**Introduction to Point Set Topology, (Part I)**
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**Lecture - 38**
**Compactness and Lindelofness**

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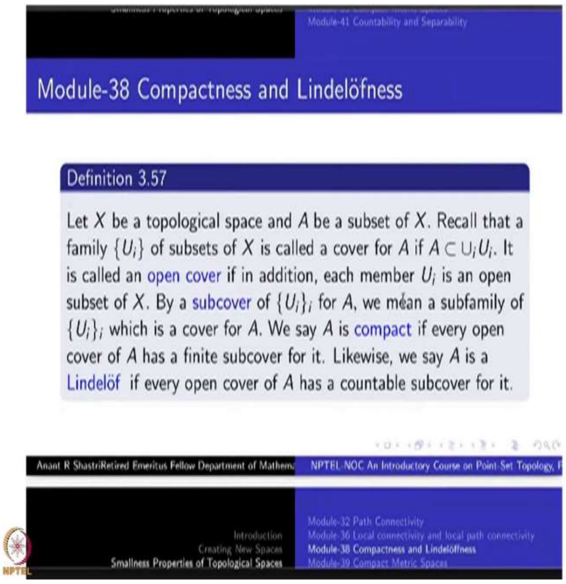


Welcome to module 38 of Point - Set - Topology Part I. So far we have studied connectivity, path connectivity, local connectivity, local path connectivity and so on. So, it is time again to take up another very important concept, namely, compactness. Tagging along with it, the Lindelofness can also be studied, which is not all that important for us. Quite often just like you study connectivity and path connectivity together, compactness and Lindelofness go hand in hand.

However, from the point of view of set topology, compactness is so important, if you try to tell the story of compactness by which I mean how the concept developed and so on, it will be half of the story of point set topology. So, so important is the compactness concept. I will not be able to much of historical remarks presently, because quite often they can be done only in an appropriate perspective, only after knowing what we wanted to do and what has been done and so on. When questions like why this one, why that one? And so on, crop up, you can go back and dig up history, historical development motivations and so on. So, let us start studying compactness in its earnest.

Start with the topological space $X$, a subset of $X$, say, $A$ will be called compact if the following happens. Suppose you cover $A$ by open subsets of $X$, some arbitrary open cover, then it must admit a finite subcover. So, let me define all these things carefully: subcover, open cover, what is the meaning of finite subcover and so on ok.

So, a family $\{U_i\}$ of subsets of $X$ is called a cover for $A$, (this I think we have defined earlier, but let us recall it anyway), if $A$ is contained inside the union of this family. It is called an open cover if every member of this family, all these $U_i$ s are open. So, one can have closed coverings also. They seem to be not so important ok?

For a given set $A$, by a subcover of of a family, we mean, first of all it must be a cover of $A$, then you take a sub family of $\{U_i\}$ which will be also a cover for the same set $A$, such a thing will be called sub cover. If this sub cover happens to be a finite or countable family, then we will say that is a finite sub cover or a countable sub cover accordingly.

So, once we have these definitions, a subset $A$ is called compact if every open cover of $A$ has a finite sub cover for it. Likewise, we say $A$ is Lindelof if every open cover of $A$ has a countable sub cover for it ok.

Deliberately I have not used the words `compact space' or `compact set' and so on. Whatever you want to call it,  if you want to think of this as a subspace you are welcome, if you want to think of this as set we are welcome, compact set or compact space both wording are allowed. What I have defined here is compactness. Similarly, Lindelofness here ok. However, you may wonder why the two wordings `compact subset' or `compact space' mean the same. Suppose this $A$ is the whole of $X$. Then all this definition makes sense right? Then we may say that $X$ is a compact space by itself. No subspace, no subset; a compact space.

So, in its own topology, take an open cover for X, any open cover you can extract a finite sub cover. You may not be able to extract it by youself, the definition gives you a finite subcover out of it. That is the whole point. So, so this is what I have repeated here namely you can call a topological space compact or Lindelof space if it is compact Lindelof as a subset of itself ok?

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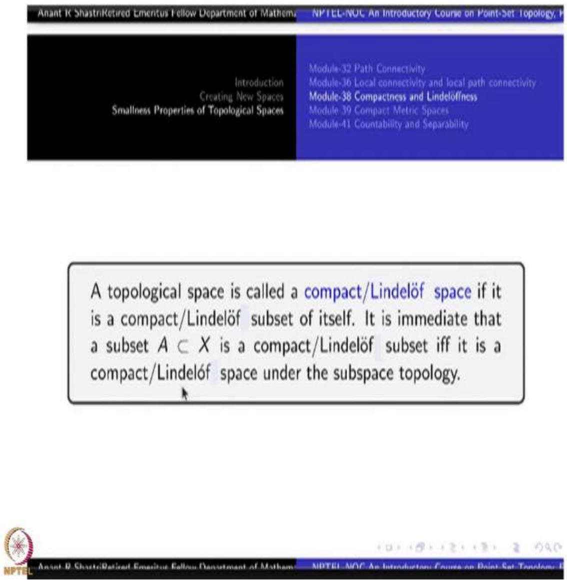


It is immediate that a subset $A$ contained in $X$ is compact or Lindelof if and only if it is compact space (or Lindelof) space in its subspace topology. You take the subspace topology. Then as a space it must be compact(or Lindelof). These two are equivalent. Because you have an open cover, the open subsets coming from the larger space, intersect them with the given set $A$, they will cover and those are the open subsets in the subspace $A$. So, these two notions are same there is no difference between them. In particular you will see that once a space is compact it does not matter where it is contained inside as a subspace, it will be always compact.

Suppose $X$ is a compact space, suppose $X$ is contained inside $Y$ as a subspace or $Z$ as a subspace. In both the cases, it will be still compact subspace ok?

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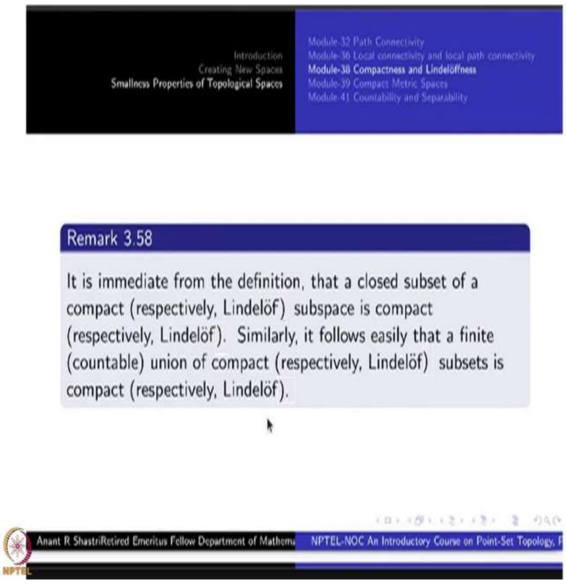


Now, if you take a closed subspace of a compact space automatically it will be compact. Similarly, if you take a closed subspace of a Lindelof space automatically it will be Lindelof.

So, how does one get it? Just add one extra open set namely the complement of the closed set. That will become an open cover for the whole space. From that you can get a finite (or countable) sub cover. Now throw away the extra set, complement of $A$, you do not need it to cover $A$. So, you have still, you know finite (or countable) sub cover. So, that is all the trick here. By extending the given cover for the subspace, which is a closed subspace by putting one extra element namely the complement of that set. And then you can come back.

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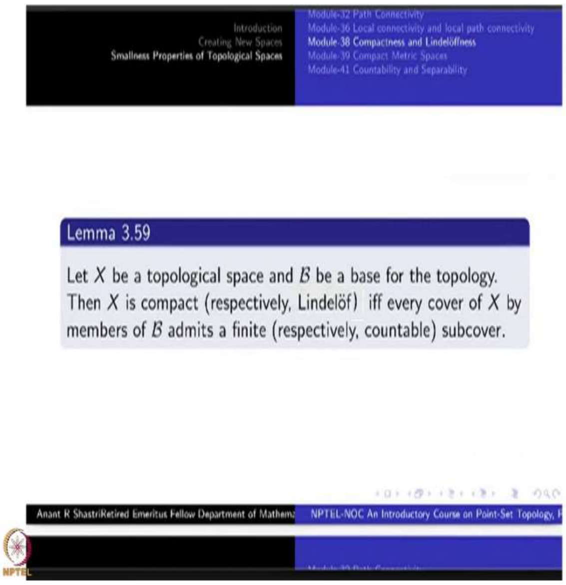


Let $X$ be a topological space and $\mathcal{B}$ be a base for $X$. Then $X$ is compact (respectively Lindelof), [see, quite often,  whatever I do for compactness I can put Lindelof etc., inside a baracket but not always, when it is not, I will mention that], if and only if every cover for $X$ from members of $\mathcal{B}$ admits a finite (countable) subcover. You see in the definition of compactness you want to have every open cover should admit a finite sub cover, but here it is a restricted thing. Here only members of $\mathcal{B}$ are used, they do not give all the open covers.

But if this condition is satisfied for members of $\mathcal{B}$ which is a sub class of open covers that is enough is the claim. Suppose you take a covering of $X$ with only members of $\mathcal{B}$ and they admit a finite sub cover that is good enough is what this lemma says ok?

So, this is the role of the base, to cut down our toil. You do not have to check it for all open covers you have to just check it for covers whose members are coming from $\mathcal{B}$. This is the gist of this lemma here ok.

The proof is very easy, one way is clear, namely if you take a cover out of $\mathcal{B}$ that will be also an open cover in the general case. So, it must admit a finite sub cover ok. Now, suppose this happens for only open covers with members coming from $\mathcal{B}$. Then why it should be true for any general covering that is what you have to prove right? ok.

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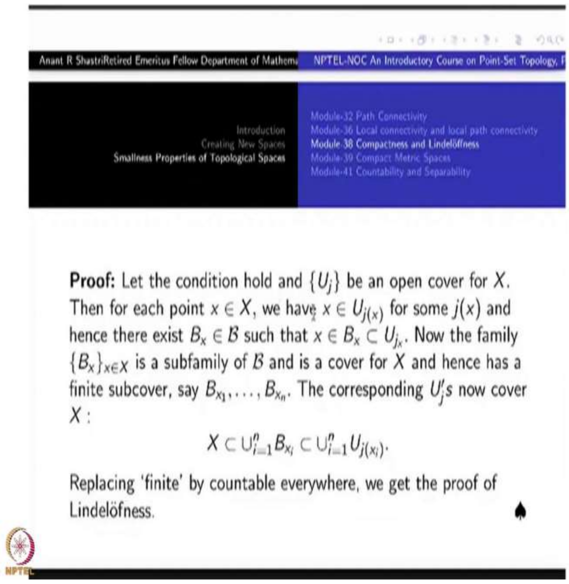


So, let the condition hold and $\{U_j\}$ be an open cover for $X$ not necessarily from members of $\mathcal{B}$. Then for each $x \in X, x$ must be inside one of the open sets here $U_{j(x)}$ ok, depends upon $x$ ok. But then $\mathcal{B}$ is a base means what? there must be an element $B_x$ inside $\mathcal{B}$ such that $x$ is inside $B_x$ contained in $U_{j(x)}$. Now if you vary $x$ over all of $X$, then $B_x$'s is will cover $X$. Now by this condition there will be a finite cover out of these $B_x$'s. Let us call them $B_{x_1}, B_{x_2}, \dots, B_{x_n}$. But each $B_{x_i}$ is contained in the corresponding $U_{j(x_i)}$. Therefore, $X$ is contained in the union of these $U_{j(x_i)}$. So, these were the members from the original cover ok?

If you replace $n$ by infinity here you will get the proof of the corresponding statement for Lindelofness. So, proof of for the Lindelof properties also same thing here ok.

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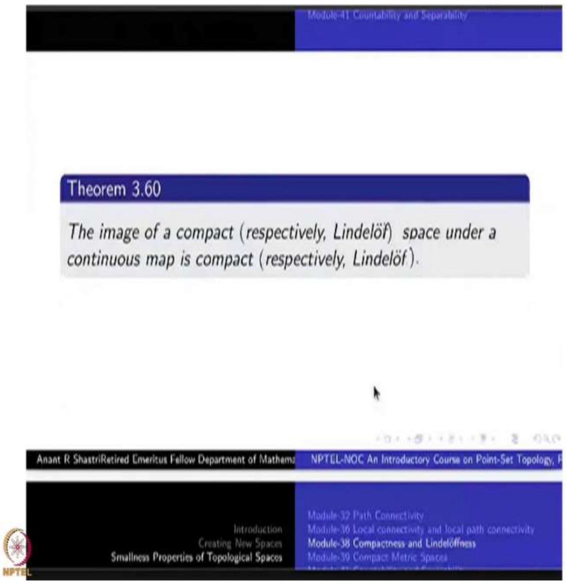


The next immediate thing is the following: Under a continuous map compactness and Lindelofness are preserved. What is the meaning of this? $f$ from $X$ to $Y$ is a continuous map, $X$ is compact implies $f(X)$ is compact. $X$ is Lindelof implies $f(X)$ is Lindelof. Just like connectivity, path connectivity etcetera right? Not local connectivity, nor local path connectivity you have seen.

What does that mean? Immediately it means that under homeomorphisms compactness and Lindelofness are preserved. In other words they are topological properties ok according to the formal definition of topologiacal properties.

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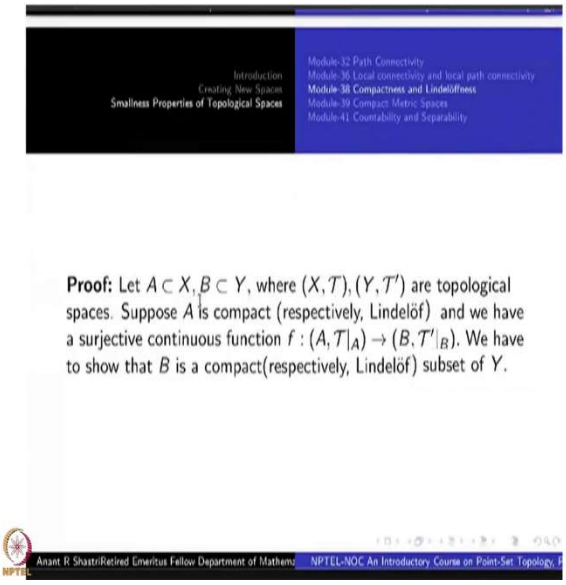


Take $A$ contained inside $X, B$ contained inside $Y$, where $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ are topological spaces. Suppose $A$ is compact (respectively Lindelof). Suppose we have a surjective function $f$ from $A$ to $B$ with $\mathcal{T}$ restricted to $A$ and $\mathcal{T}'$ restricted to $B$, are the subspace topologies and this is a continuous function here. You do not need $f$ to be defined on the whole of $X$ to $Y$. We have to show that $B$ is compact if $A$ is compact ok? (respectively Lindelofness) Alright.

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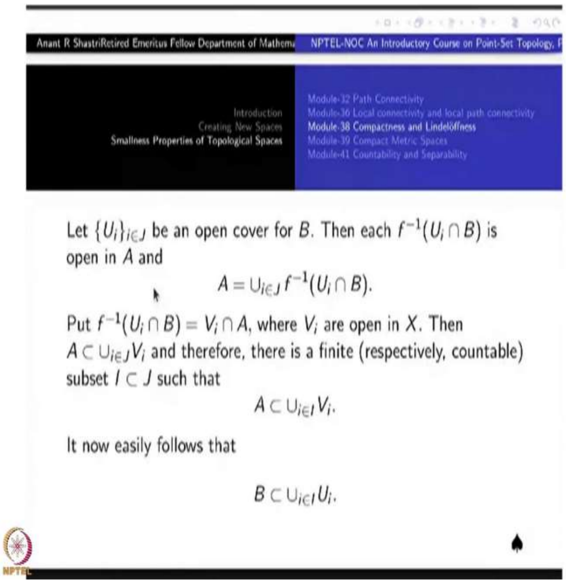

So, how do you do that? Very easy. Start with an open cover $\{U_i\}$ for $B$, take $\{f^{-1}(U_i) \cap B\}$ , see $U_i$ s are now open subsets in $\mathcal{T}'$ ok and all that you have is $B$ is contained in the union of $U_i$'s, but if you intersect $U_i$ with $B$ this is an open subset in $B$ in the subspace topology, $f$ is continuous, $f$ inverse of that will be open inside $A$ ok. And since this is a covering $f$ inverse of all these, their union is the whole of $A$.

So, I have got an open cover for $A$ ok, you can go back to topology $X$ here ok, just for fun. What you have to do, what is the meaning of these are open subsets? Each open subset $f^{-1}(U_i) \cap B$, is equal to some $V_i \cap A$ where $V_i$ is open in $X$. In any case these $V_i$'s will now cover $A$, obviously being larger than the original subsets here ok. $A$ itself is contained here. So, $A$ will be contained here also.

Therefore, there is a finite (respectively countable) subfamily, $I$ contained inside $J$, such that $A$ is contained in the union of $V_i$'s, $i$ over $I$. This is by compactness (or Lindelofness) of $A$. But now you can come back because these things are larger than $f^{-1}(U_i) \cap B$. So, $B$ will be contained inside the corresponding $U_i$'s, which is either finite cover (or a countable cover) ok.

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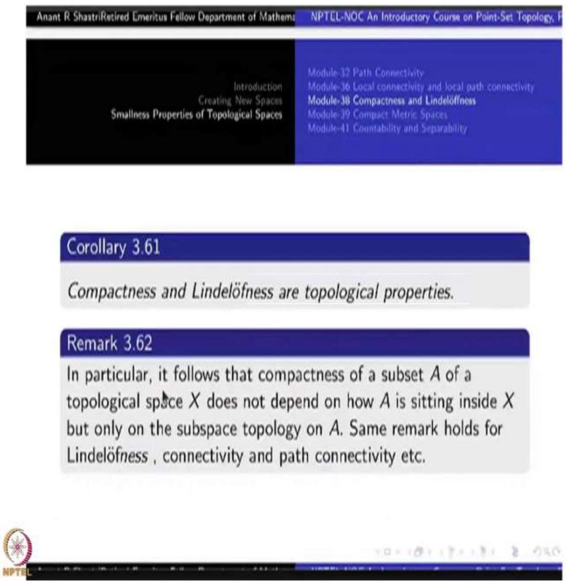


So, that is the consequence of this elementary result namely under continuous functions compactness is preserved, and therefore, it is a topological property. In particular, it follows that compactness of a subset $A$ of topological space does not depend on how $A$ is sitting inside $X$, I am repeating it again. Once $A$ is homeomorphic to another $A'$, $A$ and $A'$ are sitting wherever they like, as subspaces, if one is compact the other one is also compact. Over. This is the same thing with Lindelofness, connectivity, path connectivity etc that we have studied. They are all topological properties that is the whole idea.

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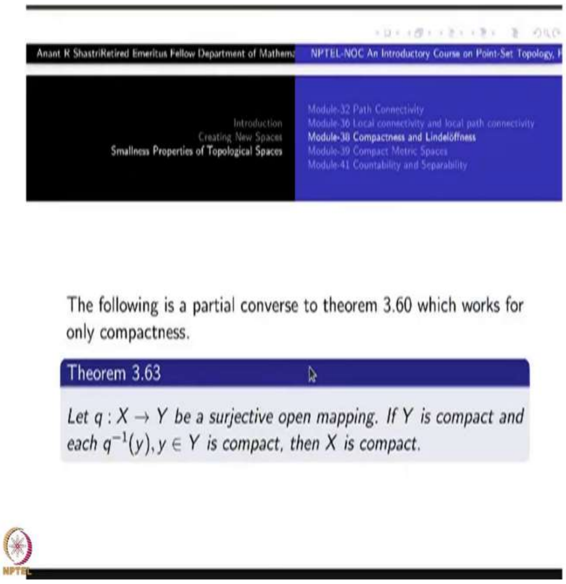

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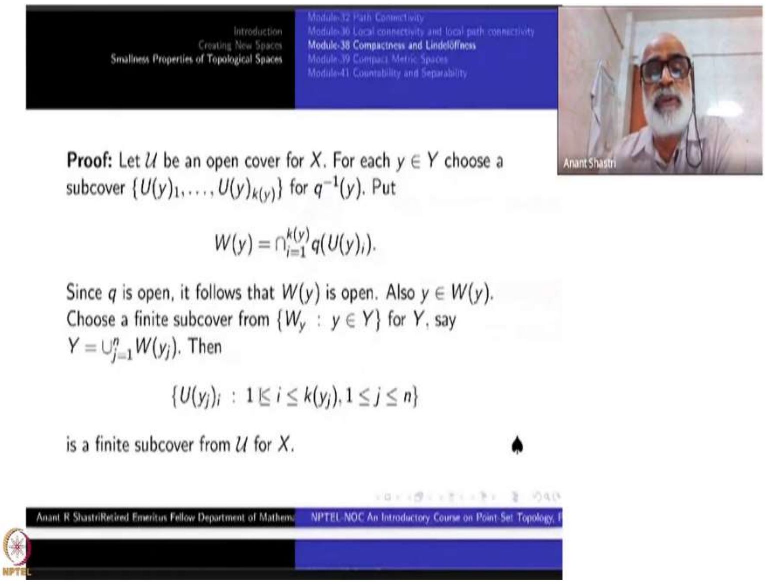


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Now another important ah landmark result finite product of compact spaces is compact. Converse is easy why because if product is compact you can take the projection maps they are open surjective therefore, each factor $X_i$ is compact if the product if compact. The other statement will all come just by this theorem itself inductively once I prove it for 2 then you can use inductively right.

For the proof of this theorem, see the modified lecture notes.

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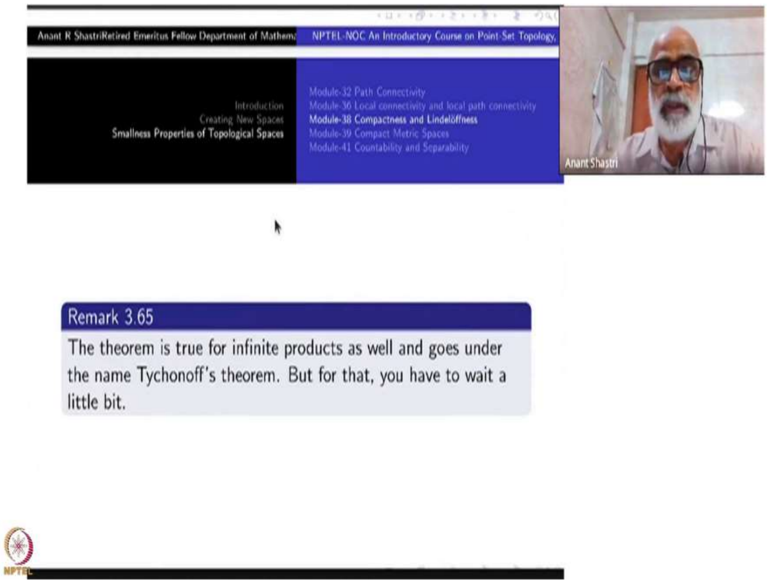


The theorem is true for infinite products also and that goes celebrated theorem of Tychonoff, but for that you will have to wait a little bit.

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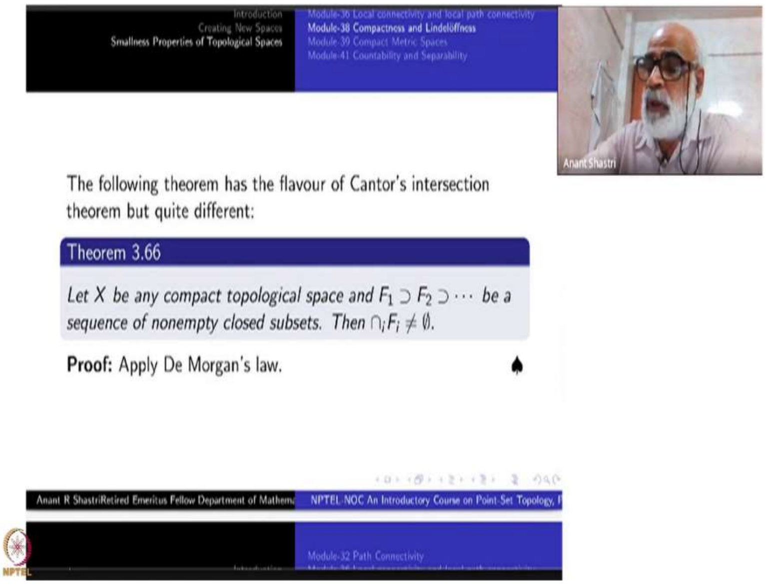


Now, another interesting diversion here. The following result has a flavor of Cantors intersection theorem for complete metric spaces. Here there is no metric, no contraction, no deltas and so on something funny happens, but you have to start with a compact topological space ok.

So, let $X$ be a compact topological space, $F_1 F_2 F_3 \ldots$ sequence of non-empty closed sets ok. Non emptyness is obviously necessary for whatever I am trying to say. They are decreasing sequences as well. They are closed subsets of the compact set. Then the conclusion is that entire intersection is non-empty.

You see in Cantor's intersection theorem finally, you had a unique point there, but non emptiness was very important.

So, the same kind of conclusion can be got out of compactness instead of complete metric space and so on. In the complete metric space you needed more stringent conditions. Herein, there are much less conditions.

Apply De Morgan law it is a one line proof that too ok? If this is empty what does it mean? the complement is the whole space. The complement of the intersection is the union of the complements. What are the complements? They will form an increasing sequence of subsets each of them open.

And they cover the whole space $X$; that means what? By a compactness at some finite stage it must be equal to the whole space right? Some $F_n^c$ will be equal to the whole space because it is a finite cover. But then what happens if you go back via De Morgan law, the corresponding $F_n$ is empty. That is a contradiction, because we assumed $F_n$ are non-empty. So, you just apply De Morgan law you get the proof ok.

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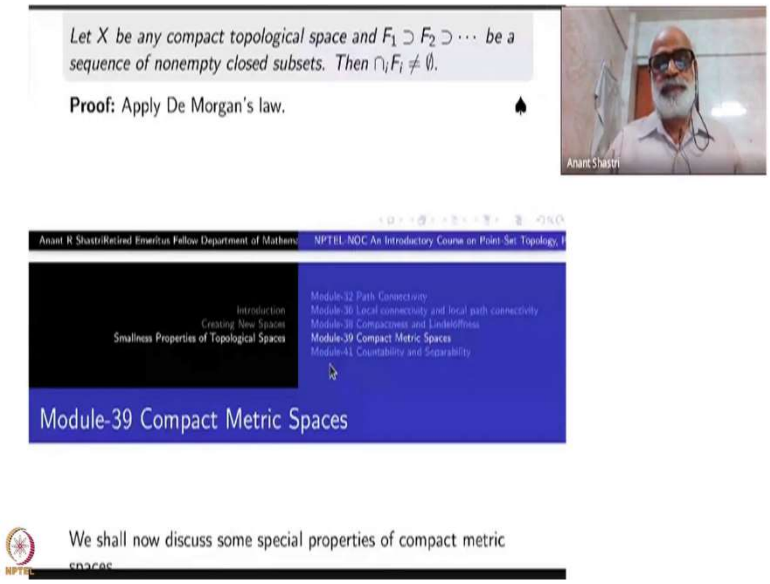


So, like this we can go on taking some glimpse of compact spaces and so on. By the way there is no such result for Lindelof spaces.

So, let us take a look at metric spaces again and get some more hints for what kind of things we can do with compact spaces ok. So, that is next time.

Thank you.