Introduction to Point Set Topology, (Part I)
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Lecture - 40
Compactness - Continued

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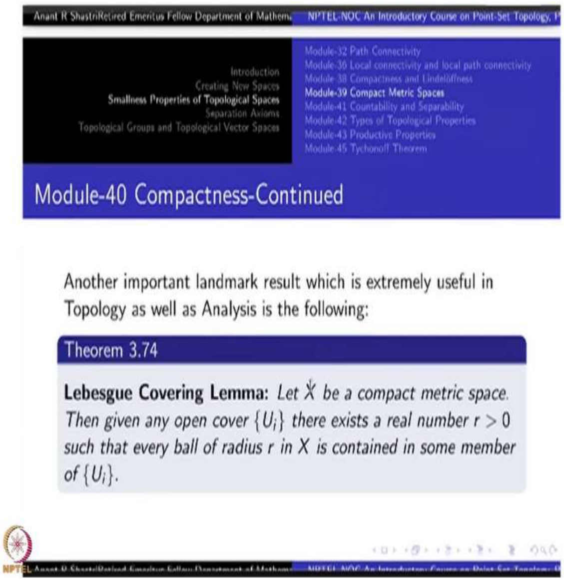

Welcome to module 40 of Point Set Topology, part 1. Today we will continue the study of Compactness. I begin with one of the most important results about compact metric spaces as important as the three big theorems that we have proved about complete metric spaces if not more, ok. This may be even more important. It is called Lebesgue covering lemma.

Start with a compact metric space. Given any open cover $\{U_i\}$, there exists a real number r positive such that every ball of radius r inside X is contained in one of the members of U_i. So, that is the statement. Several applications of this are there even at the calculus level, right in the beginning of Riemann integration theory and so on. Especially for the closed interval $[0, 1]$, in the study of functions defined on $[0, 1]$, directly or indirectly you must have used this theorem.

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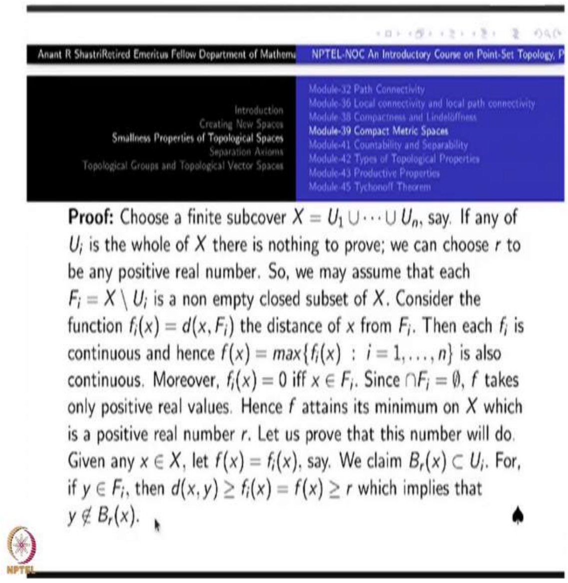

So, let us have a proof of this one which is not at all difficult. As soon as you have an open covering because X is compact, there is a finite subcovering. X is contained inside $U_1 \cup \cdots \cup U_n$ where U_i's are coming from the given open covering $\{U_i\}$. If one of the U_i's is the whole of X, then there is nothing to prove. You can take r to be any number. Every ball of radius r will be contained inside X which is one of the U_i's. So, that is nothing very great, ok?

So, we may implicitly and explicitly assume that the complement of U_i's which I will denote by F_i is non empty for i equal to $1, 2, 3$ up to n, ok? If one of them is empty, there is nothing to prove. So, we are assuming that F_i's are non-empty. Now, consider the distance function from F_i. Let us call it $f_i(x)$. Recall what is $f_i(x)$? It is the infimum of all $d(x, y)$ where y ranges over F_i.

So, that is called the distance function and we know that the distance function is continuous ok? Distance of a point x, where x varies over all of X, from a given set. This set is closed subset which we will use soon. Right now any set will do, the distance function is a real valued continuous function. Therefore, you take all $f_1, f_2, \dots, f_n$ and take the maximum. That will be also continuous ok? So, let us call that $f(x)$.

Now, I use the fact that F_i's are closed. Therefore, $f_i(x)$ is 0, if and only if x belongs to F_i. So, this is where F_i's are closed, that is used. But $U_1, U_2, \ldots, U_n$ covers the whole of X. Therefore, if you take intersection of F_i, by de-Morgan law that must be empty. So, if all the f_i's are 0 that would have mean x would have been inside the intersection, but intersection is empty.

Therefore, given any x at least one of the f_i is not 0. Therefore, this $f(x)$ which is maximum will be always positive, ok? It will never be 0 ok? $f(x)$ itself is the maximum of all the $f_i(x)$. Now as a function on X this function is continuous and will attain its minimum on X, why? Because X is compact. So, compactness is used twice here ok? That minimum will be a positive real number because f is never 0.

This number r will do the job namely take any open ball of radius r wherever you take the center. That ball will be contained in one of the members $U_1, U_2, \ldots, U_n$ ok. So, that is the claim one of the members contains the ball. Ok?

So, proof is very easy. If this is not true, what does that mean? A point y belonging to this ball is not in U_i here means it will be inside F_i; F_i is complement of U_i, right. So, y will be in F_i.

As soon as y is in F_i, the distance between x and y ok will be bigger than $f_i(x)$, because $f_i(x)$ is distance of x and F_i which is the infimum of all $d(x, y)$ ok. So, $d(x, y)$ is bigger than $f_i(x)$.

Actually what we have shown here is given any x look at that index i for which $f(x)$ is equal to $f_i(x)$ because f (x) is the maximum of these finitely many numbers; so you can select one of them, right? For that index i, the same U_i will do the job, that is $B_r(x)$ is contained in U_i. this is what we have proved. May not be very important to take note, but this is what we have proved. That is all.

Look at the way the argument goes. First you have some kind of a infimum here in the definition of f_i's. Then you have a maximum in the definition of f. Then again you take the infimum of f itself. So, the proof of this already gives you another principle which can be used in several mathematical concepts which is called mini-max principle. I have no time to explain that one further. Also it is out of our way, but this is used in all sort of analysis always ok. The proof itself is of importance here.

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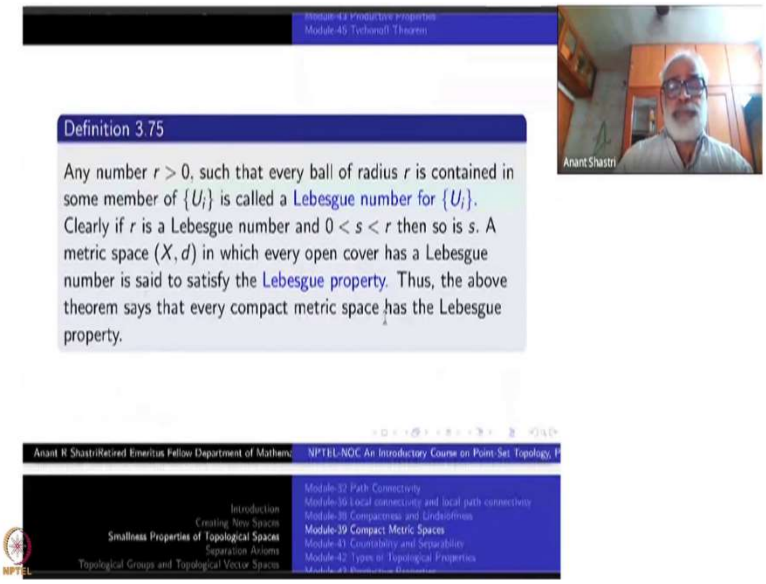

So, we can make a definition ready to use in with this kind of situation. So, that we are able to recall that concept very easily. Any positive number such that every ball of radius r is contained in one of the U_i's is called a Lebesgue number of the cover U_i. Start with any cover if there is an r like this that r will be called a Lebesgue number. It is very easy to see that if s is less than r of course, I have to take r always positive,

So, $0 < s < r$, then r is a Lebesgue number implies s is also a Lebesgue number. So, you can take the supreme of all such values and call that as Lebesgue number. But that is not necessary. So, in practice, we get any number satisfing this property and then call that Lebesgue number, alright ok. A metric space (X, d) which which this property that every open cover has a Lebesgue number associated to it, ok? That metric space is said to satisfy Lebesgue property.

Namely, If this happens for every open cover of X, then we say X has Lebesgue property. So, in this terminology what we have proved is that every compact metric space has

Lebesgue property. There may be other spaces and other metrics ok they may have this property. We do not know whether that will imply that is compact. So, that is another aspect.

So, I am not going to touch that one here, but this is the definition. So, we can just work out with this one. Lebesgue covering lemma is very important result in real analysis.

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Let us derive one immediate consequence from this one, namely another important concept in metric space theory namely, uniform continuity, ok? In real analysis you have seen that a continuous function on closed interval is uniformly continuous ok. So, that result can be extended. By the way this result is very much used in Riemann integration theory so, so uniform continuity itself is an important thing.

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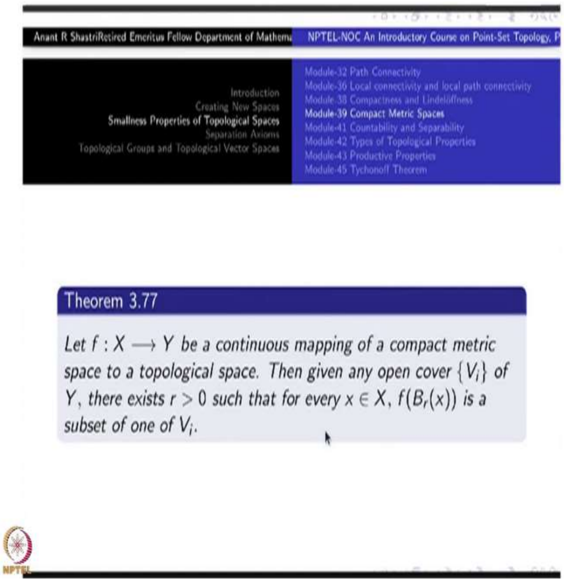

Take any function on a compact metric space to another topological space and a continuous one, ok. The assumption is X is compact metric space. Then given any open cover $\{V_i\}$ of Y there exists $r > 0$ such that for every $x \in X$, we have $f(B_r(x))$ is a subset of one of the V_i's, ok? The statement follows directly from the theorem, because all that you have to do is when you have taken a covering $\{V_i\}$ of Y, take $f^{-1}(V_i)$'s, that will be a cover for X. Then choose this r to be a Lebesgue number for it, and then you have this property. So, that is an easy consequence.

But why this is called uniform continuity? I will explain this, ok? So, if both X and Y were metric spaces then how do you define uniform continuity? Given $\epsilon > 0$, there exists a δ such that distance between x_1 and x_2 is less than δ should imply distance between $f(x_1)$ and $f(x_2)$ is less than ϵ.

That given ϵ? But now there is no metric on Y. So, I have to convert that `given ϵ' one and that is converted into an open covering $\{V_i\}$ here. If you have a metric space Y and you have ϵ here you have all ϵ balls that is an open cover for Y. So, that has been replaced directly by taking this V_i now. These V_i's play the role of ϵ ok all over the whole thing you have to talk about in one single call, ok.

Then this r plays the role of δ there is no problem. So, this r is independent of x you see, if it is depending upon x that is ordinary continuity ok. So, this is the way the uniform continuity is attained.

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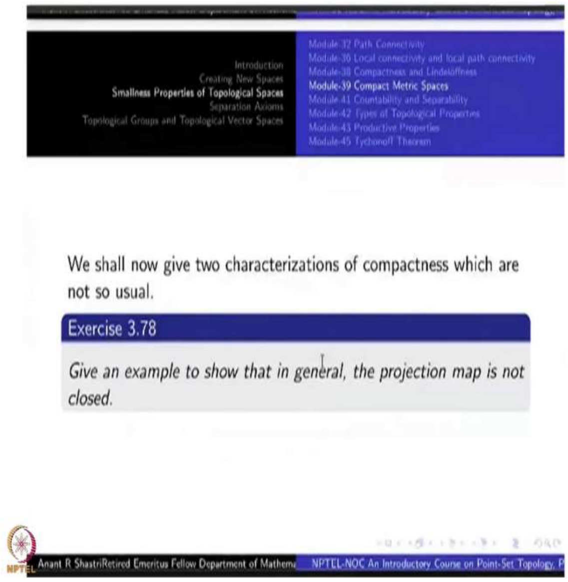

So, we shall take a break from compact metric spaces now and go back to the study of compact spaces in general again. Now, I come to one of the important results what is called the tube lemma or Wallace theorem. But, what I am going to do is, I am going to combine it with another important result, which is not so central and do a bit of circus here, to give you a characterization of compact spaces.

See our definition of compact spaces is what? Every open cover has a finite sub cover, just one definition, whereas, for many other concepts we have seen that there are several definitions, right? So, such an important thing you should have different ways of looking at it ok? So, here are two other ways of looking at compactness; compactness as a property that is. Two characterizations, I am going to give now ok?

Before that I will recall that you might have by now you might have seen such a thing, but let us look at this one. Give an example to show that in general projection map is not a closed map. So, I still put it as an exercise, but since I want to illustrate the coming theme here so, let me tell you that in general, projection maps from $X \times Y$ to X or Y are not closed maps.

They are open maps remember that, ok? While studying the product space we have seen it. All the time we have used it also. So, what is the simplest example? There are many. The simplest example is you know from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ itself. Look at the hyperbola given by $xy = 1$, ok? Its projection on the x-axis just misses the point 0, nothing else. Therefore, it is not closed right? That shows $(x, y) \mapsto x$ is not a closed map. So, having said that now we come to the characterization of compact spaces no metric now.

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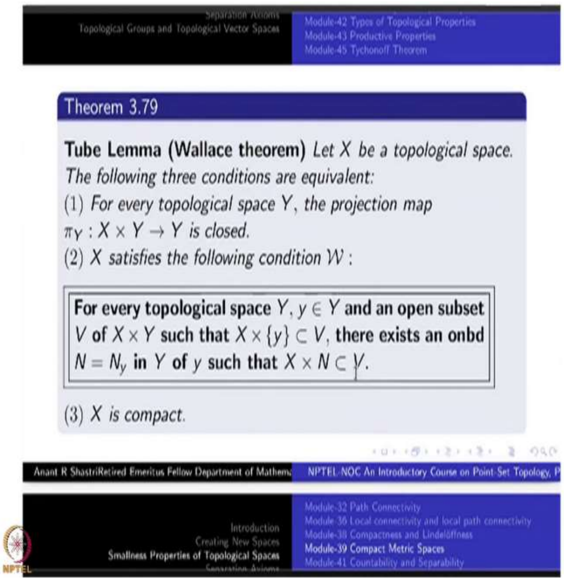

So, I would like to call it, the whole thing, as Wallace theorem, But in classical, standard books Wallace theorem is only one third or even one sixth of whatever we are doing here, ok? I will tell you what it is exactly. Let X be any topological space, then the following three conditions are equivalent. So, I have put deliberately the 3rd one here, that is, X is compact.

The other two are going to be equivalent to that. That is my aim. The first statement is: for every topological space Y look at the projection to the Y coordinate from $X \times Y$, that is a closed mapping ok? Not just some particular Y, for every topological space Y, this should happen ok?

The second statement is: X satisfies the following condition which is somewhat longer. So, I have put this one carefully and made it very much visible. So, you should know this one this is a very important thing which goes under the name tube lemma. So, for every topological space Y, again $X \times \{y\}$ pick up a point inside Y and an open subset V of $X \times Y$ such that the copy of X at y namely contained inside this open set. So, V is an open neighborhood of $X \times \{y\}$. Suppose, this is the situation. Then there exists an open neighborhood N which may depend upon y such that the entire $X \times N$ is contained inside V, ok?

So, this is also very much used in analysis. So, this is a condition I want to say which will be equivalent to X being compact. So, third statement is X is compact ok.

The proof will go through not (1) implies (2) implies (3) implies (1) as usual, but I will prove (1) and (2) are equivalent and (2) and (3) are equivalent.

So, that is why I have put this one in the center.

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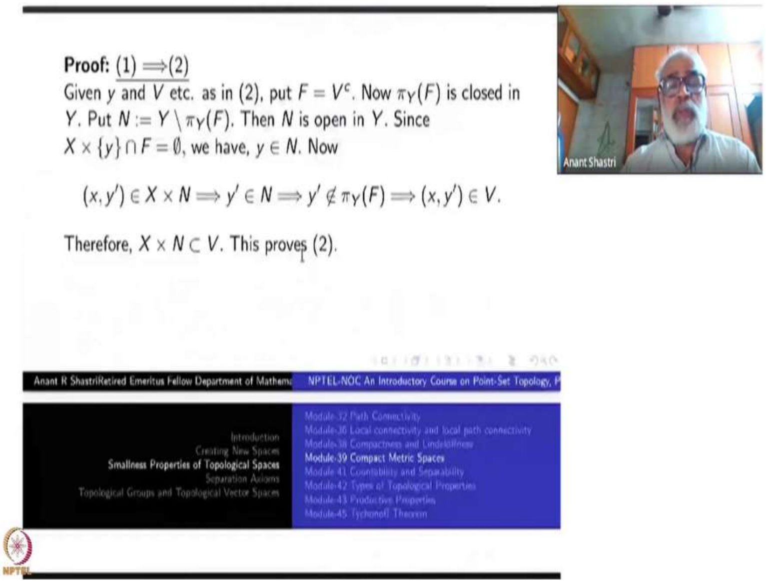

So the proof of (1) implies (2),: what is the meaning of that? Start with this condition that projection maps to every Y is closed. Then I must prove this condition ok? So, start with any

$y \in Y$, a neighborhood V of $X \times \{y\}$ etc. this part as well as statement (1), then I must produce a N with this property, right?

So, take F to be the complement of V. See, V is an open subset of $X \times Y$. So, take its complement, that will be a closed subset of $X \times Y$, ok? Now, use this property (1) and come to Y, π_Y of F is a closed subset of Y right. So, its complement in Y will be an open subset. So, where have you come? $X \times \{y\}$ was completely contained inside V.

Therefore, $X \times \{y\}$ intersection with F which is the complement of V will be empty set. That just means that the projection of F does not contain the point y. Its complement N is an open subset which contains y. This is what I start with $F = V^c, \pi_Y(F)$ is closed ok.

Now, take the complement of that $\pi_Y(F)$ inside Y, that is an open set. $X \times \{y\} \cap F$ is empty right? Because $X \times \{y\}$ was contained inside V. Therefore, this y will be inside this N, Y cannot be inside $\pi_Y(F)$ that is otherwise this would be a non empty ok. Projection of (x, y) if it is there then projection would be inside F, right. So, this is empty means now this y is inside N, ok.

So, N is a neighborhood of y right? That is what we are trying to prove here you see N is a neighborhood of Y. And what I want to prove is $X \times N$ is contained inside V, right? So, that also comes freely here: look at any point (x, y') ok; suppose it belongs to $X \times N$. That means that y' is in N ok? The second coordinate projection of this one cannot be inside F because you have taken N as the complement. But y' is in N implies y' is not in $\pi_Y(F)$ ok; that means, (x, y') is in the complement of that which is V. So, otherwise it will be in F. So, the entire $X \times N$ is contained inside V. So, that was an easy proof. Just some set theory. You have to keep going up and down the projection map. Alright. So, (1) implies (2).

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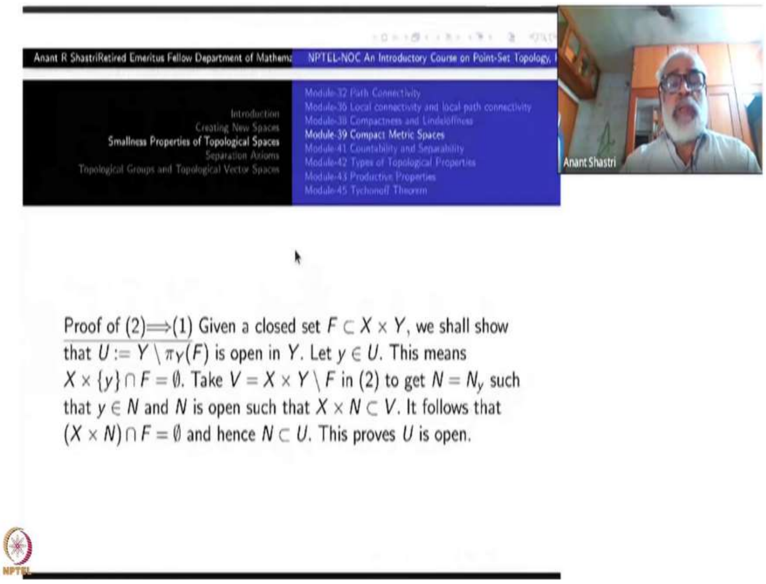

Now, we will reverse the arrow: (2) implies (1): Namely take any space Y, have to show that the projection map $X \times Y$ to Y is a closed mapping. Now, start with a closed subset F of $X \times Y$. Then $\pi_Y(F)$ is closed this is what we have to show. Take the complement U, you must show that this complement is open inside Y. In the same way did (1) implies (2), I am trying to go backwards there, ok.

Now, take y belonging to U. Again, these are all set theoretic steps, they are reversible. This just means that $(X \times \{y\}) \cap F$ is empty. Now, you take V as $(X \times Y) \setminus F$ ok? To begin with F is closed. So, V equal to $(X \times Y) \setminus F$ in (2). Once you have this is an open subset right the property (2) gives you a neighborhood N of y, such that N is open and $X \times N$ is inside V.

So, it follows that $(X \times N) \cap F$ is emptyset. Because by definition V is $(X \times Y) \setminus F$, ok. Therefore, N is contained inside U because U is the complement ok.

So, more or less you know if you choose correctly the notations every arrow can be reversed here from (1) to (2). However, since this is somewhat complicated statement, we have written an independent proof. So, (1) implies (2) and (2) implies (1) is over.

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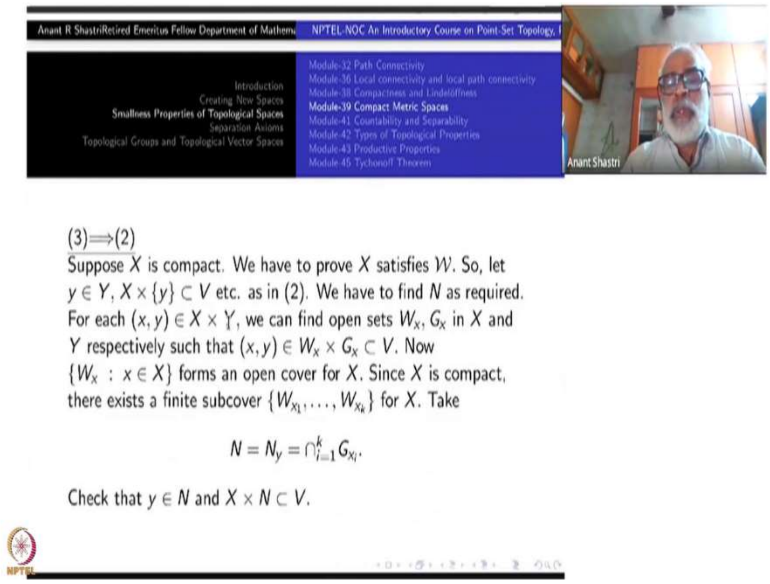

Now, come to (2) implies (3) and (3) implies (2). So, I first prove (3) implies (2): I save the last thing ok namely (2) implies (3) for the last thing that is the theorem which is called Wallace theorem. Sorry, this is the one which is called Wallace theorem or tube lemma 3 implies (2), (3) is compactness right yeah, (3) is compactness.

Compactness implies this one is a standard result which is called Wallace theorem or tube lemma, ok. So, what I have done is I have made it into if and only if along with another condition here ok. Let us prove this one.

Let X be compact. We have to prove X satisfies this property $\mathcal{W}$: Wallace property I have denoted by $\mathcal{W}$, $\mathcal{W}$ for Wallace ok. So, that entire property we have to prove.

So, what is the meaning of property proving I have to start with the hypothesis of the property? Let $y \in Y, X \times \{y\} \subset V$ open, this much we have to start with. We have to find N as required. Namely N open such that $y \in N$ such that the whole $X \times N$ is contained inside V. That is what we have to prove. For each $(x, y) \in X \times Y$, we can find an open subset W_x , G_x in because by the way y is fixed here, x is the variable here.

But, (x, y) is inside $X \times \{y\}$. So, W_x open in X and G_x is open in Y ok? Open subsets such that (x, y) is in $W_x \times G_x \subset V$. This is the definition of the product topology, right. These are the basic open subsets $W_x \times G_x$. Since V is open there is a basic open set. The basic open set looks like the product. That is all I have used: V is an open subset containing $X \times \{y\}$, ok. Here only x varies y is fixed as x varies these W_x's will cover X. That is an open cover.

Now, use the property that X is compact, get a finite cover $W_{x_1} \ldots W_{x_n}$, ok? All that you do is take N equal to intersection of all these W_{x_i}'s, no intersection of corresponding G_{x_i}'s. So, $G_{x_1}, G_{x_2}, \ldots G_{x_n}$ are all neighborhoods of y, take the intersection that will be neighborhood of y and that will do the job.

Because now take any point (x, y') here ok? Now, $y' \in N$ means y' is in all the G_{x_i}'s right. x is in one of the W_{x_i}'s which one you do not know. Suppose, it is in W_{x_1}, then (x, y) will be in $W_{x_1} \times G_{x_1}$ that is inside V over. So, that is why we have to take the intersection here ok. So, the proof of Wallace theorem as such is just that much only ok, this (3) implies (2).

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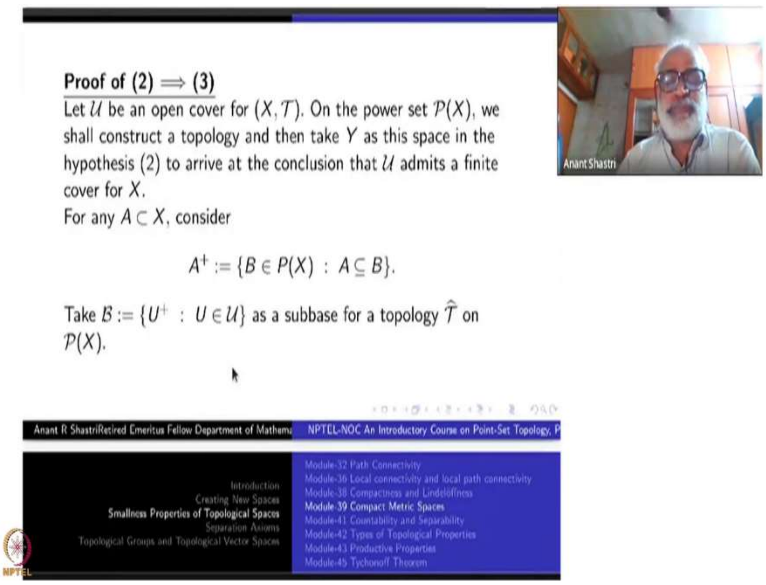

Now, finally, I come to (2) implies (3). Pay attention to this ok? Because many expositions, many books do not have this theorem, definitely not this proof, alright? Now, I have to prove that the space X is compact by using the condition (2), (2) implies (3) means that ok. So, start with any open cover for X. How to use (2), so that this is going to give me a finite sub cover? Condition (2) says for every Y something is true.

So, I must cook up some, you know, some nice space Y nice means what? A friendly one which will give you something ok? A space Y such that when I apply this condition to that Y (of course this condition is true for every Y), it will give you something. How to use that? You have to cook up some Y.

So, that when you use this condition that it should give you X is compact ok. So, having said that much I just go ahead with with the proof. How I got this one. Start with an open cover $\mathcal{U}$ for the given topological space X. Now, go to the power set of X. Now, Y is going to be the power set of X and I must give Y a topology. On this power set of X, ok?

So, the open subsets of power set of X will be what? They will be collections of subsets of X , ok? It is it is a subset of the power set of X, a subcollection of power set of X. So, you have to be careful here what is going on. We shall construct a topology and then take Y as this space in the hypothesis, to arrive at a conclusion that $\mathcal{U}$ admits a finite cover. So, this Y is going to be depending upon the covering $\mathcal{U}$. The underlying space is always $P(X)$, the power set of X.

But, the topology I choose will depend upon the covering $\mathcal{U}$, ok? That will say that this covering has finite sub covers that covers ok. So, that is the trick. For any subset A of X, let us make a notation here A^+ means the collection of all the super sets of A, collection of all supersets. Everything which contains A and of course, subsets of X. All subsets B of X such that A is contained inside B, that is the notation A^+.

For instance, if A is empty what will be A^+? It will be the whole of $P(X)$, all the subsets of X. Every subset contains emptyset, right? So, empty plus is the powerset itself. Similarly, if A is X, then what will be A^+? It will consist of only one element namely X itself. It is a $\{X\}$. You remember that it is not X, it is set containing X; whereas, A is empty, A^+ is all the subsets of X the power set of X and it is not X, ok? So, that is the convention here ok.

Now, let $\mathcal{B}$ be the family of all U^+ where U ranges over this open cover ok. This collection B, I do not know what property it has, it does not matter. Take this as a subspace for a topology $\widehat{\mathcal{T}}$. Any collection of subsets of a given set, this time the given set itself is $P(X)$ ok? That will generate a topology, the smallest topology containing $\mathcal{B}$ as a subbase.

So, I am denoting by $\widehat{\mathcal{T}}$. It is nothing, but $\mathcal{T}_B$ that we have earlier used in the notation. So, this is a topology on $P(X)$. That is what I am interested in now. Clearly it depends upon $\mathcal{U}$, ok?

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An important point to note is that $\{X\}$ is in contained non empty member of $\widehat{\mathcal{T}}$. Take an open subset of $\widehat{\mathcal{T}}$, X will be always member there, why? Because you look at U^+. Of course, empty set does not have, empty set is also there in every topology right. Take any non empty set. look at this U^+ all super sets. So, in every U^+ that X is there.

So, when you take subbase remember you have to take finitely many members in the subbase and then take the Intersection. That will also contain $\{X\}$. Then union of course, will contain $\{X\}$. So, $\{X\}$ is in every member ok? Other than the empty set ok. So, I have taken Y as $(P(X), \widehat{\mathcal{T}})$ alright.

So, having chosen Y, what is your V? An open subset of $X \times Y$? It is equal to the union of all those which look like $U \times U^+$, where U is a member of $\mathcal{U}$, ok? That is an open subset of $X \times Y$.

So, this is going to be a subset of $X \times Y$ ok, then clearly V is an open subset of $X \times Y$. Now, singleton X is always in the second factor U^+ here for all U. Therefore, what you take here $\{X\}$ is there, but when you take the union where U ranges over this $\mathcal{U}$, that covers X. Therefore, what happens this entire $X \times \{X\}$ is contained inside V. So, this is the situation of the condition $\mathcal{W}$, Wallace condition.

Given this little $y = \{X\}$ is a point of Y. Remember that ok? And $X \times \{y\}$ is contained in V, V open. What does it give you? It gives you a neighborhood N of the y here, which is $\{X\}$ here such that $X \times \{y\}$ is contained inside $X \times N \subset V$. So, that is the conclusion of (2) alright?

So, after cooking up this Y which is a topology on the power set, I have used that condition (2) to get this neighborhood N. Now, I want to say that this will give you a finite cover out of $\mathcal{U}$. Can you see how finite cover comes? That is a trick here. That is a very strange thing.

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Even now you may not see why the finite cover comes ok. The point is that $\mathcal{B}$ here, this is a sub base. What is the meaning of subbase for a topology? Given any point belonging to some open set, then there will be finitely many members from $\mathcal{B}$ such that their intersection contains the point as well as contained in the given open set, right. A basic open set out of this one is finite intersection of members of this one. So, that is what I am going to use here.

$X \times \{y\}$ is contained in $X \times N$, right. Right now I will replace this N by a basic open set, intersection of finitely many members of $\mathcal{B}$. But in this statement I can replace it in (26) this this statement I can replace it because this $\{X\}$ will be inside this one and this is contained inside N. So, $X \times N$ will be also contained.

So, instead of this one I can write this one, that is all. I am writing I can write the $\cap_{i=1}^{n} U_i^+$ as N to make it simplify. In other words I am getting another equation here namely $X \times \{X\}$ is contained inside X cross this intersection contained inside V ok? So, (26) can be read with N equal to intersection of U_i^+, where i range from 1 to n.

Now, you verify another property of this plus the supersets of A; $A^+ \cap B^+$ is $(A \cup B)^+$. Something belongs to this one means, it contains both A and B. This is a subset of $P(X)$, members of this one are subsets of X.

If it contains A, it is here it is contains B it is here. If it contains A and B, it contains $A \cup B$. So, it is here and conversely any set which contains $A \cup B$ will contain both A and B and hence in both A^+ and B^+. So, $A^+ \cap B^+$ is the same thing as $(A \cup B)^+$.

So, this happens for every member. In particular it happens for the members of $\mathcal{U}$ also. So we put G equal to union of U_i's. See this $U_1, U_2, \ldots, U_n$, you have got a finite union. Now, you take $G = \cup_{i=1}^{n} U_i$, ok? It follows that this W is equal to G^+ ok. What was W? I forgot there is no W here.

Student: W is G^+ here.

But, what is W? G is union of U_i's. Apply this one inductively. So, what do you get? For two of them you have this, you can do it for n of them ok? So, G^+ is the intersection of U_i^+'s, that is N. So, this W was N here, that is all. So, this is a typo this must be N that is all I wanted ok by this notation if you look at the intersection of U^+ is the with the same thing as taking union of these and then take the plus. So, this N is G^+, ok.

So, it remains to show that G itself is X. That will prove that this open cover has a finite subcover. Since $\mathcal{U}$ was an arbitrary open covering, that will prove that X is compact ok? So, that is the whole idea. So, I have to show that these $U_1, \ldots, U_n$, they cover X, ok.

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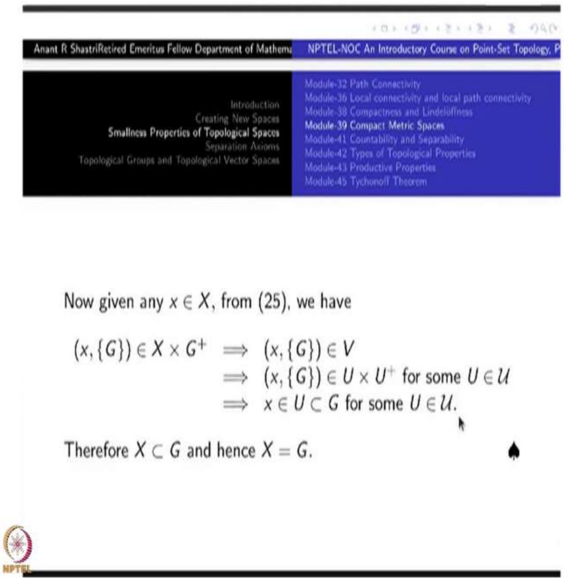

So, look at (x, G), ok? This is member of $X \times G^+$. Remember what was G^+. Whatever G is, G^+ contains all those members which contain G. In particular G is there in G^+, ok? So, this is a member here G in G^+, alright. So, what is the meaning of this? (x, G) will be inside V, because the whole thing was this $X \times N$ is $X \times G^+$, that is what I have used here. $X \times N$ is contained inside V by (26).

So, now come here, yeah. So, this implies (x, G) is inside V which is same thing as (x, G) belongs to $U \times U^+$ for some $U \in \mathcal{U}$, right? Just go ahead what was V? V is the union of all this. So, it must be inside one of the $U \times U^+$,ok, for some $U \times U^+$. What is the meaning of this? This is an ordered pair: x is in U and G is inside U^+. Means what? G is inside U^+ means G contains U.

So, x is in U and U contained in G for some U. Therefore, all of x is belongs to G. Start with any x here, ok? Then you go through this one it shows that it is inside G. Therefore, this X is contained inside G, but G is a subset of X after all. So, X is G ok.

In some sense this has the magic similar to Furstenberg's proof of infinitidue of primes. The key here is the passage from arbitrary to finite comes only because the topology has the property that such family you know generate topologies by taking finite intersections first and then go into the arbitrary union. That is all. There is no other way I can explain this one ok? Suddenly you get finite cover out of nohwhere.

In the definition of compactness, we have put the finiteness ourselves. Now, we have to produce it right? to understand why and what is going on? So, you have to choose, you know, properly thought of topology which has something to do with the open cover here you know, the given open cover, yeah.

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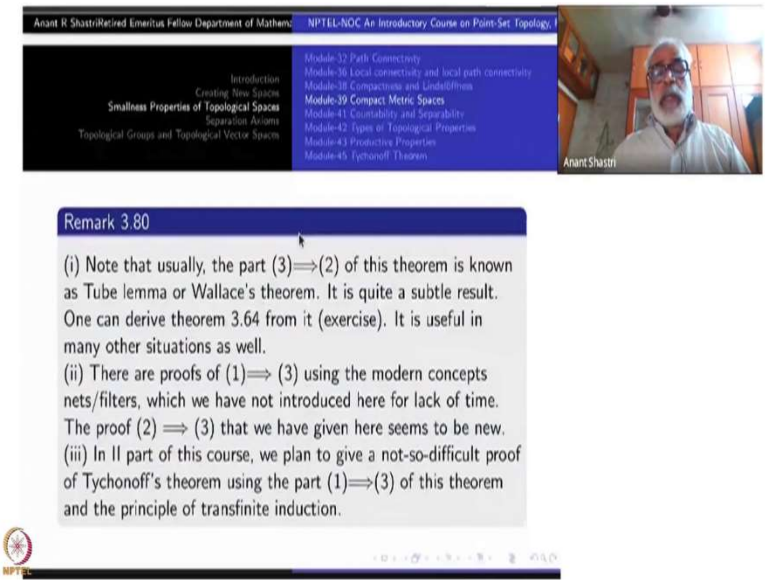

So, having said that, let me again repeat a few things which I have told already. Usually (3) implies (2) of this theorem is known as tube lemma or Wallace theorem. This itself is a very subtle result and it is very useful also. For example, we can derive theorem 3.64 from it. So, this I will leave it as an exercise to you, the uniform continuity etcetera. It is useful in many other situations as well ok.

There are other proofs of (1) implies (3). See, now we have all three are equivalent, right. So, this all the three are equivalent. How I have proved it? I have proved (3) implies (2) and (2) implies (3); and (1) implies (2) and (2) implies (1). So, some people you know classically have proved that not very classically this is a modern approach now (1) implies (3) using ideas of nets and filters. These things you can find in many books, ok.

Proof of (1) implies (3) by using nets or filters, what are called as ultra filters actually, also produces this in a magic way, just the way this proof has produced it ok. The nets also give you they pretend to explain it, but it is still some kind of a mystery only, how the proof comes.

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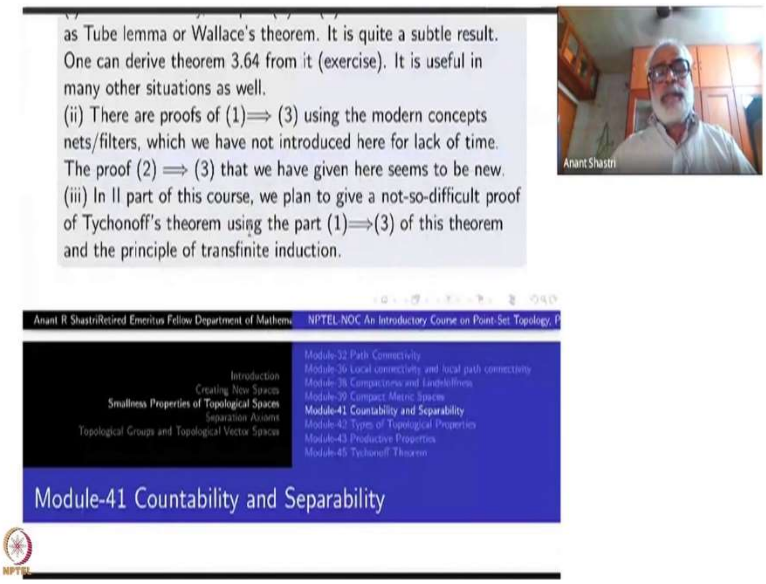

So, that is one thing I wanted to tell you whereas, the proof that I have given here of (2) implies (3) namely from Wallace condition producing the compactness, this seems to be new I have not seen it anywhere, ok.

Now, in part II of this course, I plan to give a not so difficult proof of Tychonoff's theorem namely, arbitrary product of compact spaces is compact using (1) implies (3) of this theorem along with principle of transfinite induction. So, this transfinite induction takes some time, that is why I have put it in part II. Otherwise I could have done it right now also ok. So, that is the thing I wanted to tell you.

So, next time onwards, we will take up some other properties namely countability conditions, separability and so on.

Thank you.