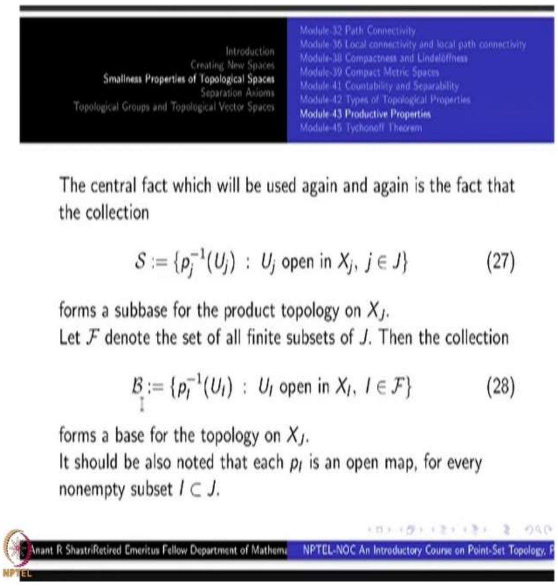
Moreover, suppose x_I is a point of X_I; everything ok. x_I is a point here and suppose y_{I^c} is a point in inside X_{I^c}. So, I and I^c are disjoint ok so, you have complementary things, we shall use the notation this ordered pair (x_I, y_{I_c}) to represent the unique element in X_J with the property that $p_I(x_I, y_{I_c})$ is x_I and p_{I^c} of this point is y_{I^c}. If I give p_I and p_{I_c}, the point is uniquely determined and that point has the property that p_I of this point is x_I and p_{I^c} of this point is y_{I^c}. So, that is the meaning of this one ok.

(Refer Slide Time: 06:38)



So, note that for any proper subset I of J and for any nonempty subset U_I of X_I, we have the j^{th} projection of $p_I^{-1}(U_I)$ is the whole of X_j for all j in the complement of I ok, complement of I will pick up all the elements X_j that is $p_I^{-1}(U_I)$. Observe this one ok? So, these are basic things which you have to be familiar with. Then only you can make sense out of the product space.

(Refer Slide Time: 07:19)



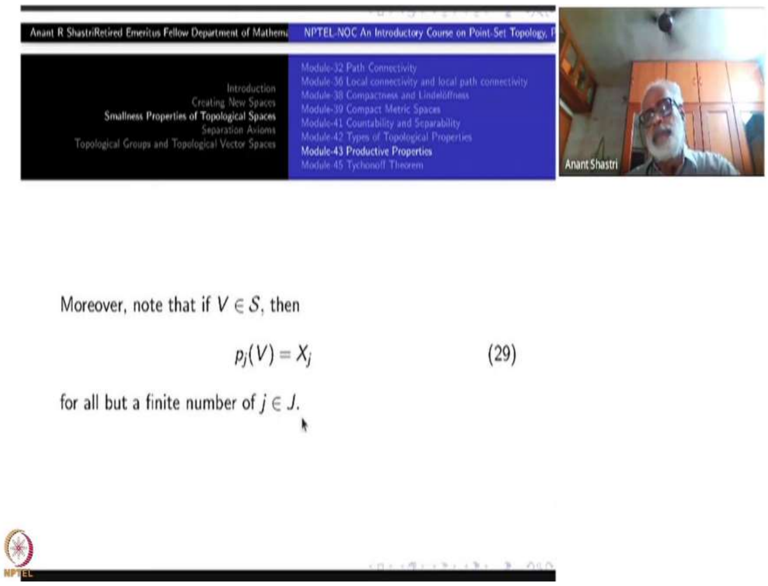
The central fact which will be used again and again is that the collection $\mathcal{S}$ of all $p_j^{-1}(U_j)$, where U_j's inside X_j are open, and j is inside J. So, these are all variables here, look at all $p_j^{-1}(U_j)$. Together that collection forms a subbase for the product topology on X_J. That is the definition of the product topology on X_J, ok? So, I am just recalling that.

Now, something more I am going to introduce here, this notation. Let $\mathcal{F}$ denote the set of all finite subsets of J, all finite subsets, I do not need empty set so, you throw away empty set, it may cause unnecessary problem that is all. Then, look at the collection $\mathcal{B}$ which is $p_I^{-1}(U_I)$ where U_I is open in X_I and I is inside $\mathcal{F}$.

So, I is a finite set, U_I is open in X_I which just means you know, you can take this as, to begin with, $U_1 \times U_2 \times \cdots \times U_n$, where U_i is are open in X_i, but there are more open sets ok than just product open sets, you take all of them take p_I inverse of those, this will automatically contain all the finite intersections of members of this so, it will be a base. It will have more open set no problem so, it will be base. So, this base is more convenient for me so, I will use this one, but this is subbase that may be also used ok, these two things I will keep using.

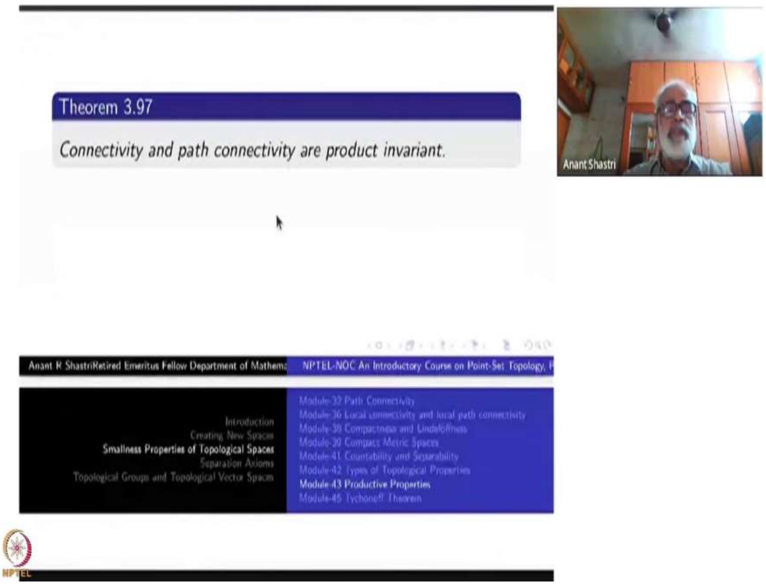
It should also be noted that each p_I is an open map, for every non empty subset of I of J, the empty subset I, this p_I does not make sense so, that is what you have to mention non-empty ok, all the projection maps are open maps.

(Refer Slide Time: 09:46)



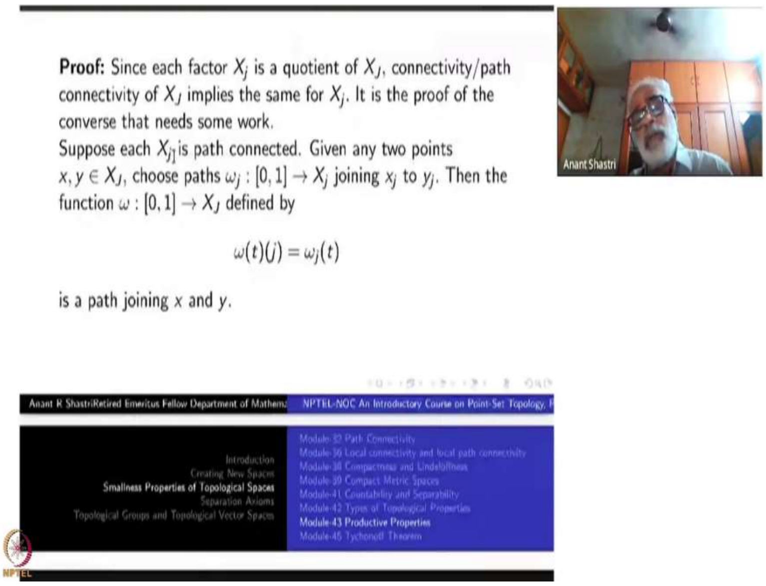
Moreover, take any member of this $\mathcal{S}$, what are they? They look like $p_j^{-1}(U_j)$ ok, for V belong to $\mathcal{S}$, $p_j(V)$ is equal to X_j for all but finite number of $j \in J$ ok, this is true actually for members of $\mathcal{B}$ also. Here it is only one j, if i is not equal to j, it will be whole thing, but here, it will be like this so, this is true for members of $\mathcal{B}$ also ok.

(Refer Slide Time: 10:34)



So, with these conventions, you will see that many proofs can be written very clearly, idea becomes extremely clear. The first thing I want to prove is connectivity and path connectivity are product invariant, which is at the bottom of our list 6 and 7 ok? But these these concepts were the first thing which we have considered in our chapter alright.

(Refer Slide Time: 11:04)



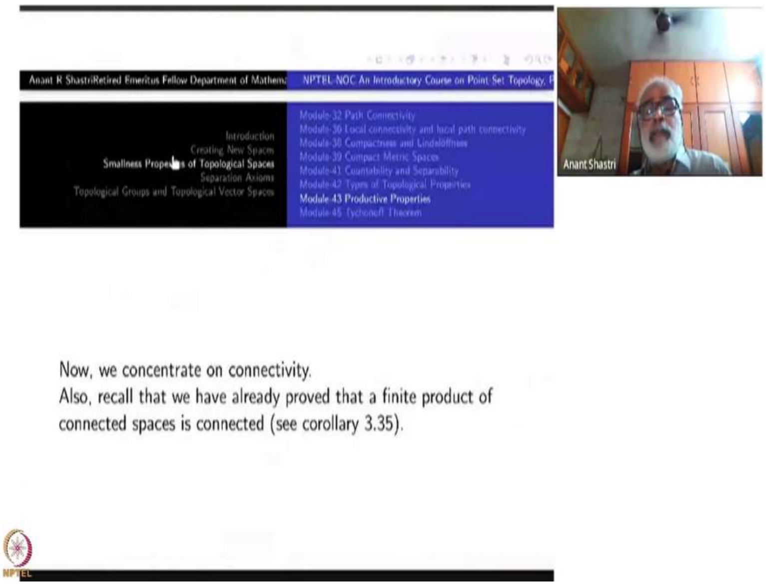
Now, each factor X_j is a quotient of X_J, right? Open quotient. Actually under any surjective map will do. Connectivity and/or path connectivity of X_J implies that of X_j, this we have seen several times ok. So, this part is alright, it is the proof of the converse that needs to be worked out, namely, if each X_j is connected/path connected, then the product X_J is connected/path connected.

Once again, path connectivity is straight forward. Suppose each X_j is path connected starting with any two points x, y inside X_J, you can choose path ω_j for $[0, 1]$ to X_j continuous function joining these points x_j to y_j. They are inside X_j right? So, there, you can join them. So, you have got all these families of paths, then you define one single ω from $[0, 1]$ to X_J such that its j^{th} component is ω_j.

See $\omega(t)$ must be a function right? From J into the disjoint union of all these X_j's. So, on the j^{th} element you take this path $\omega_j(t)$, ok? So, automatically this will be continuous function. We have verified that a function is continuous if and only if all its coordinate functions are continuous. These are continuous so, this is continuous.

When you put $t = 0$, this will give you $\omega_j(0)$ which is x_j, $t = 1$, it will be y_j, right? For each j that is the thing; that means, $\omega(0)$ is x and $\omega(1)$ is y. So, it is a path joining x and y that proves the product is path connected.

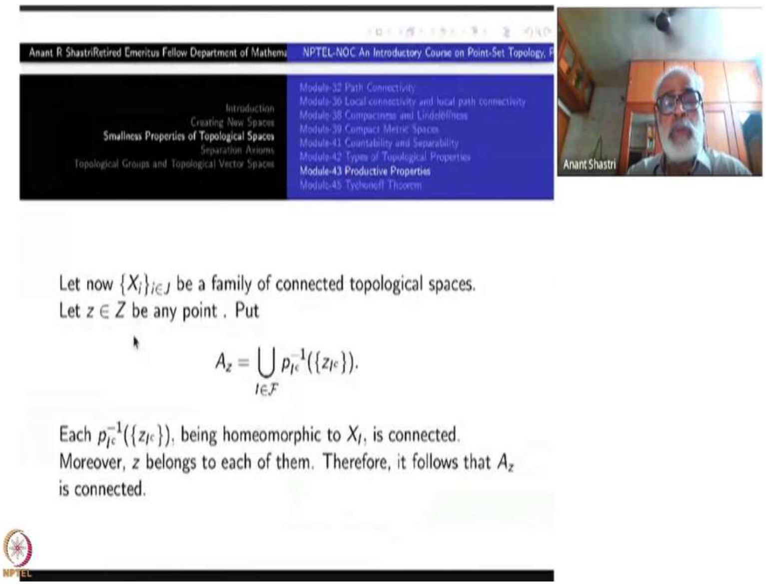
(Refer Slide Time: 13:26)



So, let us now concentrate on connectivity. Recall that we have already proved that finite products of connected space is connected. So, this is corollary 3.35 or whatever ok, I do not have to show it to you, we know this one very clearly, not very long ago, we proved this one alright.

Indeed, we proved a stronger result namely if Y is connected and the fibers are connected, then X is connected under any quotient quotient map right? And using that we can prove product of two of connected space is connected, then three of them is connected, and inductively any finite product is connected alright. So, I am going to use that one.

(Refer Slide Time: 13:28)



Now, I go to the case of product of arbitrary family. So, X_i for $i \in J$ be a family of connected topological spaces. Take any point z what you should show? If the connected component of that point z is the whole Z. Then I am done. These are my X_j's, I am taking Z, a short notation the product. z be any point of Z. I am taking a point in the product space, after all I won't show the product space is connected, but what is my idea?

Idea is to show that the connected component of any point is the whole space ok. So, this is a first step here.

Take A_z to be all $p_{I^c}^{-1}(z_{I^c})$ for $I \in \mathcal{F}$. Remember what is z_{I^c}? z_{I^c} is a point inside X_{I^c} which is the projection of z ok? Take p_{I^c} inverse of that which is nothing, but $X_I \times \{z_{I^c}$, see that is notation for this, if you are going to use the cartesian coordinate notation, I do not want to use it so, I have written it carefully like this $p_{I^c}^{-1}(z_{I^c})$.

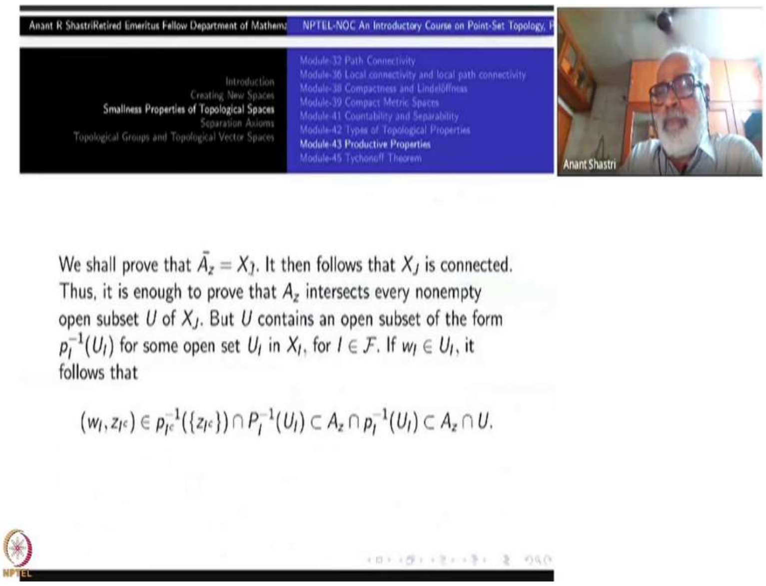
The coordinates inside I; they are dropped out, there is no mention, they could be arbitrary, but when j is inside I^c; that means, j is not in I, the coordinates of that point must be the coordinates of this point z, they must coincide so, that is the meaning of this p_I inverse of this one ok. Take union of all these where I is a finite set, I is belong to $\mathcal{F}$, finite subsets of J ok.

Each $p_{I^c}^{-1}(z_{I^c})$ is homomorphic to X_I. Just know I told you, it looks like X_I cross one single point in the complementary product namely in I^c ok, they are homomorphic X_I, they are connected why? Because I is finite and the finite product is connected we have used. So, these are each of them on the right-hand side is connected.

Moreover, z is a common point to all of them. Therefore, it follows that A_z is connected. So, this argument we have used before also ok. It is very easy to see that A_z is connected: take a separation if possible. Suppose A_z is $B|C$, where will be z? It will be in one of B and C. The moment z is inside B, all these things will be inside C ok; that means, C is empty. So, if z is inside C, B is empty, if z is inside B, C is empty so, there is no separation.

So, this is a connected set, this is not the whole of X_J, but it is very close to that, namely, what is its closure here? If you take the closure of this, it is the whole space, that is the claim ok.

(Refer Slide Time: 18:09)



We shall show that A_z closure is X_J ok? So, half the step we have proved that this is a connected set, the second half is that its closure is the whole space ok. So, that will complete the proof why? Because closure of a connected set is connected alright.

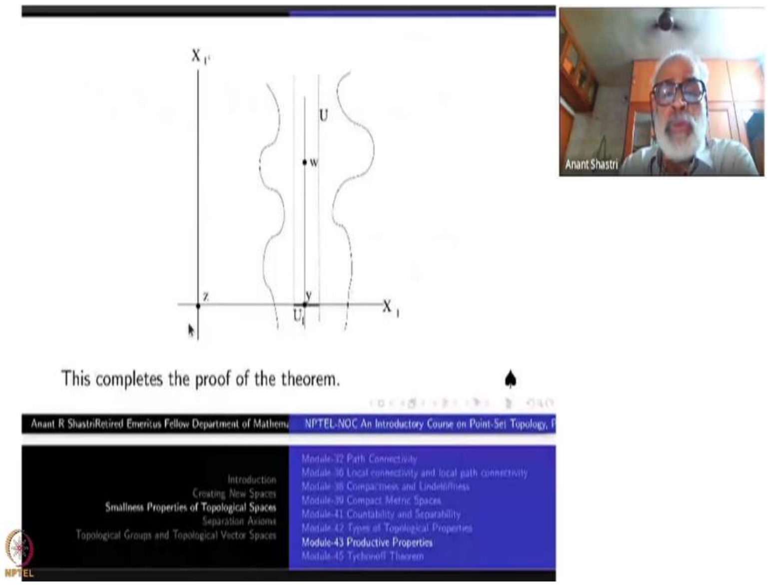
So, what is the meaning of saying that the closure the whole space? Every nonempty open subset in X_J will intersect A_z. So, that is what I have to prove that A_z intersects every non-empty open subset U of X_J.

What is the meaning of U is an open subset non-empty inside X_J? It contains an open subset of the form $p_I^{-1}(U_I)$ for some open subset of U_I in X_I. These are the basic open sets, that is the first thing I am going to use here where I is itself a finite subset of J, ok? There must be some such thing and U_I must be non-empty, open inside X_I, I must be finite.

Take a point w_I inside this U_I, this U_I is non-empty right? Now look at this point (w_I, z_{I^c}), what is the meaning of this? All the i coordinates for i inside I are w_i and if i is inside I^c, then it is z_i. So, that is the meaning of this point, we have defined this one.

This point is clearly inside $p_{I^c}^{-1}(z_{I^c})$ because its I^c coordinates equal to z_{I^c}. Also, this point is inside $p_I^{-1}(U_I)$ because this w_I is inside U_I. So, it is $p_I^{-1}(U_I)$ ok so, it is in the intersection, but the first set is contained inside A_z, the second set is contained inside where? The second set I have written just now, it is contained inside U.

So, the point is contained in $A_z \cap U$ that is the end of the proof why? Because starting with U, I have shown that $A_z \cap U$ is non-empty ok. (Refer Slide Time: 21:16)



Here is a picture if you want a picture to explain what is going on. Whatever I have told here so, this is an arbitrary point z right, then I have bunched up this X_I, I is a finite set of indexing sets right suppose it just X_1, then this looks like the x-axis and the rest of them are y -axis, the rest of them are I^c; I have put. So, you have divided the entire set into I^c; I and I^c. So, draw this this this is one plane, this is a finite product of finite things right so, this is connected.

Now, if you take an arbitrary point w inside this U_1 inverse image, this dot dot dot U_1 ok is $p_{I^c}^{-1}(U_I)$. So, if you take any point like this here $p_I^{-1}(U_I)$, you can project back here actually this y_I have started with a point here w and this is w_{I^c} of that length, this y is nothing, but w_I here in the notation ok.

Its other coordinates are just the coordinates of this point, you can think of this as y coordinate here, y coordinate of z and this point y are the same ok. So, I^c coordinate of these two are same. So, that is precisely the point here which will be in the intersection, this was my arbitrary U ok. So, every U contains such set is what you have to use namely what are the base for the product topology that is all ok.

So, you see the proof that product is connected has just two ideas, one is you look at a point and then, take all these coordinate planes, finite coordinate planes passing through the given point. So, I am using the terminologies motivated from $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$ and so on ok, that is the only way you can picture it, you can imagine coordinate planes and so on ok, there are no planes here after all, all are arbitrary topological spaces ok. So, that set A_z is what it is dense. So, that is all you have to remember. Proofs are not at all difficult ok.

So, let us do the other things next time. This result itself is something substantially good we have done. Other properties we shall do next time.

Thank you.