Introduction to Point Set Topology, (Part I)
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Lecture - 41
Countability and Separability

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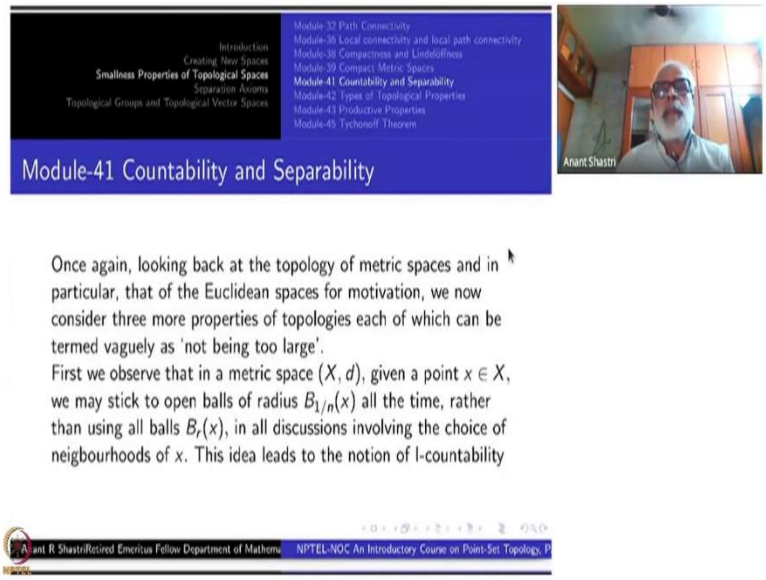

Welcome to Module 41, today we shall take up three more interesting properties; first countability, second countability and separability. Once again, we go back to metric spaces for motivation, especially the Euclidean space itself. So, these properties just like path connectivity, connectivity, compactness, Lindeloff, etcetera, they can be classified as smallness properties, but not exactly. We have given a formal definition and this first countability and second countability does not fall into that though separability does.

Now, why we are combining them? Because they are often going hand in hand. That is why you are combining them. But nevertheless there is some vague feeling that these conditions put some restriction on the size of the topology, you shall see why ok. And it is better to leave it like that instead of getting into formal definitions.

So, what are we going to observe here, first let me take any metric space ok. The base for a metric space topology is: you take the collection of balls of radius positive at all the points right?

Suppose you are fixing one point and then looking at all the neighbourhoods of that point. Then instead of taking all the balls of positive radius you can simply restrict yourselves to balls of radius $1/n$, balls of radius $1, 1/2, 1/3, 1/4, \ldots$ You know wherever the discussion of neighbourhoods is involved only these balls will be sufficient. So, this idea leads to the notion of first countability.

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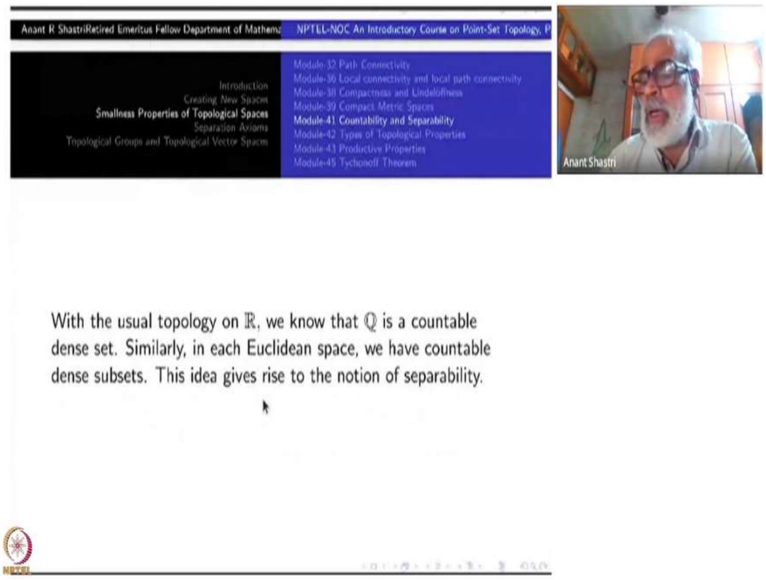

With the usual topology on $\mathbb{R}$, we know that the set of rational number is countable as well as dense subset. The same thing is true for all Euclidean spaces also, only thing you have to do is you take all points such that all the coordinates are rational. That is also a countable set and it will be dense inside $\mathbb{R}^n$ ok. So, this gives you another notion which is called separability ok. Before going to the third one, let me state a theorem here for metric spaces and then that will suggest the third definition.

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Start with any metric space with a countable dense subset A. Just like $\mathbb{R}^n$, but now I am generalizing it for any metric space. All that I want is a countable dense subset similar to $\mathbb{Q}$ or points with rational coordinates in $\mathbb{R}^n$ and so on ok? With this much of hypothesis, I want to conclude the following thing. Then there exists a countable base for the topology on X ok?

(X, d) is a metric space. So, d is a metric and the topology $(X, \mathcal{T}_d)$ has the property that it has a countable base. In particular, all Euclidean spaces have countable base. So, what does it mean, it means that every open subset is arbitrary union, but of members coming from a fixed countable set that is the beauty.

So, that already puts a lot of restriction on the topology, it does not say, remember, it does not say that number of open sets is countable no, because you are taking arbitrary union, but now all the open subsets are unions of members from a one single family of open sets that is the countable base ok. So, that is quite a strong condition.

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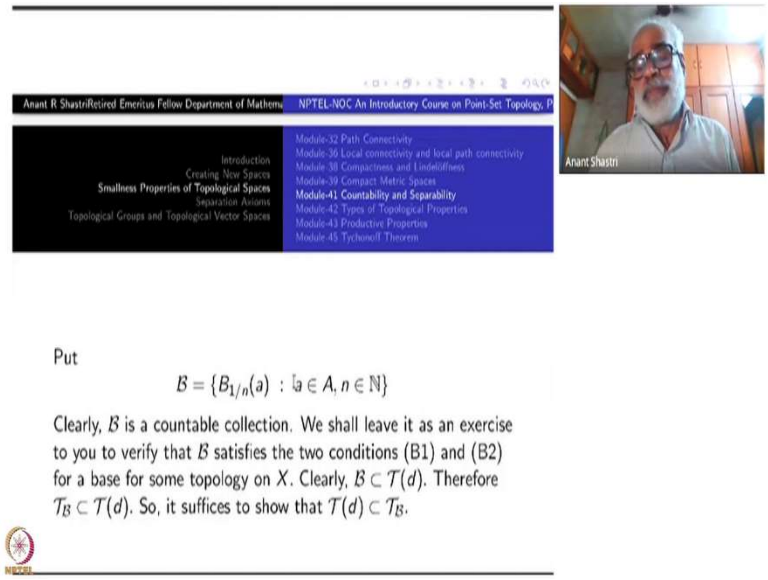

To prove this one, it is one line proof: Put $\mathcal{B}$ equal to collection of all balls of radius $1/n$ not at all the points, but only at the points of A, where A is a countable dense set.

So, a is inside A, n is inside a natural number ok, this is a clearly a countable collection, because A is a countable set and the set of natural numbers is also countable set. So, there are two different variables here, but $\mathbb{N} \times \mathbb{N}$ is a countable set. This collection $\mathcal{B}$ is a base for a topology. That it satisfies (B1) and (B2). This does not need any extra you know hypothesis you know this is this kind of things you have done.

So, you must do that one ok, that they form a base. It means what? Once you take finite intersection two intersection of two of them and a point there show that there is a smaller one contained inside that at the point and so on. Conditions (B1) and (B2) you have to verify. So, this part, I leave it as an exercise.

All these are open subsets in $\mathcal{T}_d$ right? Therefore, if you take the topology generated by $\mathcal{B}$ ok, $\mathcal{T}_{\mathcal{B}}$ that is contained inside $\mathcal{T}_d$ because $\mathcal{T}_d$ is a topology after all. So, what I want to show is that $\mathcal{T}_d$ is contained inside $\mathcal{T}_{\mathcal{B}}$. So, that this itself is the basis for $\mathcal{T}_{\mathcal{B}}$, ok? And this is a countable set. So, that will end the proof.

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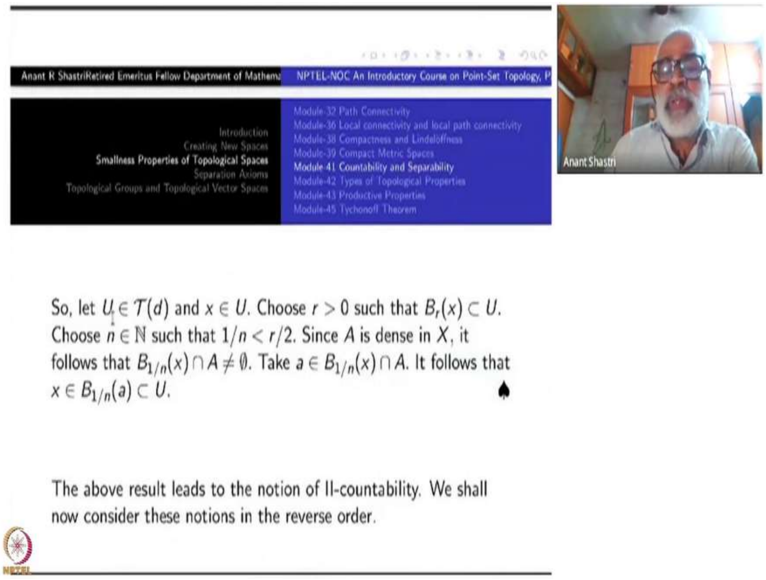

But this is one line proof again. Let U belong to $\mathcal{T}_d$, and x belonging to U, what I have to do I must produce a member from this $\mathcal{B}$ such that x belongs to that member and that member is contained inside U.

So, choose r positive first of all, such that $B_r(x)$ is contained inside U. We are used to this one because every member of $\mathcal{T}_d$ is a union of such members where r is any positive number, but once r is positive there is some n sufficiently large such that $1/n < r/2$, ok? Now A is dense in X, I am going to use that. So, instead of arbitrary x, I am going to shift it inside A ok? How? Since A is dense in X, every open ball will intersect A that is all, because A is a dense dense subset. So, $B_{1/n}(x) \cap A$ is non-empty.

Now, take a point a in this intersection. Then check that x itself belongs to $B_{1/n}(a)$ that is contained inside U. First part is obvious because x is inside this one means distance between x and a is less than $1/n$. So, it is symmetric. So, x is in $B_{1/n}(a)$.

But why this is contained inside U? Because I have chosen $1/n < r/2$, use triangle inequality. Over. The gist of this is that we are going to take this as a definition: The property of having such an A which is a countable dense subset. This is going to be separability. Having a countable local base at each point is going to be I-countability.

This gives you a big thing namely there is a countable base for the whole topology, it is a global base that gives you the third definition which we are interested in and call it second countability ok.

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So, we shall make a definition here. So, we will now study these three notions in the reverse order. We say a topological space is second countable ok? Sometimes you just write II here instead of `second' ok, if it has a countable base. Over. A topological space is set to be separable if there exists a countable dense subset over.

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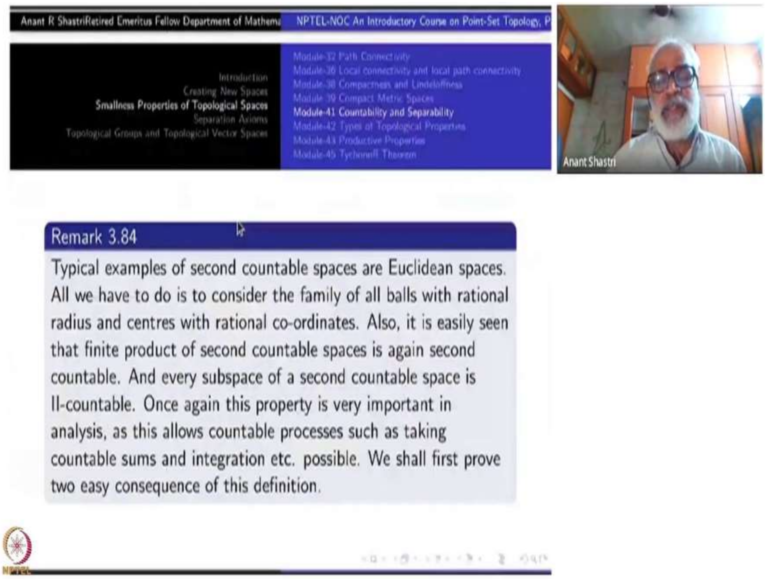

Third one we will wait for it, the third one was the first one which you started with there, ok? Let us make some observations here. Typical examples of second countable spaces are Euclidean spaces. All that you have to do is consider the family of all balls with rational radii and centres with rational coordinates, when I say rational coordinates all of the coordinates should be rational ok.

So, we can also see that finite product of second countable spaces is again second countable. Because if you take a countable base ok countable base here members from here to cross that one they will generate they will give you base for the product. So, countable cross countable is countable again. So, finite products are II countable.

Every subspace of a second countable space is second countable. If you can generate all the open subsets of the bigger space ok when you take subspace, look at the intersections. Suppose Y is a subspace of X, look at Y intersection members of $\mathcal{B}$, where $\mathcal{B}$ is a base for X. That would be a base for Y ok?

So, once again this property is very important in analysis, because it allows several countable processes such as taking countable sums and integrations etc, because every open subset now will be a union of members of a countable family right. So, you can take $U_1, U_1 \cup U_2, U_1 \cup U_2 \cup U_3$ and so on. So, any opens set can be written as an increasing union. So, all these things are possible ok. So, we shall first prove two easy consequences of this definition namely second countability.

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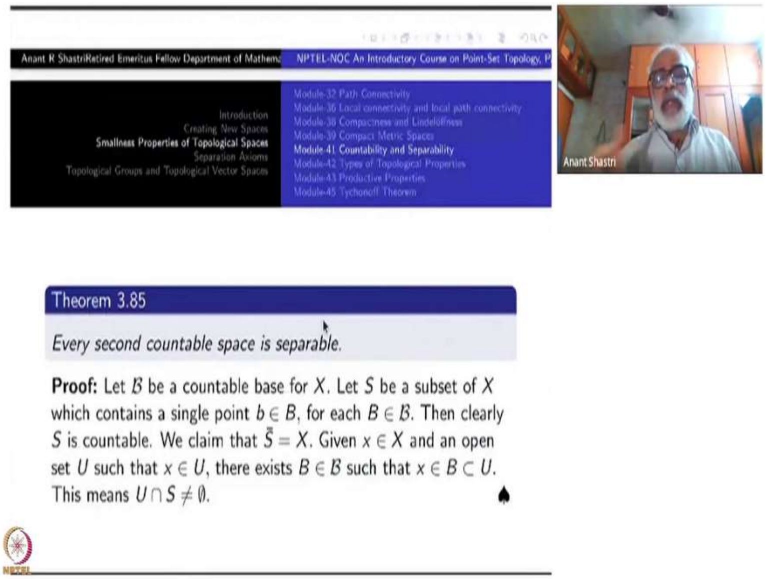

I said second countable, the separability automatically comes here. So, how? Start with a countable base for X ok. Let S be a subset of X which contains exactly one point from every member B of this $\mathcal{B}$. One single point $b \in B$ for each B inside $\mathcal{B}$. $\mathcal{B}$ is the countable base for X, ok.

Since I have chosen one point from each of them this S will be again countable. This S is subset of X by the way ok? And is a countable set ok. Claim is it is dense, $\bar{S}$ is the whole of X. What is the meaning of $\mathcal{B}$ is a base? Given any point $x \in X$ and an open set U such that x belongs to U, there exists B belonging to $\mathcal{B}$ such that x is in B and B is contained in U ok?

But; that means, $U \cap S$ is non-empty ok. So, that $b \in B$ will have a point inside S right? So, $B \cap S$ is itself is non-empty. So, $U \cap S$ is also non empty. So, what we have shown here is that S intersects every non-empty open subset and that is the density.

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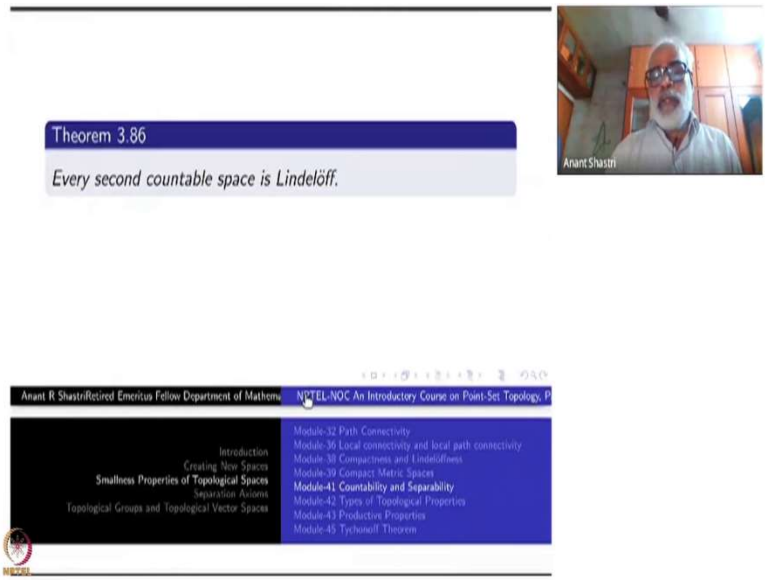

Every second countable space is another thing: it is Lindeloff. Remember what is Lindeloffness? Every open cover has a countable subcover. This also is very easy, how? Take an open cover. I must produce a countable subcover right? Yes or no? So, how do I do that.

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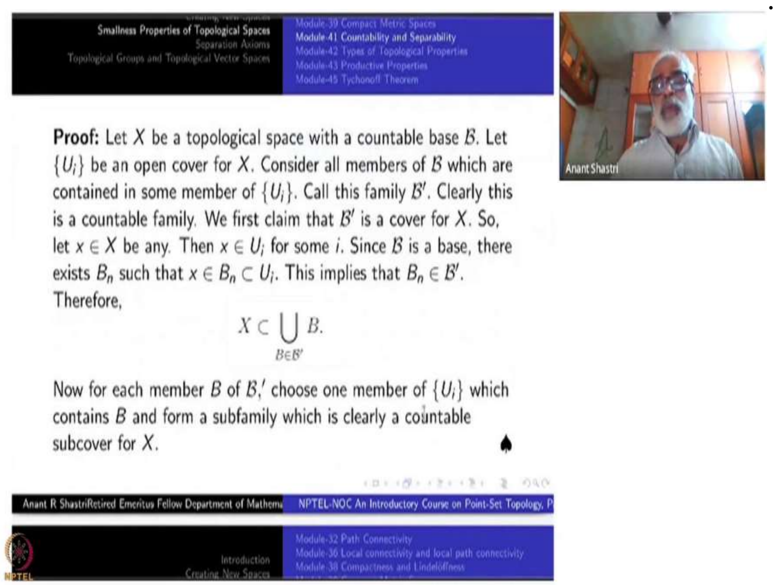

So, let U_i be some open cover for X, but let us say $\mathcal{B}$ is a countable base for X, because X is second countable ok? Consider all members of $\mathcal{B}$ which are contained in some member of U_i ok. So, that is a subfamily of $\mathcal{B}$, some of them may be too large, take only those which are contained in some member of U_i. That is all.

So, call that collection $\mathcal{B}'$. Clearly, $\mathcal{B}'$ is countable because it is a sub family of $\mathcal{B}$ ok? We claim that $\mathcal{B}'$ itself is a cover for X. Automatically that will produce a sub cover. Because we have taken only some members of $\mathcal{B}$ each of them is containing some member of U_i.

So, X will be automatically contained inside the union of correspong members of U_i, ok? So, all that I have to show that $\mathcal{B}'$ is a cover for X. So, let x belong to X be any point. Take x belonging to U_i for some i because U_i's cover X. Since $\mathcal{B}$ is a base, there will be some B_n such that x belongs to B_n and B_n is contained inside U_i. This B_n is inside $\mathcal{B}'$, by definition, because I have taken those which are contained inside U_i here. Therefore, X is contained inside only those B's which are inside $\mathcal{B}'$ ok.

Now, for each member B of $\mathcal{B}'$, choose one member U_i which contains B ok? That forms a sub family of U_i, which is clearly countable because I have chosen one for each B_n and they are bigger subsets than original $\mathcal{B}$. So, they will cover X.

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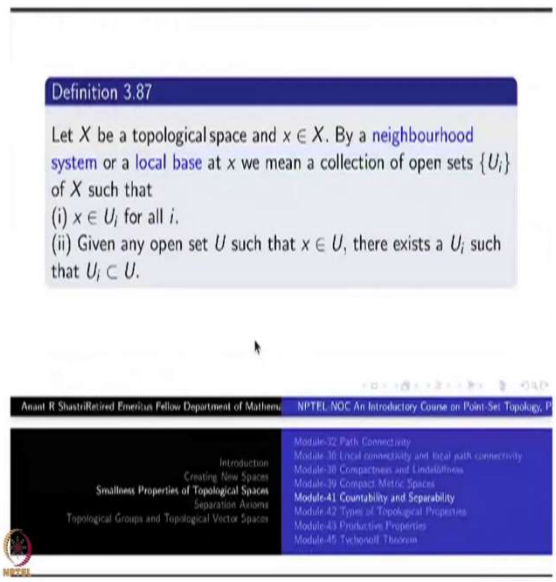

Now, I come to the first countability. Look at a topological space X. So, I want to define another concept here which I have not done so far namely, by a neighbourhood system or a local base (there are two names for the same concept ok? Some people call it a local base, and some others call it a neighbourhood system) at a point x belonging to X, we mean a collection of open subsets U_i of X such that x is inside U_i for every i ok. So, they are all neighbourhoods, and secondly, given any neighbourhood namely some open subset such that x inside U, there exists a U_i such that U_i is contained that.

So, this part is similar to base ok, but everything is happening at a single point x that is why it is called a local base that is all ok. Away from x, this x is fixed, away from x this collection may not have that property. Only for neighbourhoods of x, x belong to U you will have some U_i which is contained inside U. So, this is called local base ok?

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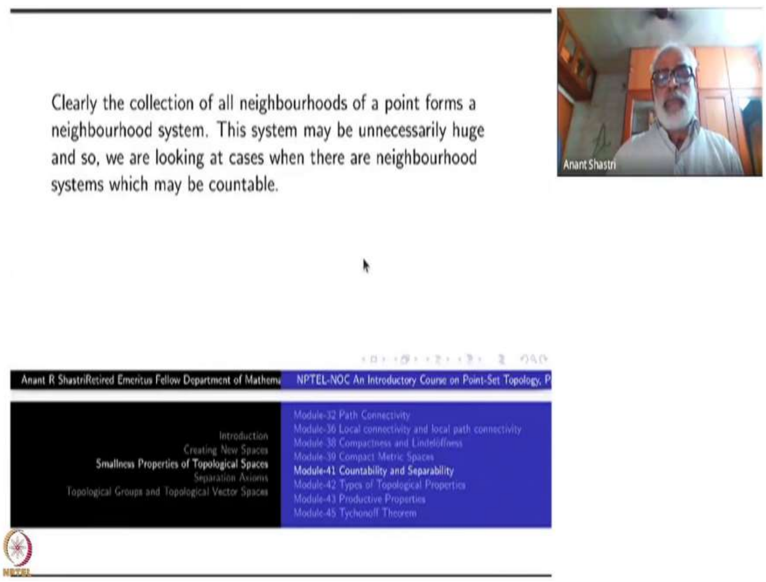

Clearly the collection of all neighbourhoods of a point forms a neighbourhood system. You are very generous in taking all of them here, then there is no problem. But this system may be unnecessarily huge and so, we are looking at cases when there are neighbourhood systems which may be countable and that is the third definition for today.

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We say a space X is first countable at x belonging to X if it has a countable local base at x. If this happens at every point then we will say X is first countable.

Now we said second countability implies separability. We can see that second countability implies first countability also. Because if we have a global base which is countable, global base is always a local base also right? Therefore, second countability implies first countability. So, out of these three concepts, second countability is the strongest.

Student: Ok.

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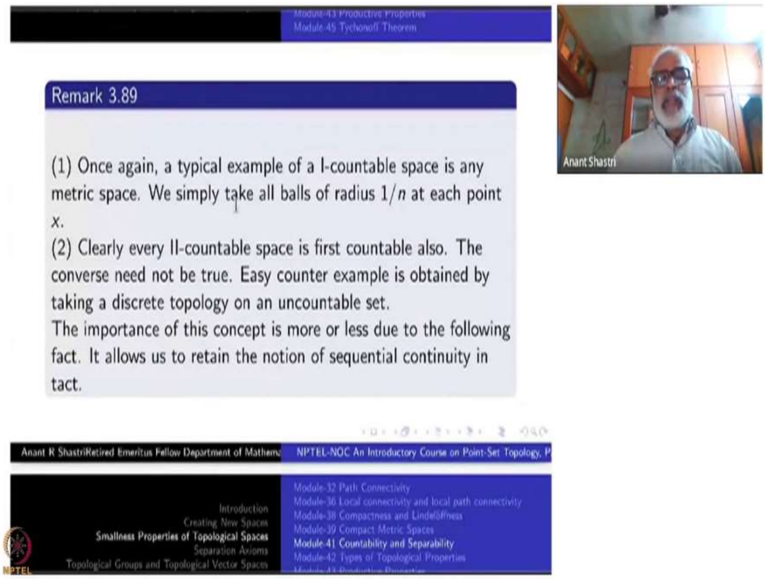

Once again a typical example of a first countable space is any metric space. No need to take a Euclidean space here. Of course, Euclidean space are first countable ok. That was our motivation. All that we have to do is to take balls of radius $1/n$ at each point. Point is fixed, take all the balls of radius $1/n$ that will be a countable base countable local base at x.

Every second countable space is first countable that I observed already. The converse need not be true. If that was the case, then we would not have two different definitions here ok? Easy counter example is obtained by taking a discrete topology on an uncountable set ok? An uncountable discrete topology cannot have a countable base. Think about that. Whereas, in a discrete topology each singleton is an open set. So, I can just take that open set in every singleton as a local base that is all. One single set will give you a local base ok.

The importance of this local base is more or less due to the following fact. It allows us to retain the notion of sequential continuity intact. Intact means what? Sequential continuity was true inside metric spaces. Now the notion of first countability is coming from the metric spaces. So, only that is the key there, not the rest of the metric space structure. The first countability is playing the role of protecting the concept of sequential continuity.

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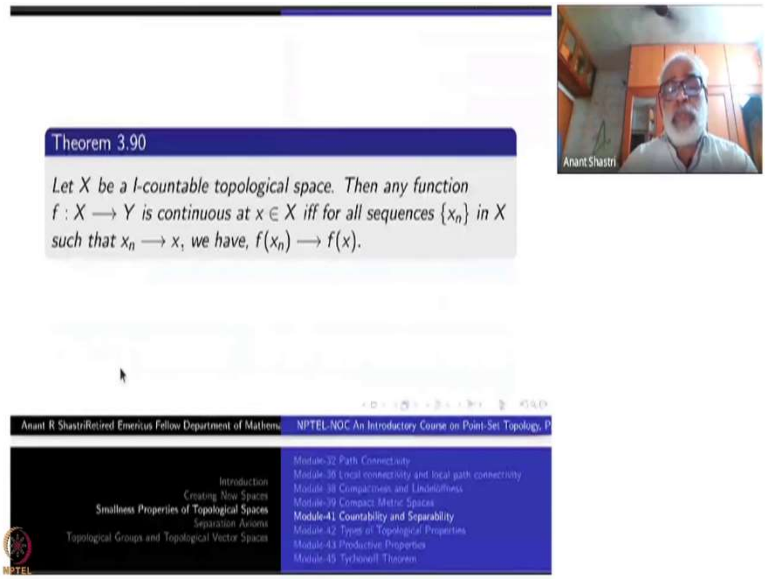

Let us see how. Let X be a first countable topological space. Then any function f from X to Y is continuous at x belonging to X if and only if $\{x_n\}$ inside X converges to x should imply $f(x_n)$ converges to $f(x)$, ok?

Proof is exactly same. One way is always possible you do not need first countability ok? Now, if x_n converges to x, f is continuous implies $f(x_n)$ converge $f(x)$. This is always true in any topological spaces. Conversely, suppose for every sequence x_n converging to x, $f(x_n)$ converge $f(x)$. Then how to show that f is continuous at x? The proof is exactly same as in the case of metric spaces ok?

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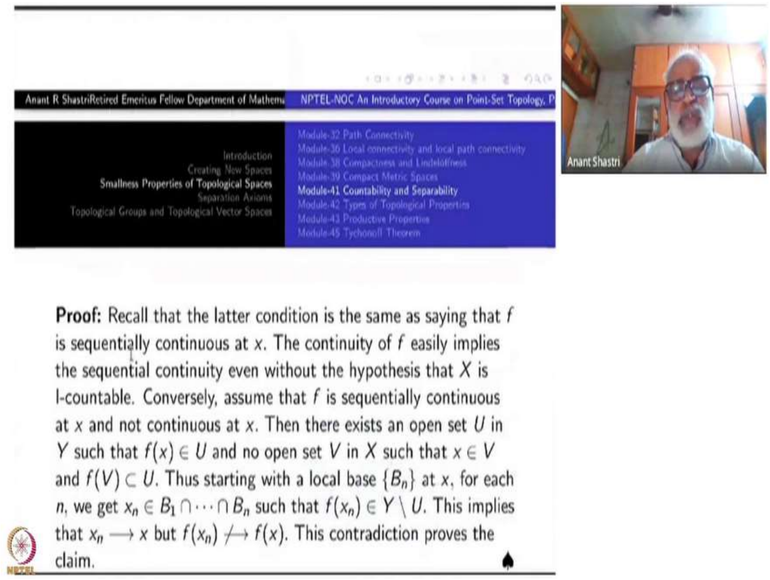

So, let us go through that one. I am trying to prove converse here. Assume that f is sequentially continuous at x and not continuous at x. Then you will get a contradiction ok? Not continuous at x means what? There exist an open set U in Y such that $f(x)$ belongs to U, but no open set V in X such that x belongs to V and $f(V)$ is contained inside U.

Now, I apply this part no open set blah blah blah to each of these elements in the local base here. $\{B_n\}$ is local base at x. Take $x_1 \in B_1$ such that $f(x_1)$ is outside U. After that you take $B_1 \cap B_2$ that is an open subset containing x. So, take x_2 inside $B_1 \cap B_2$, so that $f(x_2)$ is outside U. It may be x_1, no worries, ok? And so on, take x_n in the intersection of $B_1, B_2, \ldots, B_n$, inductively, such that $f(x_n)$ is outside U, ok? Automatically this implies x_n converges to x, ok?

But $f(x_n)$, certainly this is very poor it is far away outside U. So, $f(x_n)$ cannot converge to $f(x)$. So, this contradiction proves the claim.

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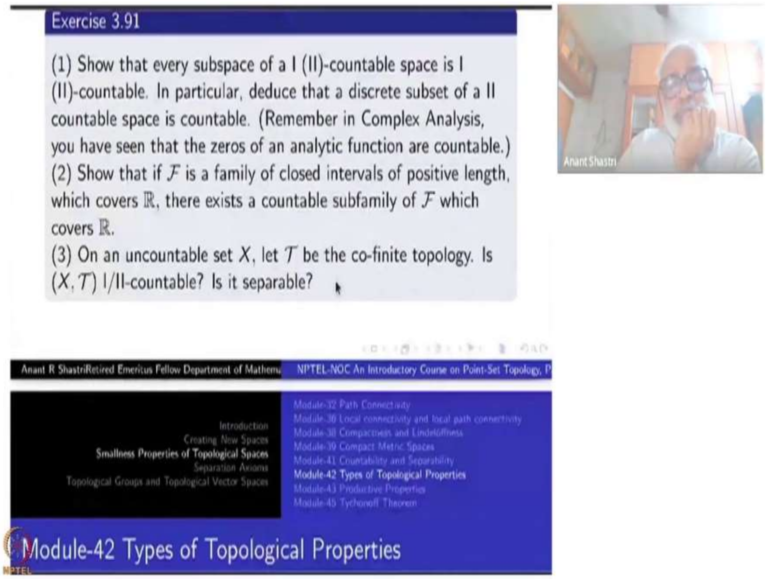

So, I have a few exercises here and a comment: Show that every subspace of a first countable space is first countable, second countable space is second countable, this I have already explained. Same explanation is there for this one also. As an exercise you write down the details.

In particular, deduce that a discrete subset of a second countable space has to be countable. This is used in complex analysis. I do not know whether you have done this kind of things, the set of zeros of an analytic function defined on a domain of $\mathbb{C}$, right? On open and connected subset of $\mathbb{C}$, it may be the whole of $\mathbb{C}$, it will be automatically countable. So, how does one prove that? By proving that it is discrete. Automatically it will be countable.

Show that if $\mathcal{F}$ is a family of closed intervals of positive length which cover the whole of X, there is a countable sub family of $\mathcal{F}$ which covers X. X is $\mathbb{R}$ here. If it is open intervals which cover $\mathbb{R}$, countable family will cover because $\mathbb{R}$ is Lindeloff, ok? We have proved that $\mathbb{R}$ is second countable and second countability implies Lindeloff right? So, open intervals covering the whole of $\mathbb{R}$ will have countable subfamily covering $\mathbb{R}$.

But here, I have closed intervals of positive length, you do not take single points. No single points are allowed, ok, intervals will be of positive length that is all. $[a, b]$ closed interval with

$a < b$. Take such things then you will get a countable sub family which covers it also. So, this you have to do is you know only in $\mathbb{R}$. This is a tricky thing.

On an uncountable set X, let $\mathcal{T}$ be the co-finite topology ok. This is our favourite example. Is X with cofinite topology a first countable space? Is it second countable? Is it separable? Check them because only if you check these things you will understand these these concepts properly ok? So, that is all for today.

Thank you.