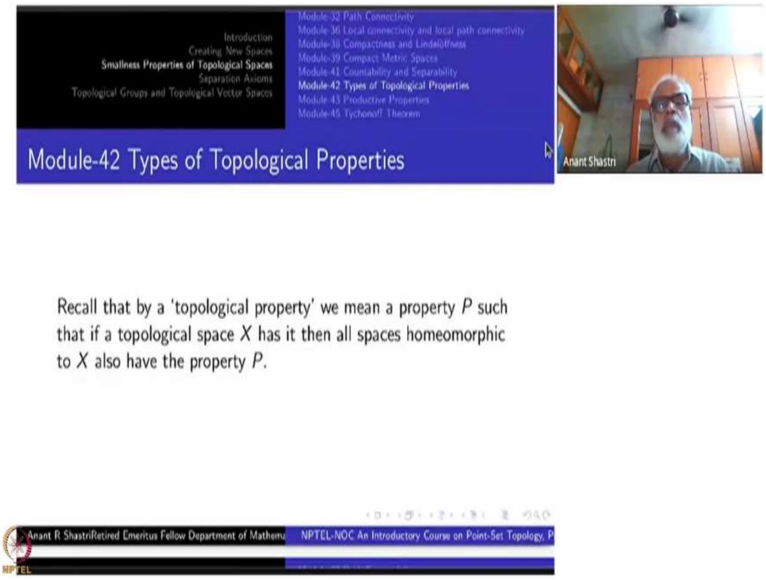
Introduction to Point Set Topology, (Part I)
Prof. Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay

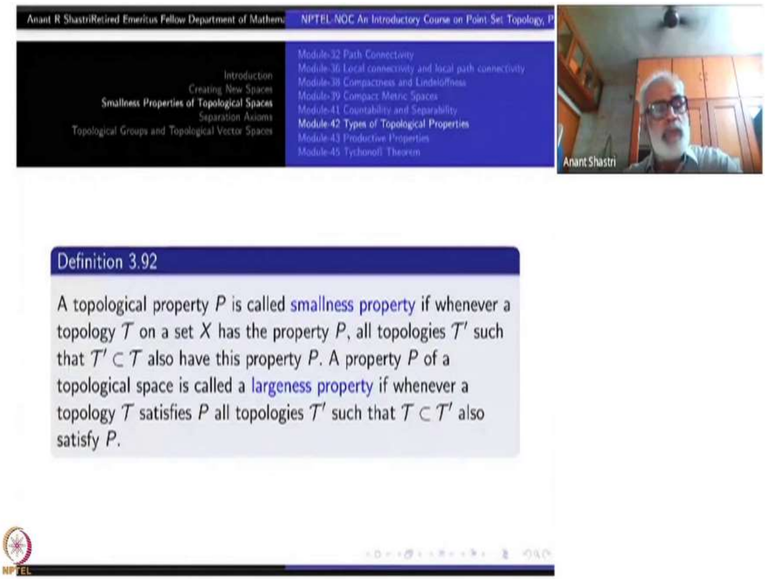
Lecture - 42
Types of Topological Properties

(Refer Slide Time: 00:16)



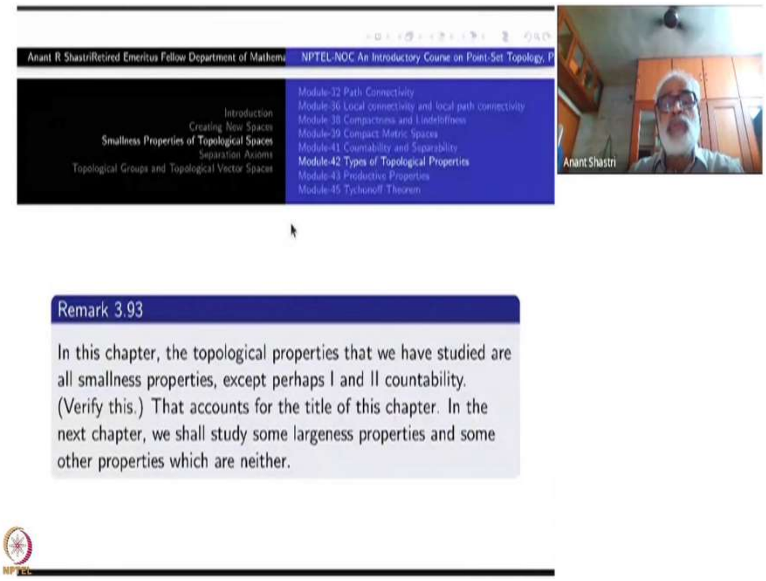
Welcome to Module 42 of Point Set Topology Part 1, so today we will take up study of Properties of Topological Properties. So, recall that we have defined a topological property by which we mean something that if it is true for one space then it must be true for all spaces which are homeomorphic that space. Such a property is called topological property or topological invariant.

(Refer Slide Time: 01:01)



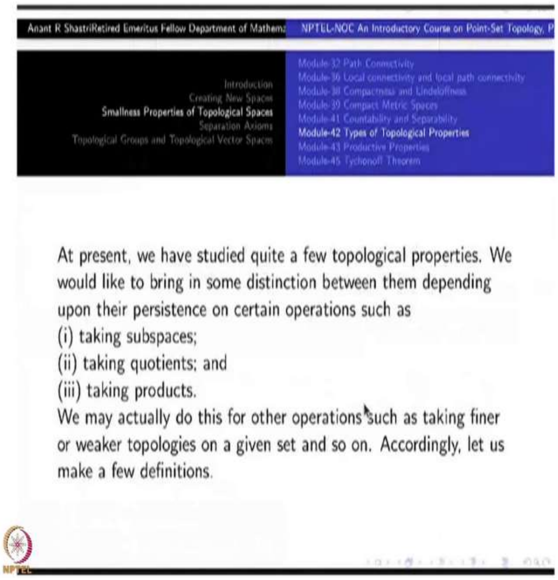
Then we made a definition, this also I am recalling, that a property is called smallness property, if whenever a topology tau on a given set X has it then all topologies $\mathcal{T}'$ smaller than $\mathcal{T}$ should also have the property. Similarly, a largest property is one whenever $\mathcal{T}$ has it and $\mathcal{T}'$ is a larger than $\mathcal{T}$, then $\mathcal{T}'$ should also have it.

(Refer Slide Time: 01:47)



Most of the topologies that we have studied in this chapter, they belong to the first type, namely, those which are smallness properties, except perhaps the first and second countability. These two are not exactly of this nature I have already told you. So, next chapter we will consider those which are likely to be called as largeness properties, some of them just like first and second countability, there will be some, which are not exactly largeness properties strictly as per our definition.

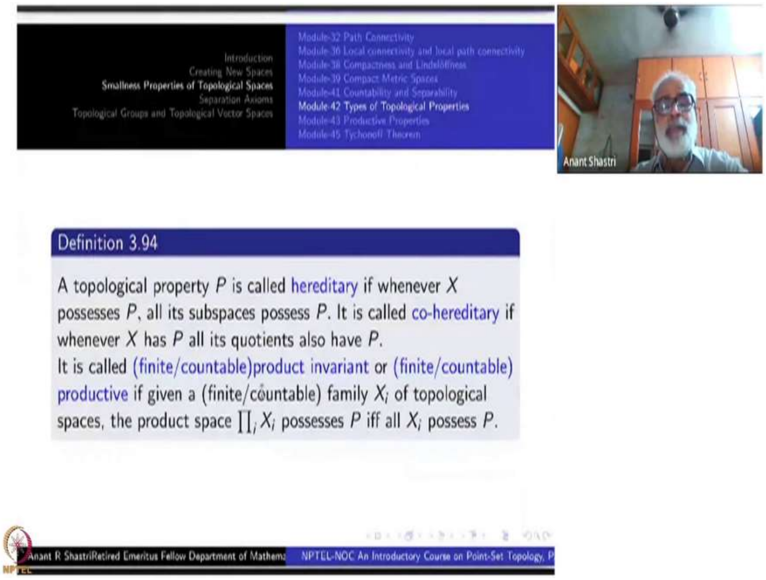
(Refer Slide Time: 02:23)



At present whatever the concepts that we have studied so far, like path connectivity, connectivity, what we would like to do is to look at them one by one, check whether the property for a space will automatically imply the same for all subspaces, whether it will be true for quotient spaces, when this property holds for several of them, whether it will be true for the product of these spaces and so on. So, these are the questions which keep bothering us, so we would like to carry out this study in a systematic way as much as possible to begin with in one single place.

Later on, we cannot do all of them at a single place anyway, as soon as a new topology comes we can keep asking these questions for that particular topology according to the time availability or our mood ok? And then there are many other kinds of questions also you can ask. Whether, it will be persistent on taking close subspaces or open subspaces instead of arbitrary subspaces. So there are modifications of such questions also. So, this is what I mean by studying the properties of the topological properties.

(Refer Slide Time: 03:58)

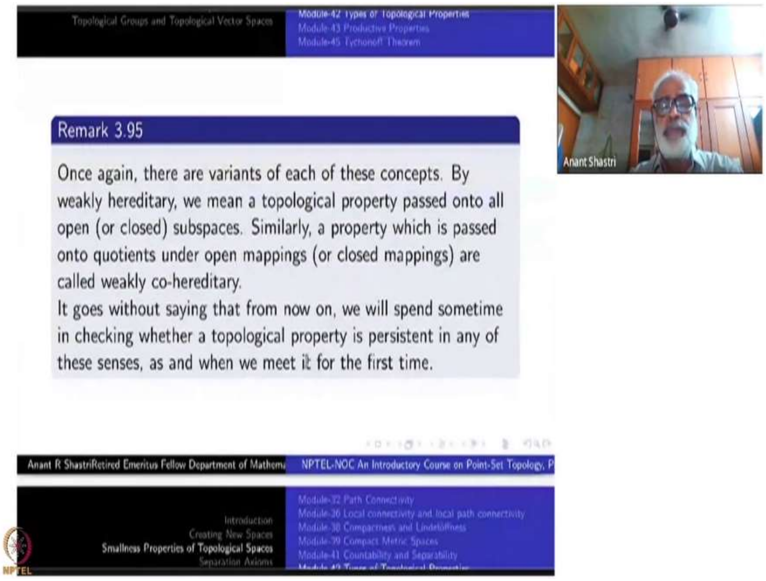


So, let us tentatively make a few definitions. If we need more and more definitions or modify these definition we can keep doing that. First property is hereditary. So, P is a topological property, P is called hereditary, if whenever X possesses it all subspace should possess the same property. It will be co-hereditary whenever X possesses it all quotients of X should also possess it.

Similarly for `product invariance, there are three different versions here. One is finite product invariance, another is countable product invariance, the third one is product invariance without any quantifier, or qualifier ok, so that is more general. What is the meaning of product invariance, whenever a family depending upon finite or countable family of spaces X_i is are given, such that if each X_i has the property then the product should have the same property and conversely if the product has it then each factor should also have it ok?

Sometimes people do not bother about the converse part--just one itself is called product invariance. And then the converse one they may call it factor invariance. Once again there are variants of these concepts that is what I wanted to tell you.

(Refer Slide Time: 05:43)



So, depending upon the author and the concepts and you know what exactly you want to study and so on ok. There are variants like I already told you, hereditariness also. Instead of general subspaces, you can just take open subspaces, you may call it open hereditary or it may be true for only close subspaces then you may call it a closed hereditary and so on.

So, they are weaker than being hereditary. So you may call these two together as weakly hereditary. But then just calling weakly hereditary, there would be still ambiguity in this definition. Similarly for co hereditary under any quotient map if the property persist then you may call it as co hereditary, but suppose it is only true for open quotients, then you may call it ah weakly hereditary or some other people may call only closed quotients, then they may call it weakly hereditary and so on ok. So, there are various versions of this one.

(Refer Slide Time: 07:03)

The following table gives you a fairly complete picture of our efforts in this chapter toward this problem.

		Hereditary	Co-hereditary	Productivity
1	Compact	weakly	yes	yes
2	Lindelöff	weakly	yes	No
3	I-countable	yes	weakly	countable
4	II-countable	yes	weakly	countable
5	Separable	no	yes	countable(?)
6	Connected	no	yes	yes
7	Path-connected	no	yes	yes

We shall verify each of them taking up one by one column-wise:

So, here is a table ok, which will give you fairly a good idea of whatever you are going to do right now and a little more in this chapter and maybe in the next section and so on. So, here I have listed seven of the properties that we have studied, so far: compactness, Lindeloffness, first and second countable, separability, connected and path connectedness. Then these are the three properties of theses properties that you would like to study hereditariness, co hereditariness and productivity.

Within productivity there are 3 types actually finite productivity, countable productivity and arbitrary productivity. So, this being little more complicated we will do it next time. Today let us see whether we will cover this much ok. So, what I will do? First I will take hereditariness, for all these things one after another. Then we go to co-hereditariness one after another ok. So let us look at compact spaces. You already know that closed interval is compact, but the open interval is a subspace which is not compact.

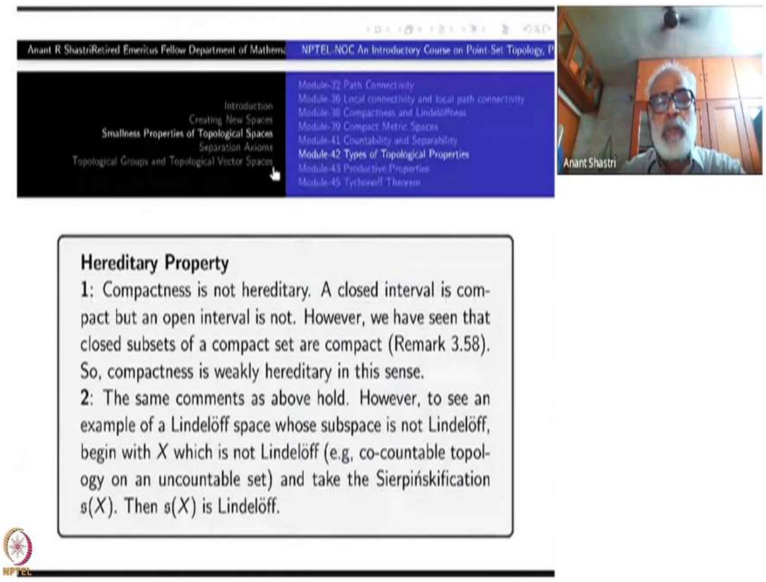
So, compactness is not hereditary on the other hand you have also proved that every closed subspace of a compact space is compact. We have already proved such a thing right? So that means, it is weakly hereditary.

Exactly same thing goes for Lindeloffness also. Every closed subspace of a Lindeloff space is Lindeloff right? Have you seen that Lindeloffness is not hereditary at all? Do you remember when we have done that? Can you see it easily with some example, what is to be done? Take any Lindeloff space, so that there is a subspace which is not Lindeloff. What is Lindeloffness? Every open cover should have a countable subcover ok. A space which is not Lindeloff, we have seen that. You can take for example, an uncountable set and a discrete topology on that. That is an easiest example of non Lindeloff space ok? Now instead of taking subspaces you construct a larger space by putting one extra point namely a Sierpinski's point ok.

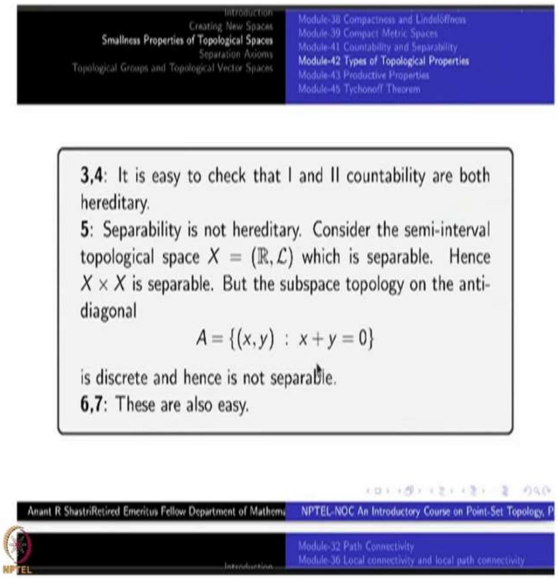
As soon as you put a Sierpinski's point what is the meaning of that? That extra point, the only open set containing that point is the whole space. Therefore, when you take an open covering, in fact every open covering of this space must have the whole space as one of the members. Therefore, that singleton subspace, the set that will be a finite subcover. You take that X member X that is a cover. So, it is automatically Lindeloff. So, this looks like as if we are cheating, but that is, you know quite legal, example of a Lindeloff space such that a subspace is not Lindeloff ok?

You can cook up more pleasant examples or more unpleasant examples also if you like ok?

(Refer Slide Time: 11:50)



(Refer Slide Time: 11:52)



So, let us keep going on. First countable and second countable now. It is easy to check that both first and second countability are hereditary. Let me do it for second countability first countable you can do in exactly same way.

Take a countable base for a topology ok? Now take a subspace. What are the open subset of the subspace? Take any open subspace in the original space, intersect it with the subspace. Suppose Y is a subspace of X, and $\mathcal{B}$ is a countable base for X. Take any member of $\mathcal{B}$ intersect it with Y, you know, collect all of them that will become a countable base for the subspace Y, that is all, ok? So, up till here we have come that countability and second countability are hereditary.

Separability, connectivity, path connectivity they are not hereditary. Connectivity and path connectivity you know already ok. You take an open interval remove a point, it is gone, the connectivity is gone right?

So, subspaces hardly need to be connected for a connected space and path connected space. But for a separability how do you do that? Why separability is not hereditary? Remember separability is what?

Student: Countable dense set sir.

Countable dense set. When you go to a subspace this is dense set may go away that may not be in the subspace right? but that does not mean that there is no other countable subset which is not dense. So, how do you give an example of a separable space that has a subspace that is not separable? So, perhaps you may try to do similar to what we did in the case of Lindeloffness. Start with a non-separable space and then cook up some bigger space which is separable. You know that may work now.

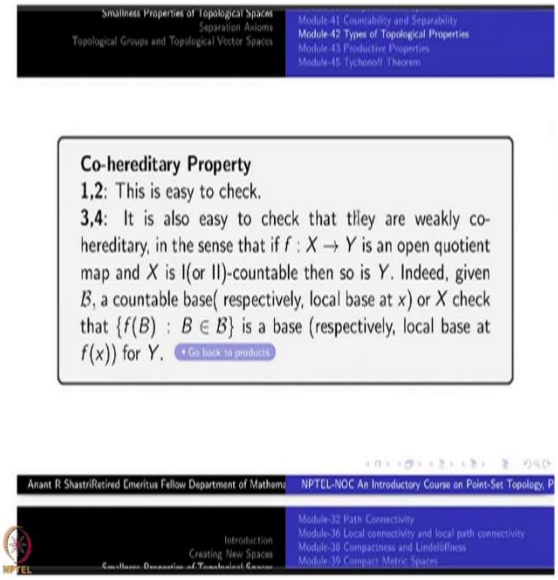
So, there are ideas you have to, you have to sometimes think about these things right. So, I have given you an example here remember we had this semi open interval topology, in which the basic open subspace are of the form $[a, b)$, a closed b open. So, this was left semi interval topology $\mathcal{L}$ on the set of real numbers ok. This space is separable. you can check that again the set of rational numbers is a dense set ok? Though this topology is larger than the usual topology on $\mathbb{R}$.

So, $\mathbb{Q}$ is a dense subset and so this space is separable. Once you have a dense set in X, say A is a dense set then $A \times A$ will be dense in $X \times X$, so $X \times X$ is separable ok. But now I look for a nice subspace here namely the anti-diagonal. The line given by $x + y = 0$ ok. In the usual topology this is homeomorphic to $\mathbb{R}$, but in this product of semi interval topologies, What happens? This becomes a discrete space. Every point is open now.

Why? Because given any point $(x, -x)$, you can take interval $(x, x + \epsilon)$ and $(-x, -x + \epsilon)$, take their product. So, this is half half closed rectangle sitting on the point $(x, -x)$. Exactly one corners of the rectangle, will be on $(x, -x)$ ok. So, this rectangle half closed rectangle will be an open subset of the product space, its intersection with the line is open in the line, but this intersection is just the singleton $(x, -x)$. So, all the singletons are open. That means it is discrete. A discrete set of cardinality uncountable right? So that cannot be separable ok?

This example is quite a peculiar one. I will be using it again ok? In the next chapter. So, as I have pointed out, connectivity and path connectivity they are not hereditary that is seen easily ok.

(Refer Slide Time: 18:01)



Now, let us go to co hereditariness compactness and Lindeloffness. We have seen that the quotients are also compact and Lindeloff ok. If you would not remember it you can just argue as follows: Take an open covering for the quotient, inverse image will be an open covering for the original space, it given a finite covering and come back ok? So that is it.

So, 3 and 4 are what? First countability and second countability right. So, these are weakly co-hereditary, in the sense that if it is an open quotient, then X is second countable implies Y is second countable. Once again it is very easy to prove. What you have to do? Start with a countable base $\mathcal{B}$, take the collection $f(B)$ where B ranges over this $\mathcal{B}$. Because f is open, $f(B)$ is open here. Now you can verify that this is a base for the topology on Y. So, same thing works for local base at a point also alright.

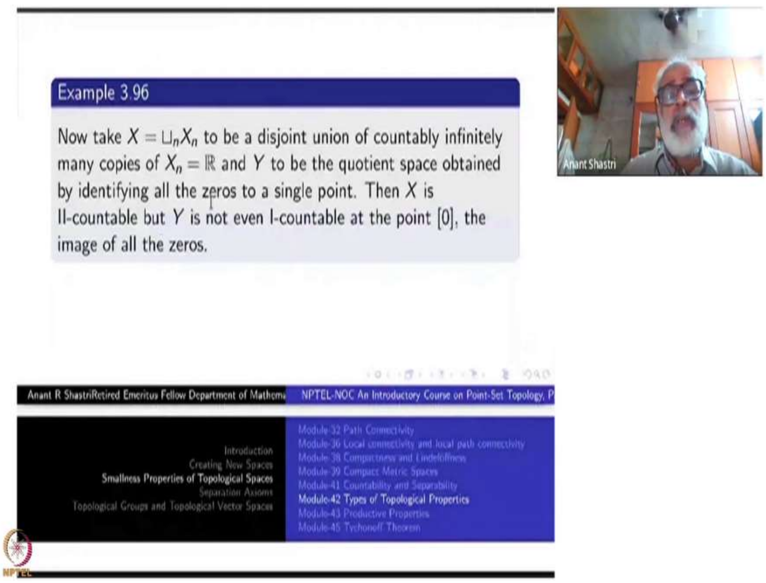
So, open quotients are preserving the first countability and second countability. So this is weakly co-hereditary alright. In general what happens? Once again we have to cook up examples here ok. If you do it for second countability, for the whole space or just I countability, for a single point, the idea and arguments will be more or less same ok. So, I will prove it for now this time first countability ok? A space which is first countable but the quotient is not first countable.

Quotient means what now? General quotient not open quotient, open quotient will be first countable ok. All that I do is to take infinitely many copies of $\mathbb{R}$ disjoint copies, is it first countable? Any first countable space if you take any number of them and take disjoint union it will be still first countable, the disjoint union does not disrupt local properties alright.

So, it is first countable, now what I do? I construct a quotient of this by identifying all the 0 in each copy of $\mathbb{R}$ to a single point. So, you can name them as $\mathbb{R}\times\{0\}, \mathbb{R}\times\{1\}, \mathbb{R}\times\{2\}, \mathbb{R}\times\{3\}$ and so on if you want ok, these are the copies of $\mathbb{R}$ right. Now 0 cross n all of them will be identified to one single point. We denote it by [0]. No other identifications. This is all ok? So, equivalence class is what? Whenever it is 0 cross something it is equivalent to 0 cross anything. All other points are singleton classes. So, that is the quotient set. Give the quotient topology.

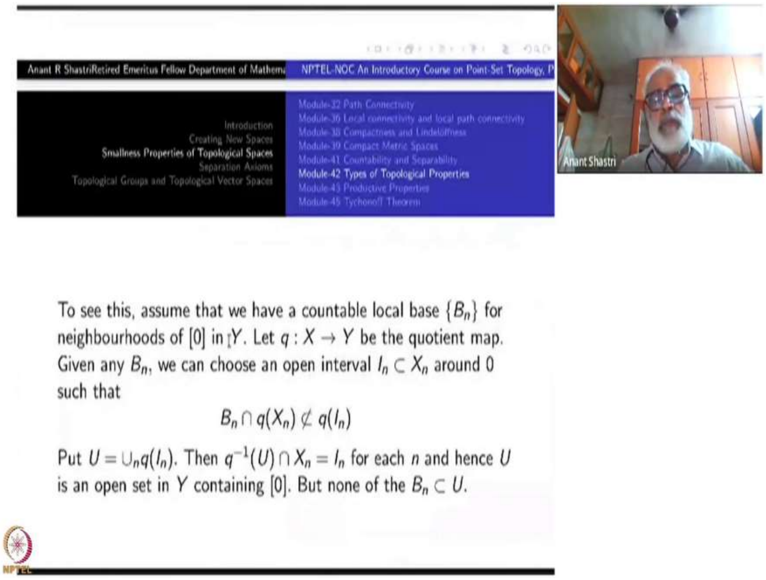
What is the quotient topology? Something is open in the quotient space if and only if inverse image is open in the disjoint union of all these copies of $\mathbb{R}$ ok.

(Refer Slide Time: 22:20)



So, that is my example here X equals disjoint union of X_n, countably many copies of $\mathbb{R}$ and Y to be the quotient space obtained by identifying all the 0s to a single point then X is second countable but Y is not first countable even. At that point [0], which is the class of all the 0 cross n, ok? The first countability fails ok? So, if it is not first countability cannot be second countable, so it this gives you an example for both of them ok.

(Refer Slide Time: 23:11)



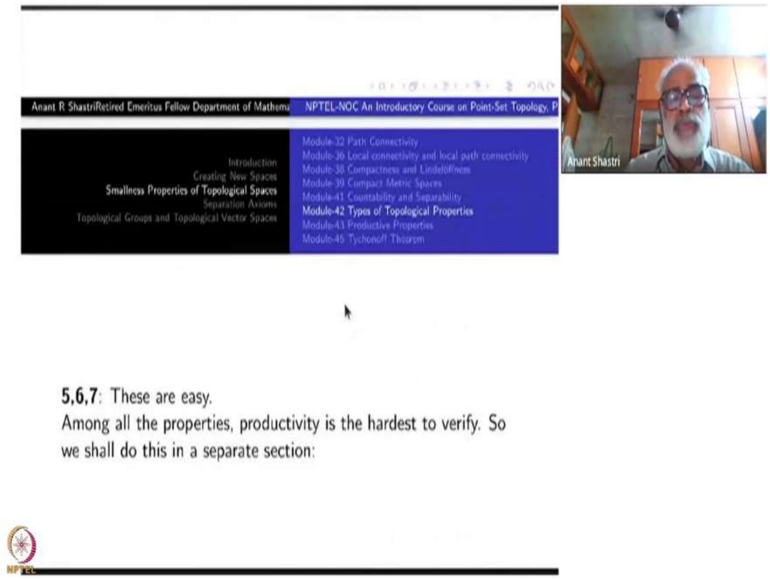
How to see that it is not first countable? That is also easy. Suppose we have a countable base $\{B_n\}$ of our neighborhoods of $[0]$, ok? Local base for this quotient space Y ok. Let us tentatively denote this quotient space and the quotient map from X to Y by q alright. Given any B_n, take B_1, then I can choose an interval I_1 in the X_1 copy of $\mathbb{R}$ around 0, so that that it is so small that this $B_1 \cap q(X_1)$, the image of that line ok, that is not contained inside this $q(I_1)$.

Do the same thing for all $n, B_1, B_2, \ldots B_n$ and so on, you look at the corresponding B_n and then go to the corresponding copy of $\mathbb{R}$, there you choose a neighborhood very small, so that that neighborhood does not contain this that is all ok. So, once you have got for each n, this interval I_n you take U to be this disjoint union of all the I_n's, in the disjoint union ok. When you take $q(U)$, that will be an open subset, because the inverse image will be precisely disjoint the union of this I_n's, ok.

So, U equal to union of all these I_n's, $q(U)$ is an open subset containing the point $[0]$, you know, the class of 0. But because of the choice, none of the B_n's will be contained inside U

ok? If it is a base then some B_n must be contained inside U. For every open subset containing $[0]$, there must be a member in the local base which goes inside that. Therefore, this is not a local base. In fact, what we have proved here is no neighborhood system at $[0]$ can be countable or take any countable set of neighborhoods, it cannot be a base that is what we have proved.

(Refer Slide Time: 26:24)



Once again 5, 6, 7: what are they? Separability and path connectivity; and path connectivity right yeah, so separability is easily seen to be co hereditary you see it was not hereditary, but it is co hereditary. Why? Take a countable set which is dense, go to the quotient take the image that will be also countable set, claim is that it is dense. Very easy. Because all that you have to do is take an open set in the quotient space you have to show that it intersects this image right?

Go to the inverse image that is an open subset in X, so that will intersect the original dense set that point will be in the intersection of these two, the image of that point will do the job, that is all ok.

6 and 7: path connectivity connectivity: we have seen that image just the image of a continuous function itself has this property: Connected image will be connected; path

connected image will be path connected. So, they are a bit more stronger than being co hereditary.

So, the 2 columns we have seen except a few things like 1 or 2 examples like this we had all these things we had seen. So, this is like a summary of whatever we have done so far, except this example and this example ok. So, so now we come to the third column here about products. What we have seen is product of finitely many compact spaces is compact. Have you seen that?

Student: Yes.

For, finite products, it is true, but productivity as such ok; means that it should be arbitrary product also. So, here I am saying yes, but we do not have a proof for arbitrary products not even countable products, finite products you know. Similarly for Lindeloff we do not know ok? For countability, first countability, again finitely many copies yes. It is ture even if you take countabily many copies ok? But arbitrary product we do not know ok? Similarly, for second countability.

Separability is more mysterious, what we have done is for path connectivity we have seen this one. Remember path connectivity is easy. For connectivity also you have seen only for finite products we have seen right? So, partly many of these things we have seen, but we do not know all the full answers, none of them properly, right? So, we shall take up this one next time alright. So, let us stop here.