Introduction to Point Set Topology, (Part I)
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Lecture - 39
Compact Metric Spaces

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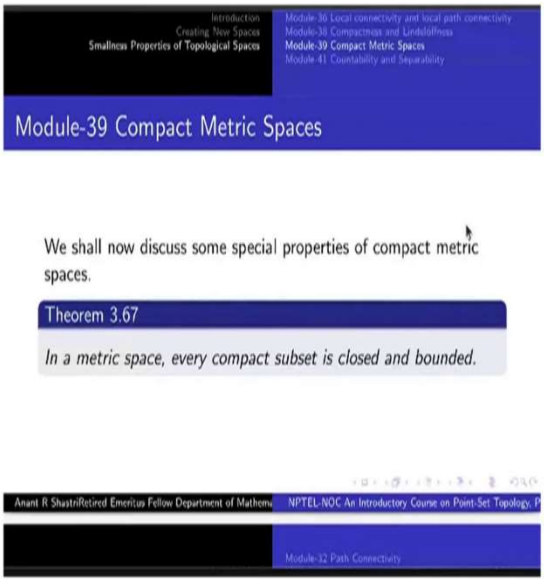

Last time we introduced compact spaces and Lindelof spaces. Today in module 39, let us discuss Compact Metric Spaces. In fact, last time we did not have any examples, why? Because, now we will have plenty of examples, naturally ok without without spending any more time. So, let us come to compact metric spaces. Earlier, whatever you have done with metric spaces, we never use the word compactness right? Now, we will bring it and get familiar with the metric spaces themselves.

In a metric space every compact subset is closed and bounded; you see I could not use closed and bounded words in an arbitrary topological space right? As soon as I have a metric space, I can use them and suddenly a compact subset is closed and bounded.

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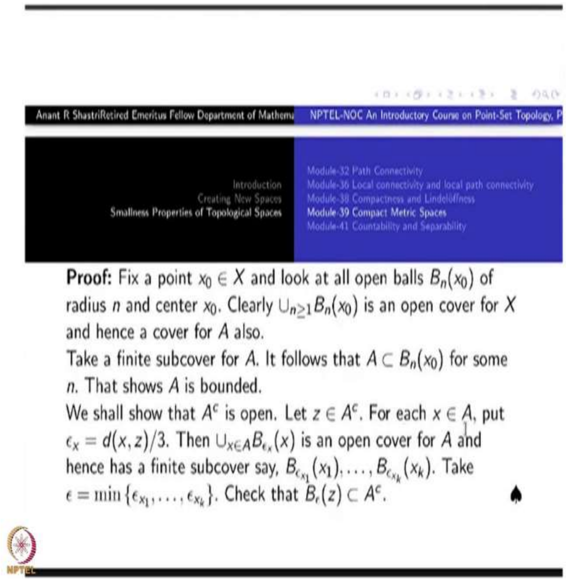

Let us go through this. Many of these things you must have seen at least for $\mathbb{R}^n$ or $\mathbb{R}$. But, now for any compact metric space. The proof will be similar, indeed almost same. So, I have to show boundedness as well as closedness right? Fix a point x_0 belonging to X and look at all open balls $B_n(x_0)$, of radius n and center at x_0.

Every point in X is at a finite distance from x_0, right? Because $d(x, x_0)$ is some finite number. Therefore, you can always choose n large enough so, that whatever x, you have taken that will be in the open ball centered at x_0 and of radius n.

This means that, X is contained in the union of all these balls. So, this is an open cover of X ok. Therefore, it is a cover for our subset A also which is compact. That means, what? There is a finite sub cover for A. So, $B_{n_1}(x_0), B_{n_2}(x_0), B_{n_3}(x_0), \ldots B_{n_k}(x_0)$.

For some n what does that mean now? If you take large enough n, bigger than all the $n_1, n_2, \ldots, n_k$ ok? That $B_n(x_0)$ will contain all those other balls smaller balls of smaller radius, because all of them are centered x_0. So, the subset A is contained inside a ball, that means it is bounded already; that is all boundedness is. That precisely shows A is bounded ok?

Now, we have to show that A^c is open; so, take a points z in the complement of A ok? For each x inside A, put ϵ_x, it is some number I am going to put what is it? It is equal to $d(x,z)/3$ ok.

For each x inside A, I am putting $d(x,z)/3$ equal to ϵ_x where x varies over A, and z is in the complement of A that is fixed. For each x, look at the ϵ_x that is a positive number, because z and x are different.

Now, you take ϵ_x ball around x ok; vary x, what you get? You get an open cover for A right. A is compact; so, you get a finite sub cover that will be $B_{\epsilon_{x_1}}(x_1), \ldots, B_{\epsilon_{x_k}}(x_k)$. Union of these finitely many balls will cover A. Now, you take the minimum of all these $\epsilon_{x_1}, \ldots, \epsilon_{x_k}$; there are finitely many positive numbers, take the minimum ok let that be ϵ. Now, look at $B_\epsilon(z)$. z was in the complement of A. Claim is that $B_\epsilon(z)$ is contained inside A^c, ok?

To work out this one, all that you have to do is, take a pencil and a paper and just plot your points here one is some A, some other point, and there is some finite cover and so on right. You have to do that for understanding this thing, I am not going to do take that kind of trouble here ok. I want to finally get the truth out of this one just by logic, no pictures ok; only that way you will learn, you know, your learning of points set topology will be strong.

So, all that you have to do is use triangle inequality to show that $B_\epsilon(z)$ is contained inside A complement. In other words, if you take some y such that distance between y and z less than ϵ, you should show that y cannot be inside A. That will follow if you show that it cannot be in one of these balls; then it cannot be inside A. So, at that level, I will leave it to you to verify that.

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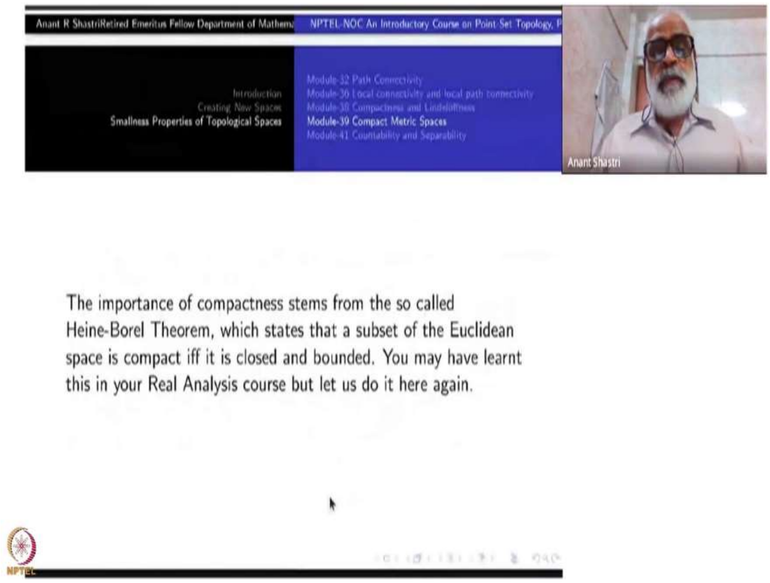

The importance of compactness stems from the so called Heine-Borel Theorem which states that a subset of the Euclidean space is compact if and only if it is closed and bounded. We proved that every compact space inside a metric space is closed and bounded; the converse holds for Euclidean spaces ok. So, that is the classical result which goes under the name Heine-Borel Theorem ok; so, you might have learned it in analysis course, but let us do it here just in case you have not learned that yet.

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We begin with $\mathbb{R}$. Inside $\mathbb{R}$, we want to say that the closed interval $[a, b]$ is compact, this also you must have learnt, but I will redo this one here.

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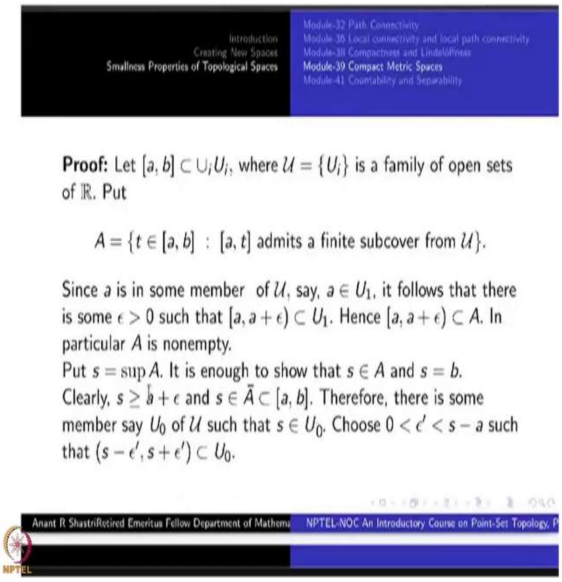

So, closed interval is compact is what I want to show. Take an open cover where all these U_i's are open subsets of $\mathbb{R}$. Nothing more is assumed ok; $[a, b]$ is a closed interval, these are open subsets of $\mathbb{R}$, union covers $[a, b]$, ok.

Now, I define a subset A of this interval all points t inside $[a, b]$, such that, the closed interval $[a, t]$ admits a finite sub cover from U. You see $[a, t]$ is also covered by the same family. So, make this hypothesis and then put all those t which satisfy this hypothesis $[a, t]$ must be admitting a finite sub cover. Put that t inside this A ok, each element in $[a, b]$ is in one of the U_i's in particular a is inside one of the U_i's right.

Say, a is inside U_1, it follows that there is some $\epsilon > 0$ such that $[a, a + \epsilon)$ is contained inside U_1 by the definition of open subsets in $\mathbb{R}$ ok. Actually, $(a - \epsilon, a + \epsilon)$ will be there, but I do not need $a - \epsilon$ part here; because, I am working in the closed interval $[a, b]$; so, this part is contained inside U_1. Once this is there ok, everything up to ϵ satisfies this property; therefore, this entire you know half open interval is contained inside A.

In particular A is non empty right? Right up to $[a, a + \epsilon/2]$, some positive part here, not just a will be already inside A.

Now, put s equal to supremum of A; since A is non empty, this is a finite number alright; the least upper bound has to be inside $[a, b]$ anyway. It is enough to show that this supremum belongs to A and it is equal to b. Understand the statement what I want to prove? $[a, b]$ admits a finite cover; if that happens what happens to A? This entire $[a, b]$ will be equal to A and conversely.

For that b itself is in A is enough; I want to show that this b is in A that is the same thing as saying that $[a, b]$ here admits a finite cover ok. So, what I am trying to say? s is inside A and s is equal to b; so, I am proving it in two stages. Finally, I want to prove s equal to b right; so, first I will show that s is inside A and then s equal to b; so, that will end the proof alright? So, let us prove this.

First of all, since $[a, a+\epsilon)$ is already in A, supremum of A will have to be bigger than $a+\epsilon$. So, s is bigger than $a+\epsilon$ and being the supremum of a subset of $[a, b]$, it will be in $\bar{A}$ ok? Also $\bar{A}$ is contained as $[a, b]$, because $[a, b]$ is a closed interval ok; it is a closed subset.

Therefore, there is some member say U_0 belonging to $\mathcal{U}$, i.e., in this family such that this s is inside one of the members that is all. I am calling it U_0, not U_1; you see U_1 was used already here. So, s belongs to U_0.

Choose $0 < \epsilon' < s - a$ such that $(s - \epsilon', s + \epsilon')$ is contained inside U_0. So, that is again the property just like this one here, some open interval around s must be inside U_0, because U_0 which is an open set and s is inside U_0 alright; nothing very great, I have done so far ok. Once supremum is inside $[a, b]$, $[a, b]$ is covered by this family, so, I am taking one of the members here to which s belongs to. Now, you make a nice observation to start with.

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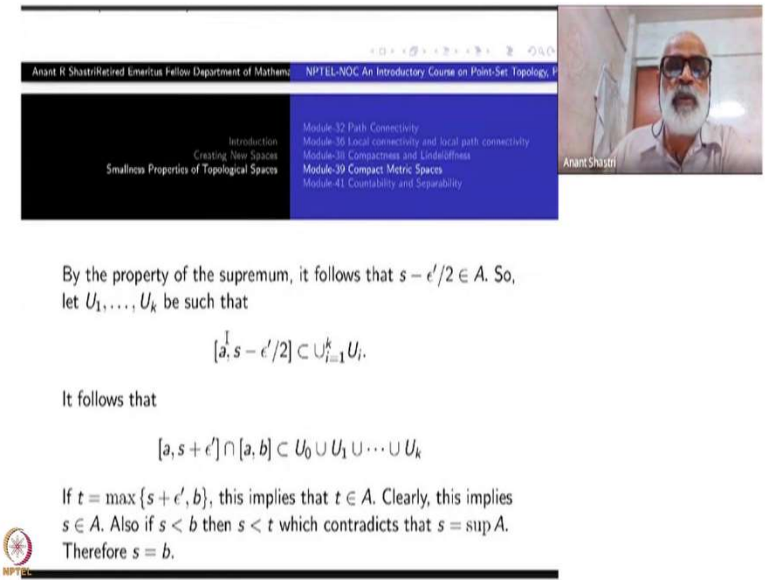

Namely, the property of the supremum; if you take anything smaller than supremum, it will be inside the set ok? There must be an element here ok. So, it follows that $s - \epsilon'/2$ must be inside A; if this is not inside A, then s cannot could not have been the supremum of A. So, this is definitely inside A ok; where ϵ' is some positive number, it has been chosen such that this open interval is inside U_0.

So, once it is inside A, what does it mean? There will be $U_1, U_2, \ldots, U_k$ ok; U_1 I have chosen for this one. I can include that member always, because it contains literally a here $U_1, U_2, \ldots, U_k$, I am calling this is the finite sub cover from this family $\{U_i\}$ for the set $[a, s - \epsilon'/2]$. That this belongs to A means, there is a finite cover like this ok. Now, all that you have to do is put the extra member U_0 also, remember U_0 covers this portion.

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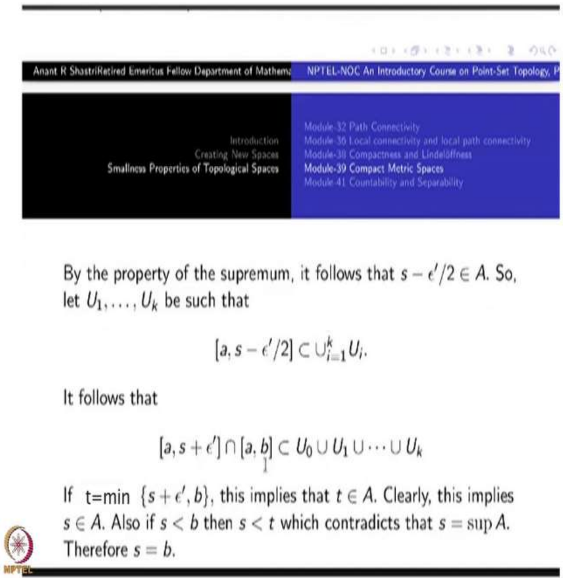

So, this portion overlaps with this one up till here and it goes up to $s + \epsilon'$. So, $[a, s + \epsilon'] \cap [a, b]$; now, I cannot say this is contained here unless I intersect with $[a, b]$ ok; so, that is contained inside $U_0 \cup U_1 \cup U_2 \cup U_k$, ok.

You can just put $\epsilon'/2$ also if you want, no problem ok. If t is the maximum of $s + \epsilon'$ and b, this implies that t itself is inside A, up to ϵ' it is there ok? If you see these two are both intervals starting from a; so, intersection I am taking right.

So, minimum of the two will be the intersection ok, look at the maximum; t is maximum of this one, I have taken $s + \epsilon'$ and $[a, b]$ maybe I should take minimum, then t will be inside a definitely ok. So, this will imply that s is inside $[a, b]$, because up to $s + \epsilon'$ is there ok, or it may be up all the way up to b; if it is $[a, b]$ and s is smaller than or equal to b, in either case s will be inside A.

So, if s is less than b, then s will be always less than t, this t is larger than s, that is the whole point. That contradicts s is supremum of A. No number bigger than s will be inside A, because s is the supremum.

Therefore, s must be equal to b. Now, that completes the proof.

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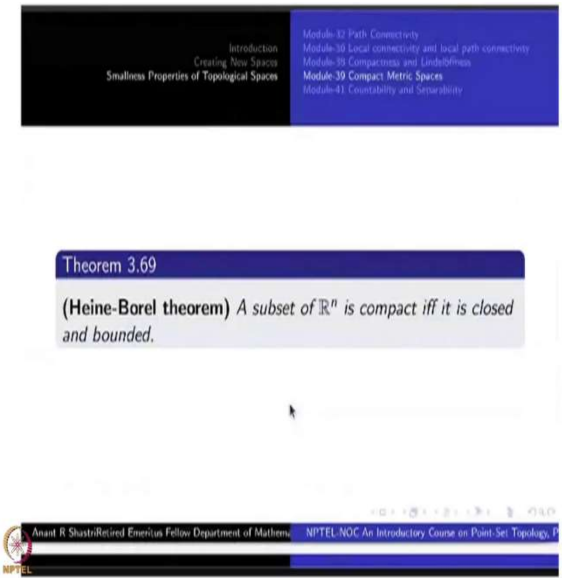

So, the next thing is Heine-Borel theorem that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. So, this is what we wanted to prove now ok.

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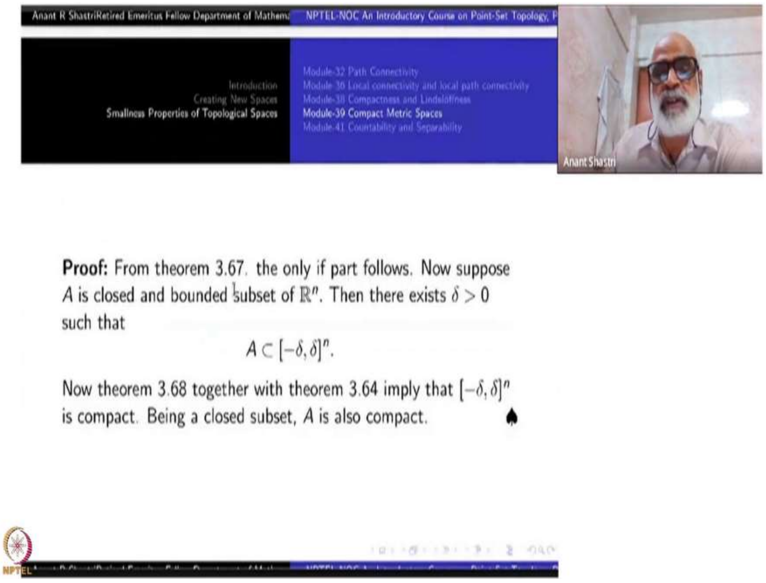

So, we are going to use the earlier theorem here; namely, closed intervals are compact. Then we have also proved yesterday that product of compact sets is compact, finite product. Therefore, you can take product of finitely many closed intervals, take the closed boxes inside $\mathbb{R}^n$, they are all compact; so, that is the thing that I am going to use now. See now suddenly you have a lot of compact subsets in $\mathbb{R}^n$, right.

Once you have closed and bounded subsets of $\mathbb{R}^n$ all of them are compact; so many examples you have now. A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded ok. From 3.67 the only if part follows, once it is compact it is closed and bounded over.

Now, suppose you just have a closed and bounded subset ok that is the way Weierstrass started with hypothesis, because he was working only inside $\mathbb{R}^n$ anyway. So, he called by the way he called these sets `limited sets'. So, there are so many different words by different authors. And so many, you know, dozens of people around the same time were working to develop topology.

So, then there exists $\delta > 0$ such that A is contained inside $[-\delta, \delta]^n$, a large square or a cube whatever any cube I am taking inside $\mathbb{R}^n$ ok. So, this is another way of looking at what is the meaning of bounded set.

You can take a ball also centered at the origin, but balls are always contained inside the squares and squares are contained inside the balls larger and larger or smaller and smaller that is what we have seen in a picture right. So, A is contained inside some $[-\delta, \delta]^n$, right? Beacuse it is bounded. Now, A is closed, this is compact; so, A is compact. The proof is over.

Life was not so easy for people who started these concepts, but now for us these things look so easy.

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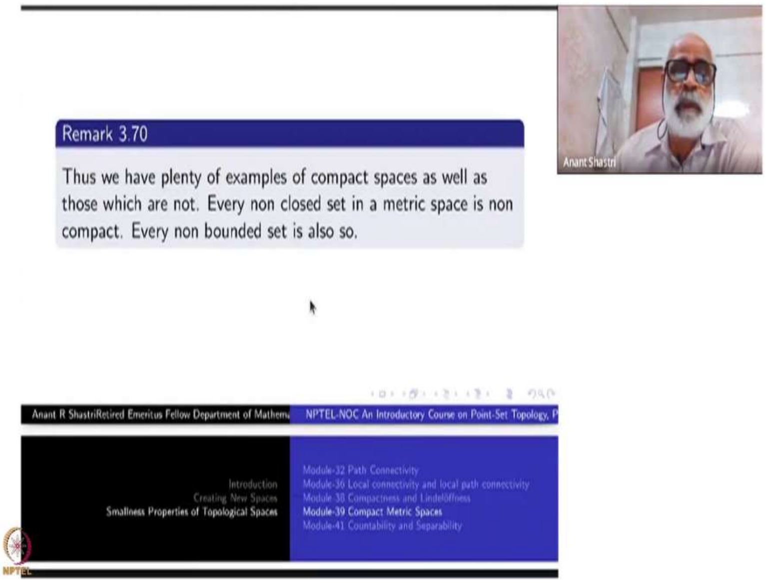

Thus, we have plenty of examples of compact spaces as well as those which are not; all that you have to take is a non-closed set, all that you have to take is a non-bounded set. Plenty of non-compact spaces and plenty of compact spaces; so, the shape may vary whatever norm you may like to take in $\mathbb{R}^n$. So, outside $\mathbb{R}^n$ of course, for other metric spaces and so on you have to to be careful. All metric spaces one way will do, any non-closed subset of a metric space cannot be compact, any non-bounded thing cannot be compact ok.

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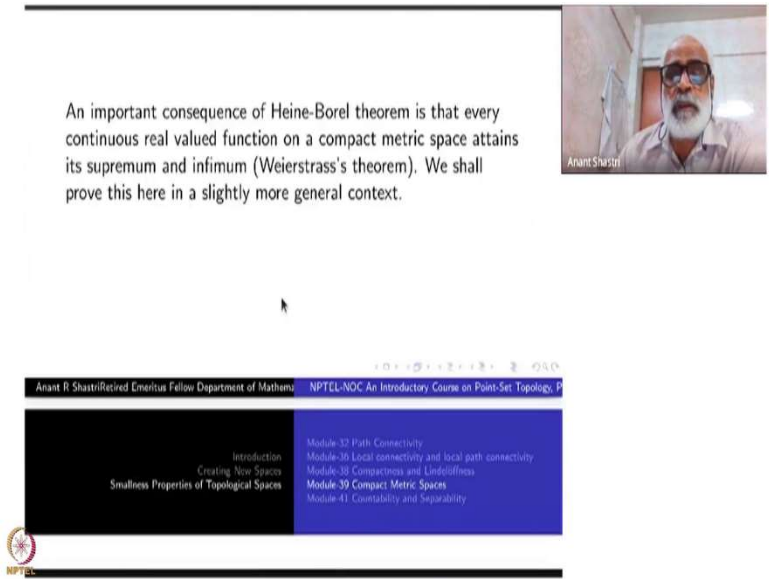

An important consequence of Heine-Borel theorem is that, every continuous real valued function on a compact metric space attains its supremum and infimum; so, this is known as Weierstrass theorem. So, this could have been actually the motivation, you know, for Borel to come up with this thing. Heine was independently working on his own. And he had the correct ideas and Borel expanded on them and came up with all this.

So, we shall prove here slightly more. You know slightly general result instead of just what is stated?

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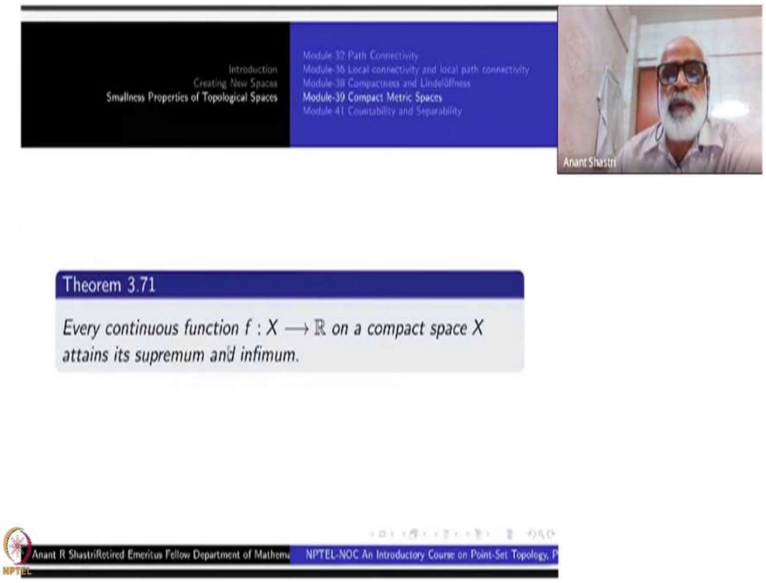

Every continuous function f from X to $\mathbb{R}$ on a compact space X attains its supremum and infimum. See Weierstrass theorem was inside $\mathbb{R}^n$ and closed and bounded, we do not need that. Now, we use the word compact and then we can go more general, any space, not necessarily metric. You see I am mixing now metric spaces and general spaces, I have told you that, I want to study both of them simultaneously.

You do not assume that X is a metric space. It is compact. Of course, $\mathbb{R}$ is a metric space in its usual topology. Any continuous function from X to $\mathbb{R}$, where X is compact attains its supremum and infimum. To make sense supremum or infimum you have to come to $\mathbb{R}$ or into some order topological space ok? And of course, least upper bound, greatest lower bound such thing should be there ok?

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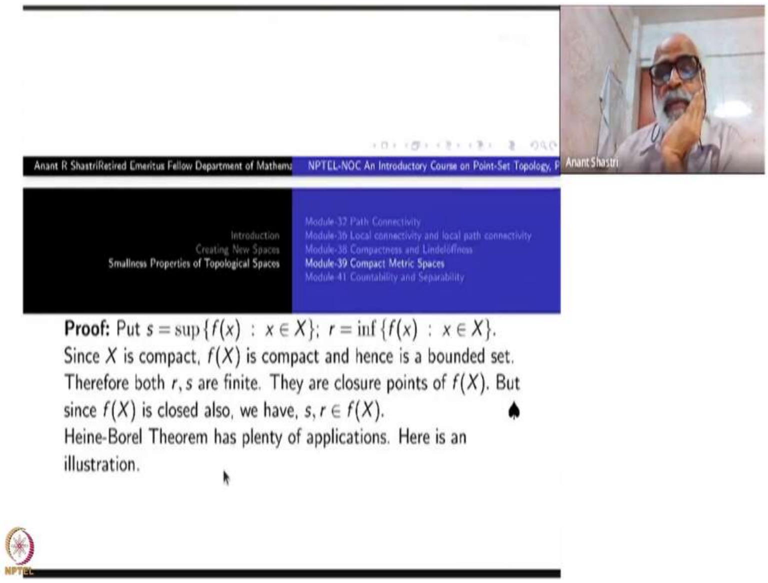

So, supremum is attained is the same thing as saying that it is maximum. The word maximum is used only after the supremum is attained. Attained means what? It is actually a value ok; supremum is thus the upper bound, least upper bound of all the values; it need not be a value. So, here it is attained means, it becomes maximum, the image of one of the maxima points. Similarly, infimum when it is attained it becomes minimum ok.

So, put s equal to sup of all $f(x), x$ belonging to $X; r$ equal to inf of all $f(x), x$ belonging to X, the supremum and infimum are defined for any set of points inside $\mathbb{R}$, including $-\infty$ this may be ∞, this may be ∞, this may be $-\infty$, that is also allowed here ok. Since X is compact, $f(X)$ is compact. By Heine-Borel theorem it is bounded right. Therefore, s and r are finite numbers; they are the closure points of $f(X)$, every supremum is a closure point of the corresponding set.

So, this is a property of real numbers. what is the meaning of supremum ok? But $f(X)$ is also closed, because it is a compact. Therefore, both s and r are in $f(X)$; that precisely means that they are values; s is equal to $f(x_0)$, it becomes a maximum; r equal to $f(y_0)$. So, it becomes minimum alright?

Heine-Borel theorem has plenty of applications ok; so, here is one illustration I cannot go on doing everything.

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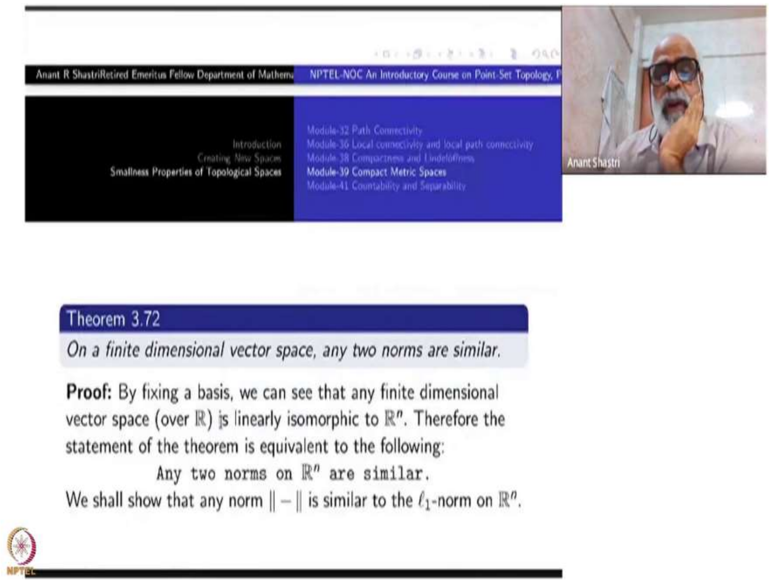

On a finite dimensional vector space any two norms are similar. This was one of the theorems that I promised you that I will prove. So, now, we can prove it.

Remember on $\mathbb{R}^n$, we had lots and lots of norms, ℓ_1 norm, ℓ_2 norm, ℓ_p norm in general where $p \geq 1$. And then we have the ℓ_∞ norm also. So, we had seen that they are all similar, but there may be many other norms. There are in fact, lots and lots of norms. This theorem says that on a finite dimensional vector space any two norms are similar ok? So, they will have all geometric properties-- similarity.

By fixing a basis, we can see that any finite dimensional vector space over $\mathbb{R}$ is linearly isomorphic to $\mathbb{R}^n$. This is just linear algebra. Therefore, the statement of the theorem is equivalent to the following. Now, instead of arbitrary vector space I can just assume we are working in $\mathbb{R}^n$ and proceed to prove that any two norms on $\mathbb{R}^n$ are similar ok.

So, do not worry about arbitrary vector spaces and so on. In $\mathbb{R}^n$, you can use coordinates etc, everything you can use now. We shall show that any norm, I am just denoting it without any suffix, is similar to the ℓ_1 norm ok? If everything is similar to ℓ_1 norm any two of them will be also similar to each other, because similarity is an equivalence relation alright? So, let us prove that this arbitrary norm on $\mathbb{R}^n$ is similar to the ℓ_1 norm.

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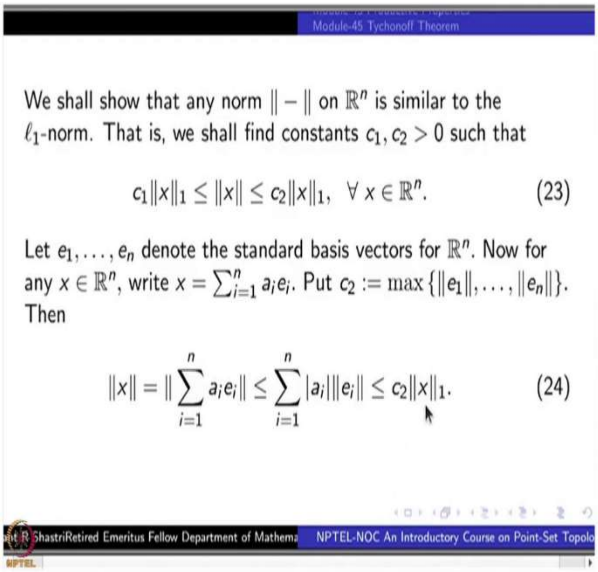

We shall show that any norm on $\mathbb{R}^n$ is similar to the ℓ_1 norm. That is we shall find constants c_1 and c_2 positive such that $c_1\|x\|_1 \leq \|x\| \leq c_2\|x\|_1$, for every x in $\mathbb{R}^n$, ok?

So, start with a basis $e_1, e_2, \ldots, e_n$, vector space basis for $\mathbb{R}^n$. Now, for any x in $\mathbb{R}^n$, you can write x as a linear combination of these standard basic vectors. So, let $x = \sum_{i=1}^{n} a_i e_i$; put c_2 equal to the maximum of the norms of $e_1, e_2, \ldots, e_n$ with respect to the new norm that we are going to estimate ok.

So, there are these n elements, you take the maximum of them none of them is 0. So, maximum is some positive number that is going to be our c_2 Now, norm of x, I have written x as $\sum_{i=1}^{n} a_i e_i$. Norm of that is less than or equal to $\sum_{i=1}^{n} |a_i|\|e_i\|$.

Each of these $\|e_i\|$, I will replace by the maximum number c_2 here; the $c_2 \sum_{i=1}^{n} |a_i|$ that is nothing but the ℓ_1 norm of x ok? So, new norm is always less than or equal to this $c_2\|x\|_1$; so, one side inequality is established already ok.

On the other hand, as we have already observed that the unit sphere with respect to ℓ_1 norm which we have denoted by S_1 that is compact. And this inequality already implies that this norm is continuous with respect to the ℓ_1 norm; therefore, we can apply Weierstrass theorem ok?

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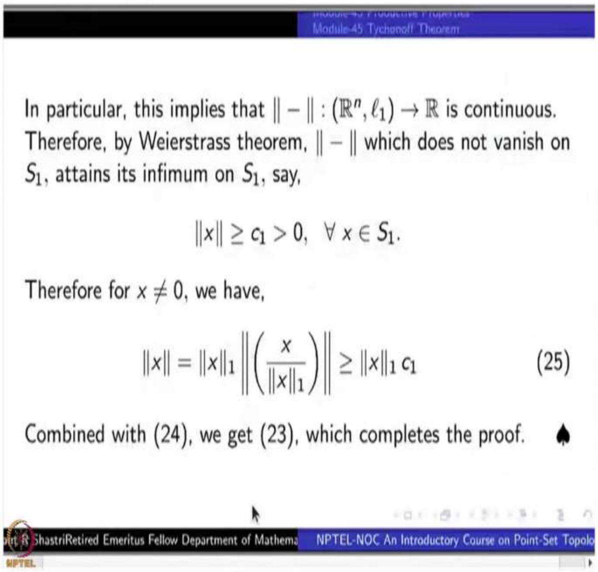

So, I repeat because of this inequality (24) what we get is that norm from $(\mathbb{R}^n, \ell_1)$ to $(\mathbb{R}, \ell_1)$ is continuous. Therefore, Weierstrass theorem whatever we have proved ok, this norm attains its infimum also on S_1; in any case the norm will never be 0 on a non-zero set of vectors that is S_1. So, this infimum will have to be strictly positive. In other words, what we have is $\|x\| \geq c_1$ positive for every x in this unit sphere S_1 with respect to the ℓ_1 norm.

Now take any $0 \neq x \in \mathbb{R}^n$, you can write norm x equal to, (you know, you can divide and multiply by $\|x\|_1$ ok, because $\|x\|_1$ is not 0), is equal to $\|x\|_1 x/\|x\|_1$. So, this is a constant and this is inside S_1. What I am telling is the norm of the given element x is equal to $\left\| \frac{x}{\|x\|_1} \right\| \|x\|_1$.

Now, the inside thing is in S_1; therefore, I can apply this inequality. So; that means, that this is bigger than or equal to than c_1; so, this is $c_1\|x\|_1$; so, combining (24) and (25) we get whatever you wanted namely, (23).

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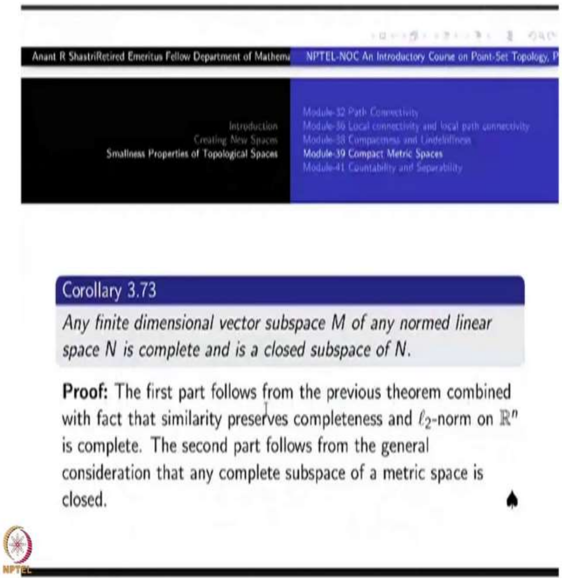

So, what is the corollary now? There is a corollary here. Any finite dimensional vector subspace M of any normed linear space N is actually complete and a closed subspace of N. Start with N which is a normed linear space, take a finite dimension subspace M. That will be automatically complete and a closed subspace ok?

See vector spaces are never compact, but now they become complete and closed, how? The first part follows from the previous theorem combined with the fact that similarity preserves completeness and ℓ_2 norm on $\mathbb{R}^n$ is complete. You start with any norm on N, you restrict it to M, but M is finite dimensional.

Therefore, the norm coming from this N is equivalent to say let us say ℓ_2 norm, but ℓ_2 norm on a finite dimensional vector space what is finite dimensional? Some $\mathbb{R}^n$; so, it is complete right?

So, its completeness follows, because similarity preserves completeness; that we have seen. The second part follows from a general principle; namely, if some subspace is complete then it must be closed in any metric space ok? Completenes means what? You take the closure point there is a sequence converging to that, but every convergent sequence is a quasi-sequence. But by completeness, this sequence is convergent in the subspace itself.

You cannot have two different limits of a sequence inside a metric space. You started with a closure point of the subspace, may be in the larger space right. But there is a sequence inside the smaller space that is converging to that closure point, that sequence will be Cauchy sequence. So, the subspace is complete means what now? The Cauchy sequence must converge inside the subspace; so, that closure must be inside the subspce itself. So, this just means that closure is equal to M itself.

So, this part is a general property, it has nothing to do with M being finite dimensional. Finite dimension vector spaces are similar to $\mathbb{R}^n$; therefore, they are complete. For that part, you needed this theorem ok? Heine-Borel theorem or whatever alright.

So, we shall continue a little bit of study of compact metric spaces and then come back again to just compact a topological spaces ok next time.

Thank you.