Introduction to Point Set Topology, (Part I)
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Module - 04
Lecture - 04
ε – δ definition of continuity

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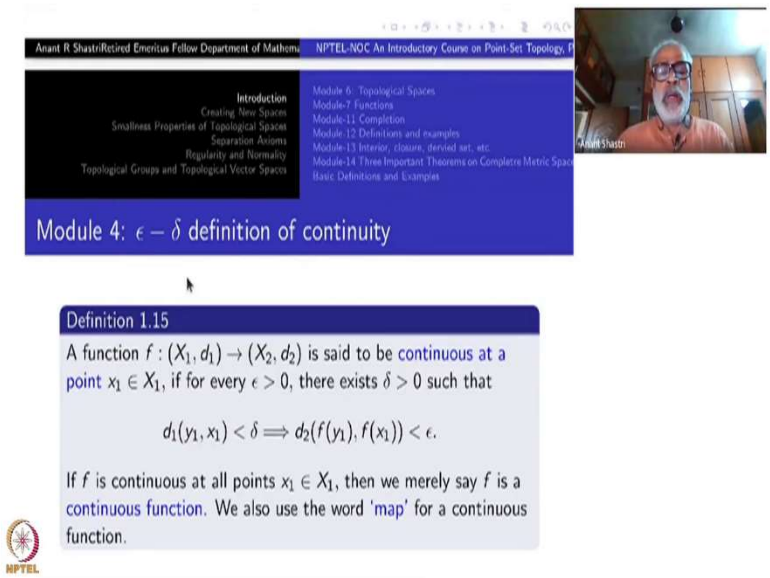

Welcome to module number 4 of Point Set Topology course. So, today we shall take up the one of the important concepts namely continuous functions on metric spaces. Starting with the idea or whatever we have learnt our basic lessons in real analysis of functions real valued functions of real number how you defined continuity there using $\epsilon - \delta$ definition without any effort exactly same definition can be adopted here for continuity.

So, that will be called $\epsilon - \delta$ definition of continuity. You know it already. So, I am recalling it. Here, take a function f from (X_1, d_1) to (X_2, d_2). What does that mean? That means that X_1 is a set and d_1 is a metric on it, X_2 is a set and d_2 is a metric on it. The function is a set theoretic function from X_1 to X_2. Take a point x_1 in X_1. The function f will be called continuous at x_1 if the following happens. What is that?

So, that is the $\epsilon - \delta$ part. For every ϵ positive there exists δ positive such that $d_1(y_1, x_1)$ less than δ implies $d_2(f(y_1), f(x_1))$ is less than ϵ. Here, x_1 and y_1 are inside capital X_1 obviously, and I am applying d_1 to the pair; the distance is less than δ implies the corresponding distance between $f(y_1)$ and $f(x_1)$, this time I have to take the d_2 distance in X_2, that must be less than ϵ. If this happens for all points $x_1 \in X_1$ namely f is continuous for all $x_1 \in X_1$. then we say f is continuous on X_1 or just f is continuous. Thus f from X_1 to X_2 is continuous means that it is continuous at all the points ok. We also use the word map which is a smaller word compared to our continuous function, that is all.

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One of the important thing is that, you know what is called sequential continuity, using the sequences and that is again you know very much available to us in the case of metric spaces also exactly same kind of definitions same kind of techniques and so on. So, let us first make one more definition what is the meaning of convergence of sequences in a metric space.

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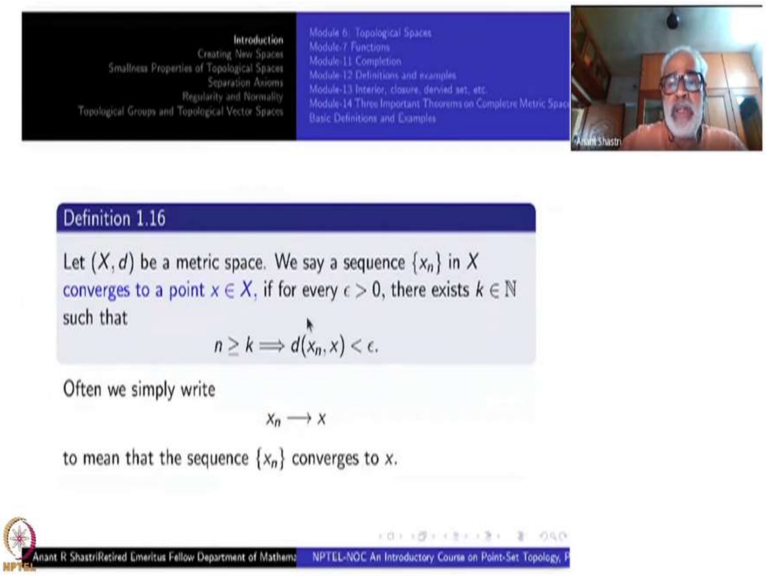

Take a metric space (X, d), take a sequence $\{x_n\}$ in X. It is supposed to converge to a point x ; I am making a definition here. So, $\{x_n\}$ converges to a point $x \in X$ if the following thing happens once again in terms of this ϵ and number k here. For every $\epsilon > 0$, you must have a natural number k such that all $n \geq k, d(x_n, x)$ the distance must be less than ϵ.

So, in other words, after a certain stage all the points of the sequence are close to x. What is this closeness? The distance is less than ϵ if you remove this d and replace it by modulus, $|x_n - x|$ what you get is the definition of continuity for $\mathbb{K}$ valued functions real or complex valued functions, ok? So, we have just done the other way around you took the definition from complex analysis or real analysis whatever you have learned the modulus you have replaced by the distance function here that is all, ok.

So, whenever a sequence converges we just write it in a very simple way $x_n \to x$. So, this I can read it as x_n converges to x. Over. Then all these conditions are inside my mind ok. So, by just this symbol we are expressing all this idea.

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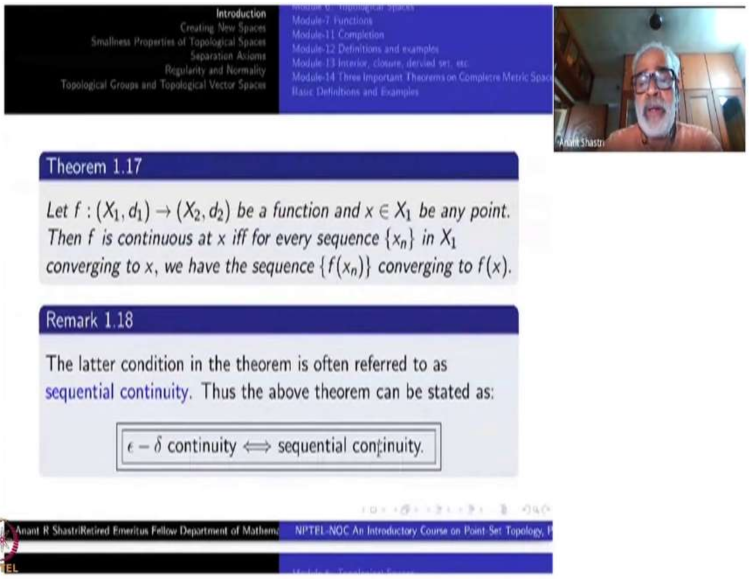

Now, I am going to give you the equivalence of this $\epsilon - \delta$ definition with that the other one called sequential continuity ok? So, start with a function f from X_1 to X_2 again take a point $x \in X_1$, f will be continuous at x if and only if for every sequence $\{x_n\}$ converging to x the sequence $f(x_n)$ must converge to $f(x)$ ok. So, that is the condition for f to be is continuous. Take a sequence it must converge to some point x. Then take the sequence $f(x_n)$, it must converge to $f(x)$. If this happens for every sequence then f must be continuous to x ok?

So, these two things are equivalent. This is displayed here namely epsilon delta continuity is equivalent to sequential continuity.

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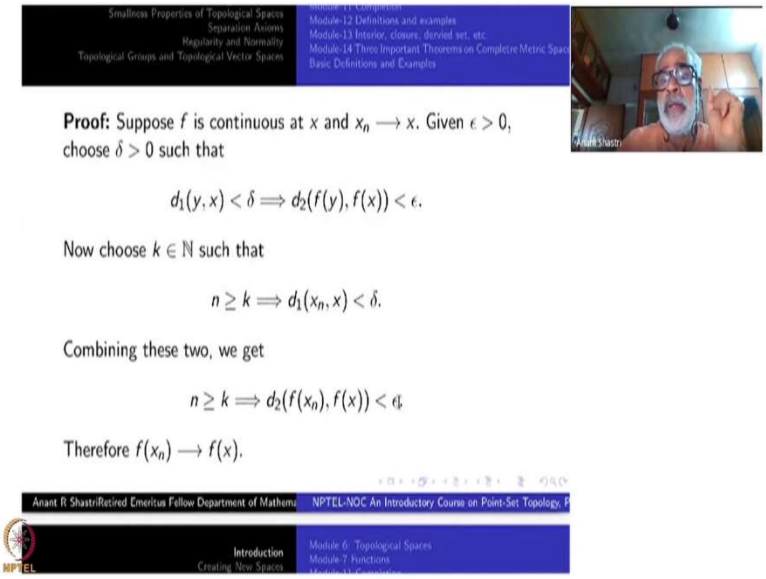

So, let us go through the proof here, so, that these things become solid in your mind. Suppose f is continuous at x and x_n converges to x. Then I should show that $f(x_n)$ converges to $f(x)$, right? So, to show that I take an $\epsilon > 0$, then I must produce a number k such that for $n \geq k$, then th distance between $f(x)$ and $f(x_n)$ is less than ϵ. This is what I should produce.

But what is the hypothesis? f is a continuous at x and x_n converges to x. So, given $\epsilon > 0$, using the continuity, I take a $\delta > 0$ such that the distance between y and x is less than δ implies $d_2(f(y), f(x))$ is less than ϵ. So, I have used the continuity here now I will use the convergence also, but this time I will replace ϵ by δ for this one. So, δ is also a positive number for that I must get a number k some natural number such that $n \geq k$ implies distance between x_n and x is less than δ.

So, this is now playing the role of ϵ this δ is playing the role of ϵ for the convergence here. So, both the conditions I have used now you just combine them. The moment $n \geq k$, this distance is less than δ, but now I can put this y equal to x_n then what happens this is less than δ. So, $d_2(f(x_n), f(x))$ will be less than ϵ that is this condition. So, one way we have proved namely continuity implies sequential continuity.

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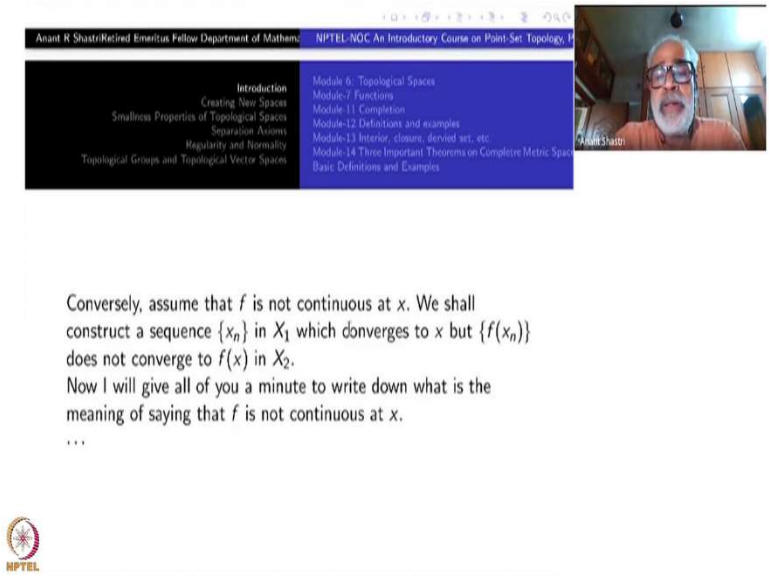

So, now we should prove the other way around. The other way around is not all that straightforward ok soon we can say these things are also straightforward --- straightforward itself is not all that straightforward.

So, here you have to use the contrapositive you want to prove the contrapositive of the statement. That is, assume that f is not continuous. Then produce one sequence which will violate the condition that is all. Produce one sequence x_n converging to x, but $f(x_n)$ does not converge to $f(x)$. So, what I am going to prove here? Assuming this is not true I am going to prove this is not true what is the meaning of this is not true this statement says that for every sequence something happens.

So, what is the negation of that? There is one sequence for which this does not happen. So, what is that? That is one sequence x_n is convergent to x, but $f(x_n)$ does not converge to $f(x)$. So, that is the final statement I have to prove. So, I have to produce such a thing now ok. So, this is where you have to learn what is the meaning of negation what is the meaning of contrapositive and so on. So, I will give you one minute all of you think about it and tell me what is the meaning of f is not continuous at x just write it down I can check it of course, I will give you a minute and then I will proceed, ok?

So, if you have done it correctly congratulations. In any case you will compare your answer with the one that I am going to write down here. I am going to prove that there is a sequence x_n which converges to x and $f(x_n)$ does not converge to $f(x)$ starting with the assumption that f is not continuous, ok.

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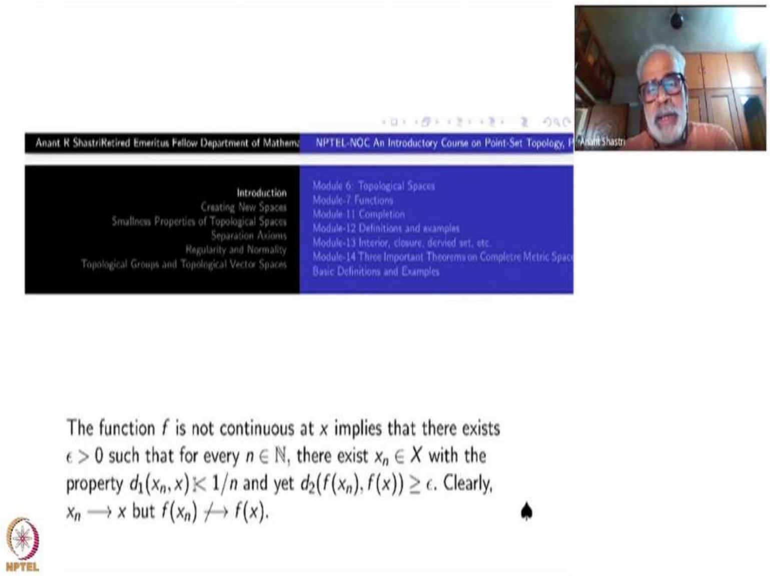

So, what happens f is not continuous implies there is one $\epsilon > 0$ for which blah blah blah those things are not true. So, what is not true? For every delta you must get a point such that something does happens.

So, that is for every δ positive. So, I will choose δ to be $1/n$. So, that is what I am going to do for every $n \in \mathbb{N}$, there will be some x_n, this depends upon n, I have chosen, such that the distance between x_n and x is less than $1/n$. So, this δ I am taking as now $1/n$, I can take every any δ because for every δ it is supposed to be true there is for every δ there is something. So, for $1/n$, there must be something and that time I am calling it as x_n. $d_1(x, x_n)$ is less than to $1/n$ but the distance between $f(x_n)$ and $f(x)$ is bigger than or equal to ϵ. If that is the less than ϵ, there is nothing. So, I have to say this the whole condition is violated namely f is not continuous at x_1 that is what I have to. So, $d_2(f(x_n), f(x))$ is bigger than equal to ϵ. If this happens for one single $\epsilon > 0$, then that is same as saying that f is not continuous at x. What I have got here, fixing one ϵ here, I do not know how big it is, it is bigger than 0 that is all.

This sequence x_n automatically converges to x. But $f(x_n)$ on the other hand, does not converge to $f(x)$ because it is keeping away from $f(x)$ a constant distance bigger than equal to ϵ. Over, ok..

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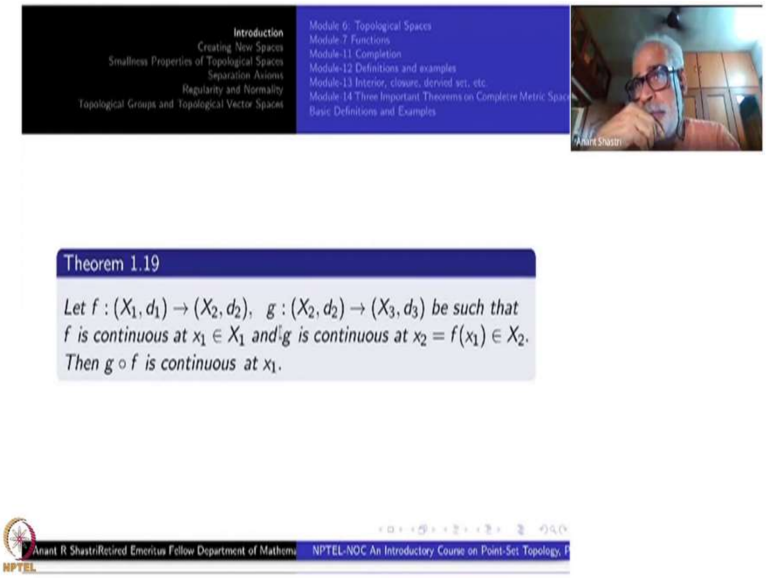

The next theorem is one of the profoundest thing that we have to learn, whereas, in reality it is just very very simple. You will see that this proof is very simple and if you use this one later on, lot of things becomes very simple ok?

So, this maybe the one of the things that you know why topology as a whole comes into picture after all, but right now there is no topology, it is the metric spaces here. f from X_1 to X_2 and g from X_2 to X_3. So, I can talk about $g \circ f$ right? suppose the function f is continuous at x_1 and g is continuous at $f(x_1)$, which I write as $x_2 \in X_2$. Then $g \circ f$ is continuous at x_1.

So, this is also a theorem in real analysis as well as in complex analysis. Proofs were identical there, the proof I am going to give here is also exactly same. The point is that it does not depend upon all that strong structures real and complex numbers have. It is just the metric that we are going to use and that is why it is simple, ok?

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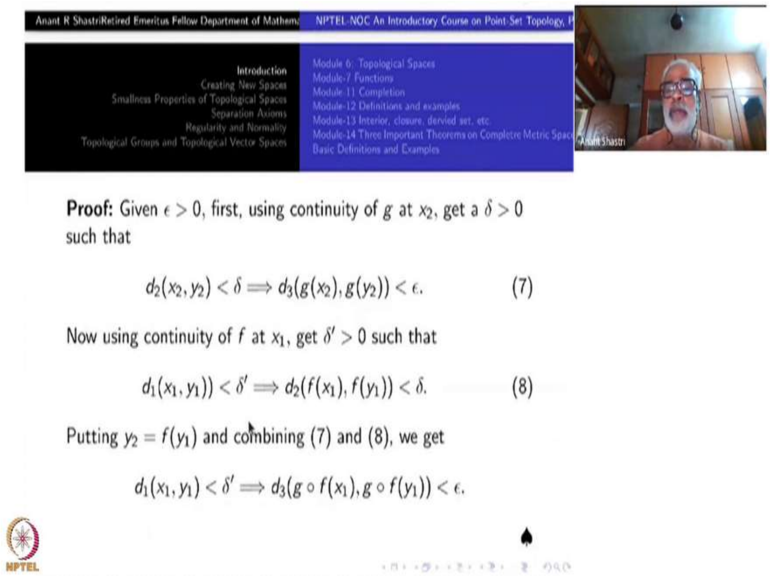

So, look at the proof. I want to show that $g \circ f$ is continuous at x_1. So, I start with an ϵ positive. Now first I use the continuity of the $g(x_2)$ that will give me a $\delta > 0$ such that d_2 distance between x_2 and y_2 is less than δ implies $g(x_2)$ and $g(y_2)$ the d_3 distance is less than ϵ ok. So, this is the continuity of g at the point x_2, x_2 is what? It is $f(x_1)$. Now use the property that f is continuous at x_1, this time using this δ again in the place of ϵ.

So, then I get a $\delta' > 0$ such that d_1 distance between x_1 and y_1 is less than δ' implies d_2 distance now between $f(x_1)$ and $f(y_1)$ is less than δ. Now just like in the previous theorem you combine them. Start from here you get here, they will play the role these things, $f(x_1)$ will play the role of x_2 anyway it is actually x_2; put y_2 equal to $f(y_1)$ here and use this equation number three, this is the implication.

Now, all the way come here, but what is $g(x_2)$? It is $g \circ f(x_1)$ similarly what is this one it is $g \circ f$ of x because I have put y_2 equal to $f(y_1)$ over ok. So, this implies that $g \circ f$ is continuous at the point x_1 ok.

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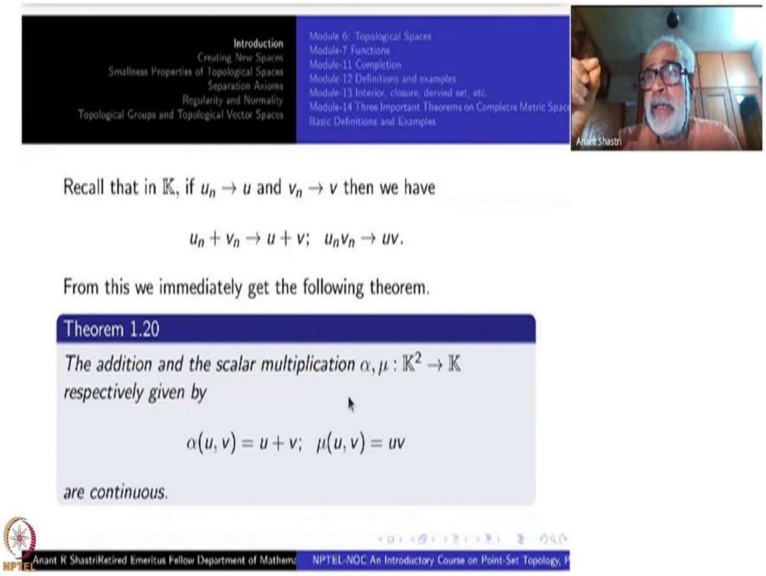

So, let us now do some kind of examples, but I will state them as theorems because they are very profound results ok? Once again we go back to real numbers or complex numbers take a sequence u_n converging to u and another one converging to v then if you take the sum sequence $u_n + v_n$, it is the sum of this sequence with that one right? $u_n + v_n$ will converge $u + v$.

And the products $u_n v_n$ will converge to uv. Hope you remember this result from real analysis or complex analysis? Now, here this is what I am not going to prove. In real analysis you have this one, now you can use this one to prove many things. Slowly ok? This you could have done in real analysis course also if you have not done it there now I am going to do that because those things are crucial for me ok?

The addition and scalar multiplication inside $\mathbb{K}$, function from $\mathbb{K} \times \mathbb{K}$ to $\mathbb{K}$, you take the addition or scalar multiplication. They are continuous this is the statement now. From these results about convergence of the sums and products, I am going conclude this one. How do I do that? Because continuity can be converted into sequential continuity, ok. $\epsilon - \delta$ continuity is equivalent to sequential continuity.

So what happens? I have to show that this α is continuous. Means what? Suppose you take a sequence here ok converging to (u, v). Then α of that should converge to $\alpha(u, v) = u + v$, is what I have to show right.

So, similar for example, let us let us do this one. Here u_n converging to u, v_n converging to v, that is the same as saying (u_n, v_n) converges to (u, v). Then the product sequence $u_n v_n$ converges to uv therefore, this is continuous when you apply a product that is f of that whatever here it is μ. So, here it is α ok. So, use this theorem for both of them namely this theorem. Then you get these results that scalar multiplication and addition are continuous, ok.

So, I will explain this one once more. Look at how to how to show that $\alpha(u, v)$ is $u + v$. Why it is continuous? I have to take continuity at a point $(u_0.v_0)$. Now how do I do? I have to take 2 sequences, u_n converging to u_0 and v_n converging to v_0, ok.

Then I have to apply α to these sequences what is this? It is $\alpha(u_n, v_n)$ where does it converge it converges to $u_0 + v_0$. So, I am using that sequential convergence criterion to prove this one. So, for every sequence converging to that I have shown that α of the sequence converges to α of the limits right therefore, α must be continuous. Similarly, for μ, μ is multiplication ok exactly same only thing is instead of plus you have take a product here, ok.

So, let us stop here next time we will do all these things we will use them and do a little more examples of continuous functions.

Thank you.