Introduction to Point Set Topology, (Part - I)
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Module - 37
Lecture - 37
More Examples

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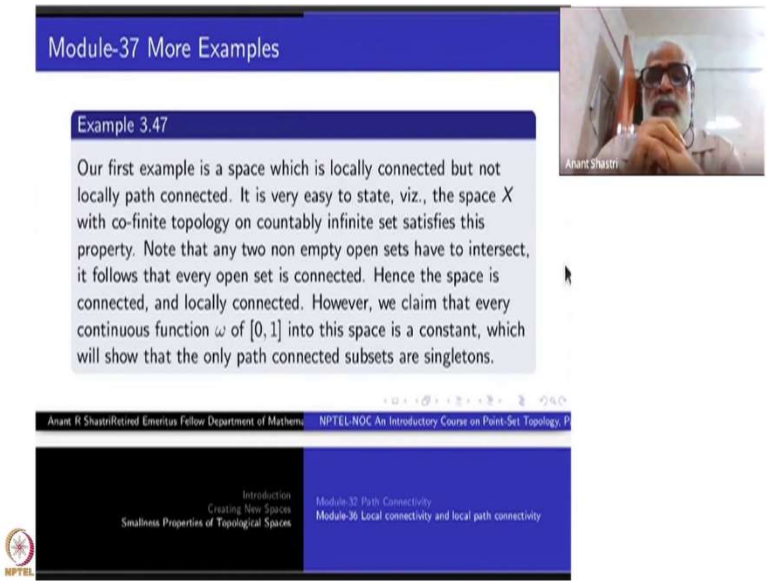

Welcome to module 37. Last time, we introduced the notion of local connectivity and local path connectivity and proved a few results there. So, today, we will illustrative examples, or counter-examples, or they are positive or negative examples and so on.

The first example is what you have already met ok? It is very easy to check that it is locally connected, but not locally path connected. This is very easy to check, namely, the co-finite topology on, say, the set of integers or natural numbers, or any, countably infinite set ok?

Countability is important, infinite also important ok. If you took a finite set the co-finite topology discrete topology. Discrete topology with more than one point is neither connected nor path connected, but surprisingly, because points are open, this is both locally path connected as well as locally path connected so. I am not interested in discrete topology alright.

So, the first thing is every nonempty open set will intersect every other nonempty open set in the co-finite topology because the complements are finite right? So, a finite set cannot contain another set which has its complement finite. That is not possible when the whole thing is infinite. So, any two open sets intersect. Therefore, there is no separation ok? That means, that the space is connected. Similarly, it is locally connected also because every open subset also has the same property, you cannot separate it out.

You take any open set that itself will serve as a connected neighborhood inside that you have to take a connect open set, but its already connected right? The subspace topology on any subset is again the co-finite topology only if it is uncountable sorry, it is countable and infinite. Therefore, this space is locally connected and connected. However, we claim that every continuous function omega from closed interval $[0, 1]$ into the space is a constant.

Once you prove that, it follows that two distinct points cannot be connected by a path ok. So, we are going to prove this strong property that the path connected components of this space are singletons ok? Strongly path-disconnected in that sense ok.

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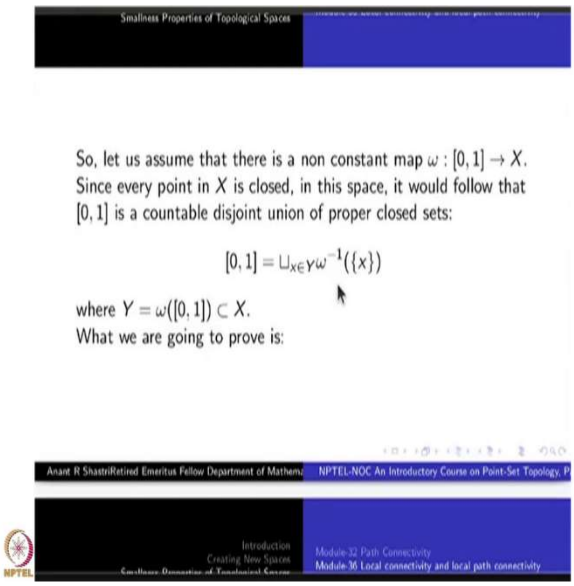
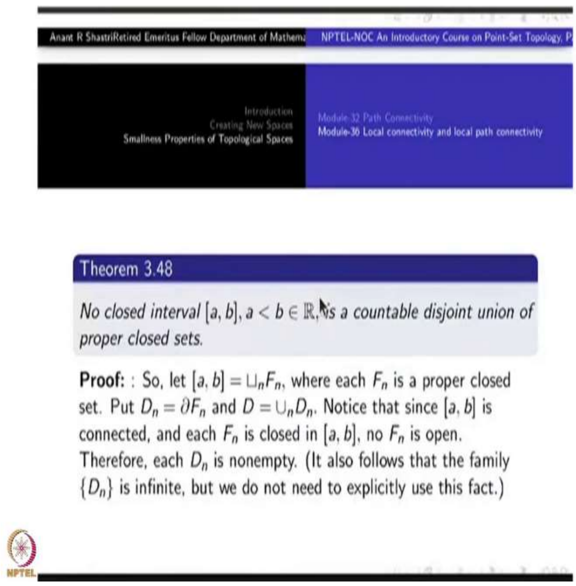


So, let us see how we prove this one. Suppose, you have a path which is non constant. That means what? The image should contain at least two points ok? Every point in X is a closed point, singleton $\{x\}$ is closed why?

Because the complement is open that is all ok. It would follow that the closed interval $[0,1]$ is a countable disjoint union of proper closed sets, what are those closed sets? Just take ω inverse of singletons, singletons are closed. ω is continuous, $\omega^{-1}(x\})$ is a closed subset as x ranges over X. The image of ω, Y contained inside X take the disjoint union, these are the fibers after all. So, they disjoint right, disjoint union each of them is closed, what is Y?

$Y$is $\omega([0,1])$ the image, the points are varying over Y ok, they are at least two of them ok? $[0,1]$ has been written as disjoint union of closed sets ok. One thing is clear namely, this Y cannot be finite why? If this Y is finite, this will give you a separation of $[0,1]$. But $[0,1]$ is connected so, there is no separation ok?

Because once Y is finite, you take one of them, everything is closed, the finite union of other things also closed so, you can write as disjoint union of two nonempty closed subsets. Over. But if Y is infinite, then there is no contradiction ok. So, $[0,1]$ may be connected, but you can always write it as disjoint union of singleton sets for example, singleton sets are closed so, that is not a contradiction ok.(Refer Slide Time: 07:01)



So, what are we going to do now? Somehow I want to get a contradiction to such a description: The first thing is that I notice Y has to be infinite, but countable. What I am going to prove is the following. A general statement no closed interval $[a, b], a < b$ is a countable disjoint union of proper closed sets. Y being a subset of X, X being countable, this Y will be countable ok therefore, immediately this description is not possible is the conclusion, when I take $[a, b]$ to be the interval $[0, 1]$. So, this general statement I am going to do.

In this statement, note that countable means finite also which we have seen that finiteness is not possible, that is easy because all intervals are connected. So, the point is now countable infinite is not possible is what we have to prove. So, here we will have to use Baire's category theorem in a clever way so, that gives you an opportunity to use Baire's category theorem so, watch out.

Suppose you have $[a, b]$ equal to a disjoint union of F_n's where each F_n is a closed proper subset of $[a, b]$, ok. You see if F_n's are empty and one of them is $[a, b]$, then there is no contradiction so, assuming that they are proper that is important. Proper means what? Nonempty as well as not the whole space. Take D_n equal to the boundary of F_n by this I mean the boundary points of F_n inside $[a, b]$, they are all subspaces of $[a, b]$ now, in the usual topology of $[a, b]$. Take D_n's equal to boundary of F_n and take this D equal to union of all these D_n's ok?

Note that since $[a, b]$ is connected and each F_n is a closed subset of $[a, b]$, no F_n is open. If F_n is open, what happens? If one of them F_n is open, what happens? It is also closed I have assumed right, each of F_n is close to begin with. So, proper nonempty closed and open subset will contradict. So, since they are closed, they are not open is a consequence ok?

So, when you take interior, interior is a strictly smaller subset of F_n. When you throw away the interior from F_n, what you get is the boundary, because F_n's are already closed, I do not have to take the closure. Remember, what is the boundary points? Boundary points are closure minus the interior. So, closure is F_n itself, interior is not the whole space.

So, interior may not be empty, but you throw away the interior, so, D_n, the boundary of F_n's are nonempty. Boundary F_n's would have been empty if F_n is open also. Therefore, each D_n is nonempty. Boundary of a subset is always a closed subset of the set, the set is closed so, therefore these are all closed subsets of $[a, b]$ ok. D_n's are nonempty closed subsets ok, it also follows that D_n is infinite, but we are not actually using this fact ok. D_n's are infinite means what?

Each of them I have proved that they are nonempty and to begin with, I have taken these things are infinite, if this is finite, then we already know that the case is over ok, but this fact we are not going to use ok. Right now, our argument will include this also anyway.

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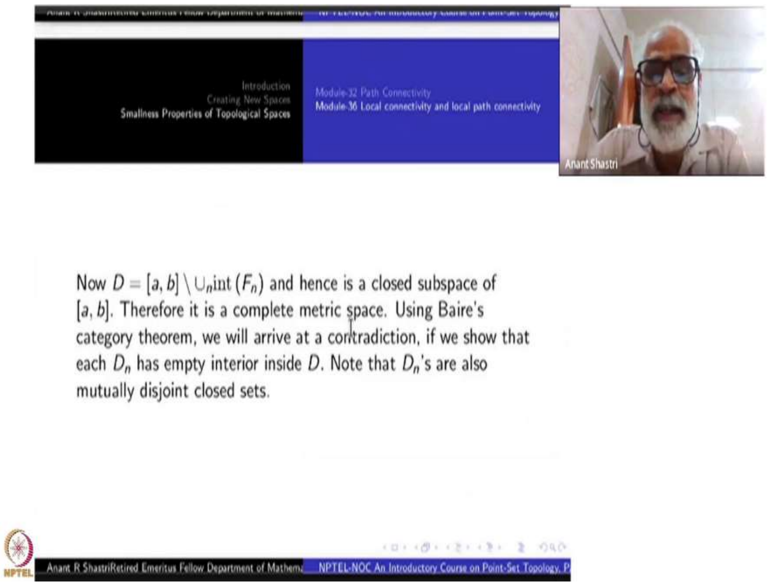

So, each D_n is nonempty close subset right, but now we have lost something because these D_n's do not cover $[a, b]$ right, the F_n are covering $[a, b]$ now, I have taken smaller subsets so, what do I do with then?

Take D which is equal to union of all D_n's right, how did I get D_n's? By deleting the interiors of each F_n right. Therefore, this union is $[a, b]$ minus union of interior of all F_n's is precisely equal to D ok. Therefore, this is a close subset of $[a, b]$. Therefore, it is a complete metric space. Now, using Baire's category theorem, we will arrive at a contradiction. How do we use

Baire's category theorem? Namely, if we show that each D_n has empty interior inside D ok? then the proof will be over. This is very important because D_n's are boundary of F_n, I have removed the interior points there. So you may say it is over no?

No. These interior points were inside $[a, b]$. Now, I have a different subspace. We have, this subspace namely D, I am applying the Baire's category theorem to this complete metric space D so, I have to show that each D_n has empty interior inside D ok? then the Baire's category theorem says that countable union of such things cannot be the whole of D ok. So, that is the end of the proof right? So, let us go ahead.

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How to show that D_n's have empty interior? Take any point in D_n; by the way what is the topology? Topology is coming from $\mathbb{R}$ right, from $[a, b]$, everything is a subspace of $\mathbb{R}$ right so, take an open interval around a point, intersect it with D, if it is contained inside D_n that will be a point in the interior ok. Take an interval, intersect it with D, show that is not contained inside D_n, then this will prove that, interior of D_n is empty. So, this should be done for every point of D_n.

What is D_n? Boundary of F_n. Take a point and take an interval J around that one. By the definition of boundary ok? Now I am using F_n, a point of D_n is a point of F_n also, but it is a boundary point of F_n. First you use that property that D_n is a boundary of F_n ok by the definition of boundary, J must intersect both F_n and complement of F_n right? So, it intersects the complement of F_n where? inside the closed interval $[a, b]$ ok.

Hence, it must contain some point of some other F_m. Because the entire $[a, b]$ is written as disjoint union of F_n's. If a subset is not contained inside one particular F_n, it must intersect some other F_m, $m \neq n$ ok? Now, suppose J does not intersect the D_m, boundary of F_m, D_m is the boundary to F_m ok? that means that it intersects interior of F_m, alright. It intersects F_m, it does not intersect the boundary so, what is left out?

It must be interior of F_m, but then, what happens? Interior of F_m, this interior is taken inside the closed interval $[a, b]$, remember that, interior of F_m, it is an open set, complement of F_m is also an open set. If you take J intersection this and J intersection that, that will be the whole of $J \cap [a, b]$. I have not written $[a, b]$ here because I am taking the interval J itself inside the interval $[a, b]$ ok. So, therefore, this will give you a separation of J and that is a contradiction. Therefore, this J must intersect D_m, the boundary of F_m ok?

So, what has happened? I started with an interval around a point in D_n that intersects D_m, $m \neq n$, but these are disjoint sets right. Therefore, this J is not contained inside D_n. So, there is a point D after all, all of D_m is also contained inside D so, J intersection D, there is a point here, which is not inside D_n. This shows that D_n has empty interior inside D.

That proves that no closed interval can be written as a countable union of disjoint you know, of closed subsets, proper closed subsets. In particular, the co-finite topology on a countable set is not path connected, every path component is a singleton.

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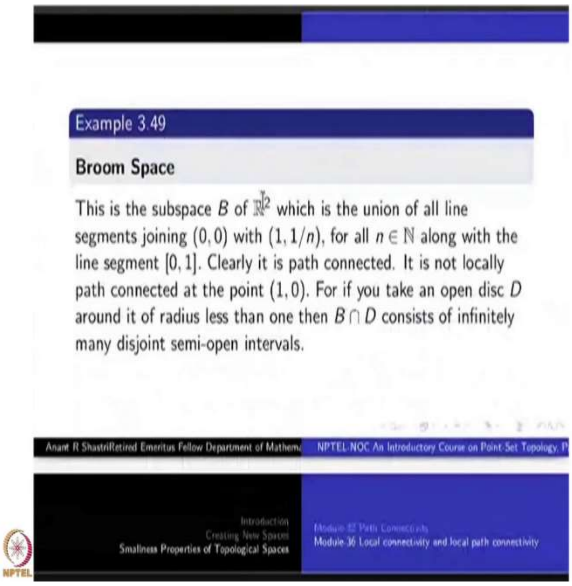

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Let me give you some other example now. This is called the broom space. Let me show you the picture here and then explain. So, I start with a closed interval. Let us call this $[0,1]$ ok. Here, I take the line segment between $(0,0)$ and $(1,0)$ in the plane. So, this is $(1,0)$. This is $(1,1)$, this is $(1,0)$. So, $(0,0)$ and this is $(1,1)$.

This is $(1, 1/2)$, this is $(1, 1/3)$, this is $1/4$ and so on. So, $(1, 1/n)$ approaching $(1,0)$ here, all those points are joined to $(0,0)$. So, it looks like a broom eh, this is the broom space. This is the closed subset of $\mathbb{R} \times \mathbb{R}$, it is closed and bounded also ok? Whole thing is contained inside the square $I \times I$. Clearly, it is star shaped at this point $(0,0)$. Take any point, there is a line segment to this point. Therefore, this is path connected. Therefore, this is connected.

So, why this example is there? For the reason that you know take a point here on the $x-$axis, between 0 to 1 say let us take this itself namely, $(1,0)$. If you take any neighborhood of that, take a small ball around that and intersect with this space, you will have lots of these line segment, disjoint sets no matter how small this ball you are taking (radius smaller than 1), intersection with this space will be disconnected. Infinitely, many line segments will be there along with a small line segment on the $x-$axis.

So, at this point, it is not locally connected or locally path connected, the same thing holds for all the points here except the point $(0,0)$. At $(0,0)$, it is clear that if you take any neighborhood, then you can take a small ball around that, then it will be star shaped again so, there is no problem ok. So, that is the broom space here. So, it is path connected, it is not locally path connected at any point on the x axis except at $(0,0)$. At least at $(1,0)$ we have seen it easily alright.

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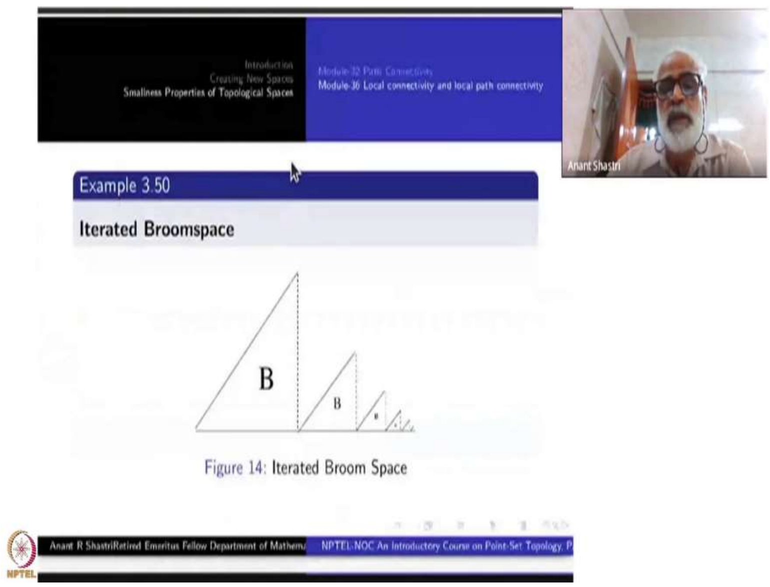

So, let us go to slightly more complicatedness spaces namely, what I do? I take a small copy of this broom space here again, I have not drawn all these lines here, they are there inside ok so, this is a broom space B. So, instead of $[0,1]$, I have made it $[0,1/2]$ here ok, this is $(0,0)$, this $(1/2,0)$, this is three-fourth, 0 and so on, take half each time. Take the side also, like this also make it half, keep making half, half like that. So, this will go on ad- infinitum, infinitely many of these things will be there ok?

Along with this point $(1,0)$, just the point only at the end, there is no broom there left out. If you come slightly out of that point to the left, you will have lots of brooms here, infinitely many brooms ok. So, this is again, the whole thing is contained inside $I \times I$ ok and it is a closed subspace. Remember I have not drawn these, there are from $1/2$, 1 this is 1, this is $(1/2,1/2)$, you can join them, you have joined them all those things are there. So, this is iterated boom space.

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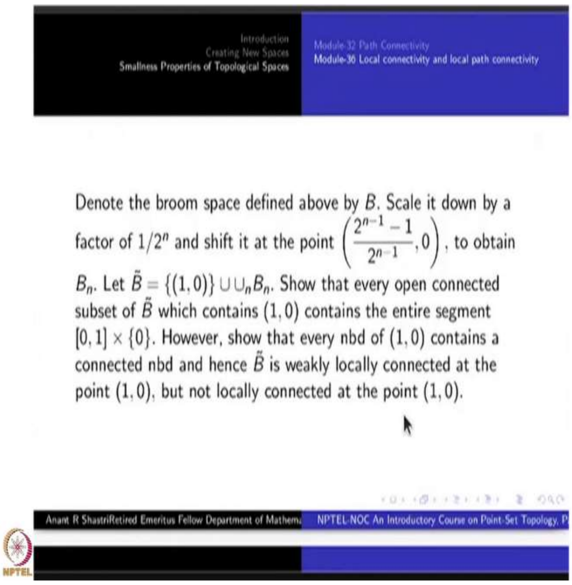

So, here I have defined it. Denote the broom space defined above by B in the first one. Scale it down by a factor of $1/2^n$ and shift it at the point $(1-1/2^{n-1},0)$, so all these points approaching $(1,0)$, to obtain B_n. Let $\tilde{B}$ be the point union of B_n and contains point $(1,0)$ has to be there ok, you take that one union along with all the B_n's ok.

Now, show that every open connected subset of B twiddle which contains $(1, 0)$ contains the entire segment $[0, 1] \times \{0\}$ ok, the entire line segment, on the $x-$axis. However, show that every neighborhood of $(1, 0)$ contains a connected neighborhood and hence, $\tilde{B}$ is weakly locally connected at the point $(1, 0)$, but not locally connected at the point $(1, 0)$.

If you want an open connected set, you will have a problem. If you do not want open sub connected set, just a connected neighbourhood, they are there, this is the whole idea. So, let us try to see this one, at least some part well, after all I have left this as an exercise to you, but let me just explain. So, first part is to show that every open connected subset of $\tilde{B}$ which contains $(1, 0)$, contains the whole line segment and therefore, it cannot be arbitrary small that is all. You cannot have a connected open subset which is arbitrary small, if you show that one.

So, how do you show that? Go to this picture, you have taken an open connected subset U around the point $(1, 0)$. There is an open ball around this point, right? Intersect with $\tilde{B}$, it is contained in U. This open contains infinitely many of these triangles here, representing the copies of the broom, ok?

Nevertheless, what happens? Suppose you have come up till some point which the apex of the nth broom. Then U will contain infinitely many segments of the $(n - 1)$ broom, line from here-to-here half going so, this is precisely what I have. At some point ok, suppose this point is there, then a small neighborhood of that point must be there in your open set U, then all those line segments should be there. So, in order that those line segments are connected, you have to go to the apex of $(n - 1)$ broom you have to go to the apex of that broom.

That means suppose the point $(2^{n-1} - 1/2^{n-1}, 0)$ is in U implies $(2^{n-1} - 1/2^{n-1}, 0)$ is also there. That means the line segment $(2^{n-1} - 1/2^{n-1}, 1)$ is contained in U. So, you just keep repeating this till you conclude that this whole line segment $[0, 1]$ is contained in U. All this because it is connected and open. There is no other way to connect those successive points on the $x-$axis.

Now, once this point is there, a small neighborhood around that will be there because I am assuming openness, open connectedness, then there will be some line here, some point here and the line all the way up to this one. So, this way you will end up here, these entire line segment will have to be there in any connected open subset containing any of these points in particular $(1, 0)$, ok.

You will have to keep coming backwards; forward you do not have to worry. If you take a neighborhood here, you do not go forward, but you go all the way here that is a point. So, for $(1, 0)$, the whole line segment will be there alright. So, connected open set is a problem. On the other hand, take any point here on $[0, 1]$, take a neighborhood, now do not go backwards all the way, just take the single boom to which this point belongs or just the two to the brooms that will be connected neighborhood.

That is not an open set, you just take the broom only, do not take all these points up to broom you take ok, everything forward so, that will be connected neighborhood. It is a neighborhood because a smaller open subset will be there, a small open subset will be there which is not connected, but this is connected, and it is contained in the original neighborhood. Therefore, this is low weakly locally connected at this point ok and not locally connected alright.

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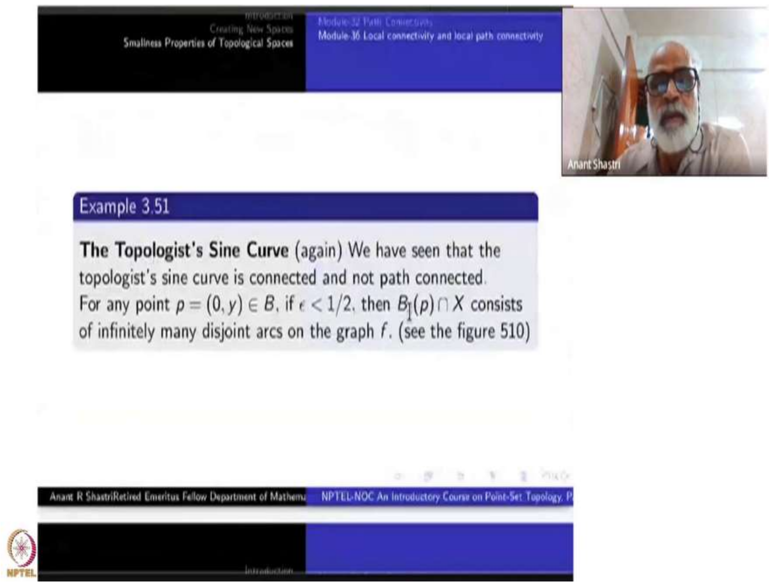

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I come back to the topologists sign curve now, it will be very easy, but we will have to just sum it up. The topologists sign curve. We seen that it is connected, but not path connected that we have seen. Now, take any point p on the y-axis part ok? x-coordinate $0, y$-coordinate to be $\epsilon < 1/2$ just for being safe. Then the ball $B_\epsilon(p)$ intersect with the topologists sine curve consists of infinitely many disjoint arcs on the graph, along with one open segment on the y-axis.

But the same reason that we have discussed for the broom space what happens is the topologists sine curve is not locally connected at any point on the y-axis ok?

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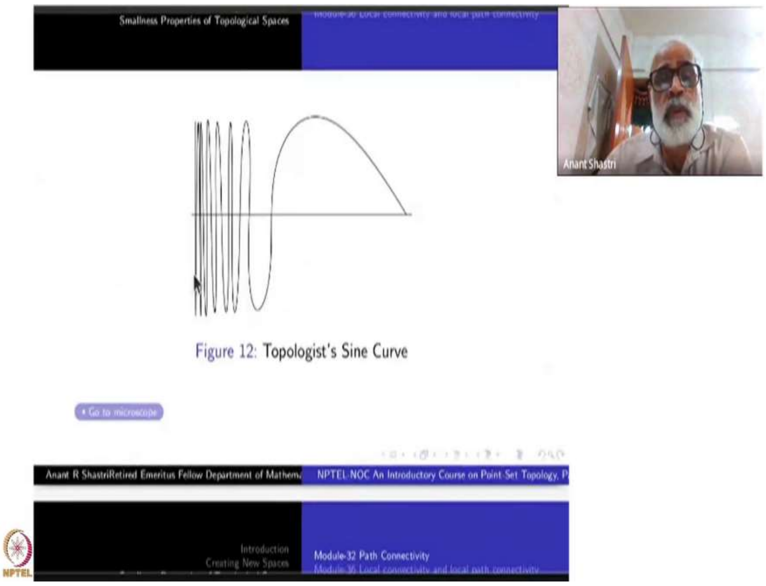

Let us have a picture of this one first here. So, on the y-axis here, you take any ball around here, the ball should not be such that it contains the points $y = 1$ and $y = -1$. So, I take the ball of radius say $\epsilon/2$.

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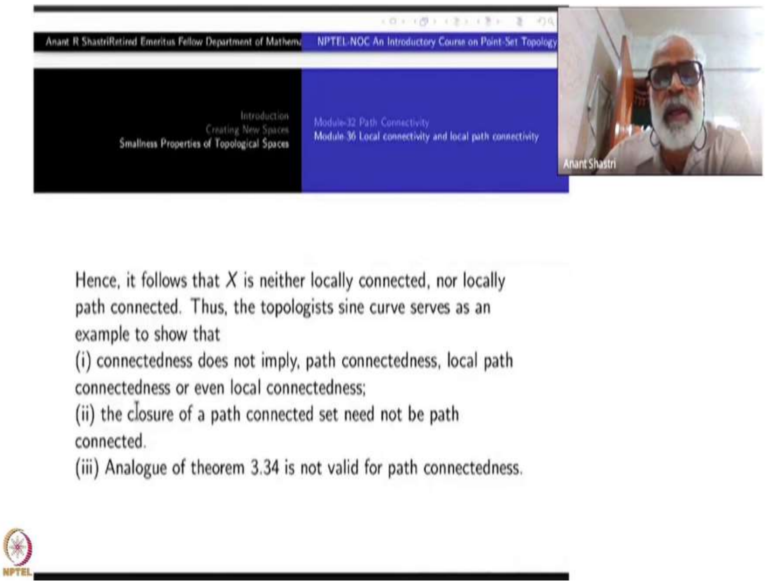

Take a ball here, let us magnify it ok, to magnify it and look at it. what do you? This may be your point and that means, epsilon ball, you let all these curves ok part of the curves, what curve? $\sin(\pi/x)$ curve going up and down like this. I have drawn this one thick line, this is not a thick line, this is just the union of infinitely many arcs like this coming close to the y-axis, that is why it looks thick that is all.

If you magnify further again, this taking a small ball here, you will have the same picture again, there will be infinitely many components there. So, this is not even connected so, it is not locally path connected either for every point on the y-axis. So, this is similar to the broom space there ok, but it has this property also is what I wanted to tell you.

Thus, let us consolidate what are the things that we have done for topologist's sine curve. It is neither locally connected nor locally path connected, just now we saw. Thus, topologists sine curve serves as an example or you can say counter example to show that connectedness does not imply path connectedness, local path connectedness or even local connectedness because it is a connected space, but it is neither of any of these things.

The closure of a path connected set need not be path connected because the sine part sine graph, is path connected part and its closure is the whole space and that is not path connected. Finally, analogue of 3.34 is not valid for path connectedness. What was 3.4? It says that if the bottom space is connected, the fibers are connected of a quotient map, then the topological space is connected.

So, if you replace path connected everywhere, if connectedness is replaced by path connectivity in this theorem, as such it is false. So, that is the meaning of this one. So, all these things we have seen ok.

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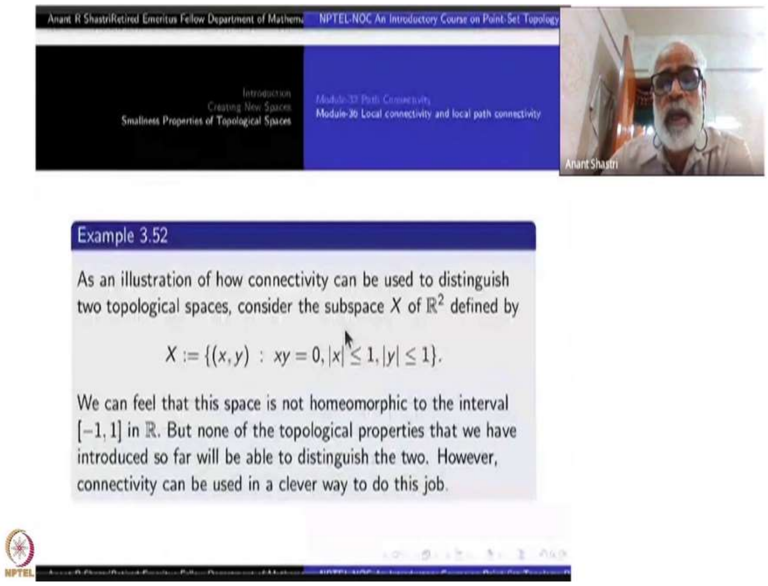


Now, I will come to another important part here, maybe that is the last part for today. How to use connectivity positively to derive some interesting results ok. So here is something.

Look at the space X which is the union of the union of the two axis, x-axis and y-axis. If you are bothered about to be too big and so on, cut it off: $|x| \le 1$ and $|y| \le 1$, ok? So, this is just x-axis and y-axis, -1 to 1 and -1 to 1.

We can feel that this space X is not homeomorphic to any closed interval. You know that this is a compact metric space, apparently whatever topological properties that you have seen so far will not be of any use in distinguishing them. Both of them are connected also, both of them are path connected ok. So, what is that?

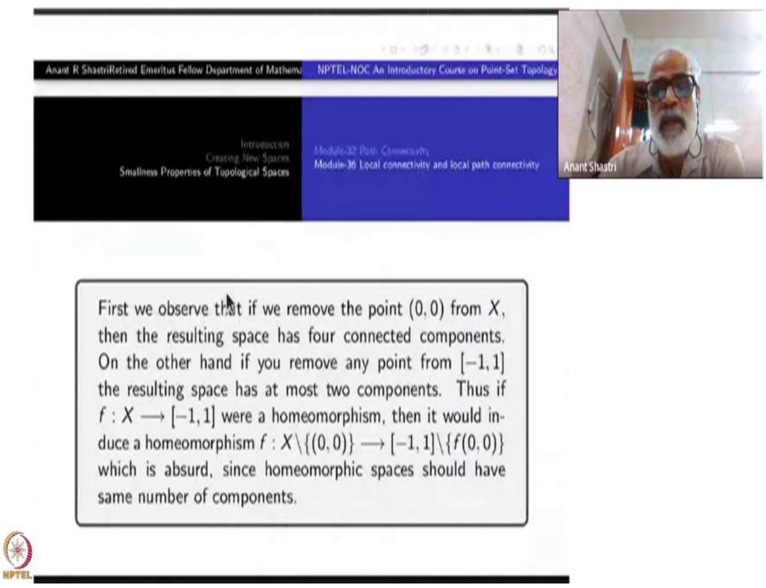
How to use connectivity to show that they are not homeomorphic? It is not homeomorphic to a closed interval, $[-1, 1]$. This $[-1, 1]$ I take because just model, you cannot take any closed interval, they are all homeomorphic to each other, but they are not homeomorphic to these space X ok.

So, how to see that? Suppose you have a homeomorphism from here to here. Now, take a point here, take its image, it may be any point here.

Remove both of them from from the respective spaces, namely from the domain you remove this point, from the co-domain remove the image of that point. Then the homeomorphism whatever it is, let us call it f, restricts to the subspaces and gives a homeomorphism again. A homeomorphism from Y to Y restricts to a homeomorphism from A to $f(A)$ for any subspace A of X ok? So, that is easy to see, that is what I am going to use here. If I remove a point in the interior of $[-1,1]$, then we know it is disconnected, has two components right?

If I go here, will it always have two components that is the point right? So, what I want to do? Immediately I see one nice way of looking at it, namely remove $(0,0)$ from X here, the point of intersection of the two lines. Immediately you see that there are four path connected components for the complement.

So, take a homeomorphism this way other way around f from X to remove the origin here, remove the image of that here, image of that may be any point I do not know which point, I am just removing one point from $[-1,1]$, what do I get? If the point I removed is one of the end points, then the space is still connected, path connected. If it happens to be some somewhere in between namely $-1 < f(x) < 1$, then it will have two components. But here in the domain, I have four components. A homeomorphism has to induce what? Bijectivity of the connected components, bijectivity of path connected components. So, what is wrong? Namely our assumption is wrong that there is a homeomorphism from X to $[-1,1]$ ok? (Refer Slide Time: 37:30)



So, that is the gist of this one, $X \setminus \{(0,0)$ to $[-1.1] \setminus \{f(0,0)\}$, that will be homeomorphism, this has four components, this has at most two components, in some cases it may have only one component so, that is the contradiction ok?

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Now, I have listed a number of exercises here. One of the exercises here let us say 3.35 for example is directly from this one, from the last example that I gave you.

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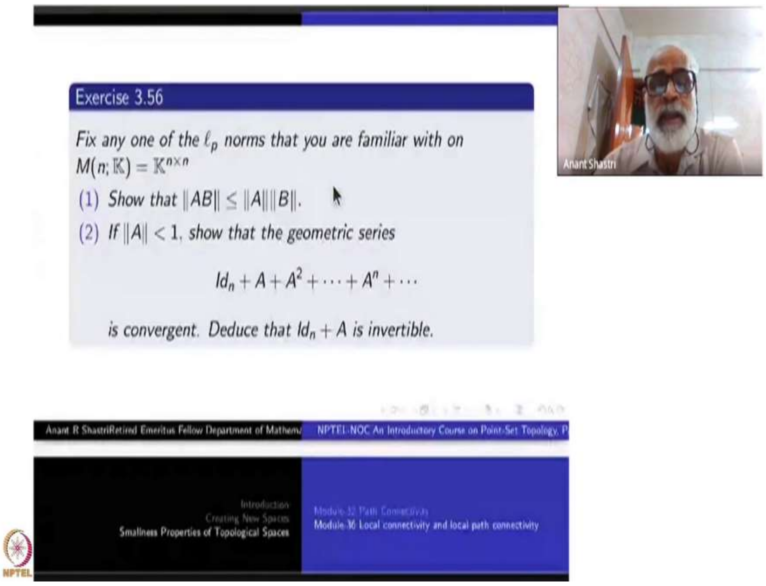

What does it say? Take the English capital letters ok? As subspaces of your plane namely $\mathbb{R} \times \mathbb{R}$, say, M, N, Z etc., Look at all these, I claim that they are all homeomorphic to each other in fact, they are all homeomorphic to an interval, easy to see that right, you can straighten out these things that is the meaning. But none of them is homeomorphic to the letter O, can you see why? If there is a homeomorphism here, you remove a point here from here ok, remove a point from here, what happens? That is still connected right.

But here, if the point has to be removed only from the interior so, what you should do? Start with homeomorphism here to here, take an interior point here, remove it, there will be two components. No matter what point you remove here, it has only one component from the circle, from the shape O, shape O you can think of a circle, it is homeomorphic to a circle, remove one point is still connected. So, none of them is homeomorphic to this one.

So, this is another example I am giving you here. So, using this idea, what I am telling you is to classify all the letters. the 26 letters up to homeomorphism ok? Enjoy this exercise.

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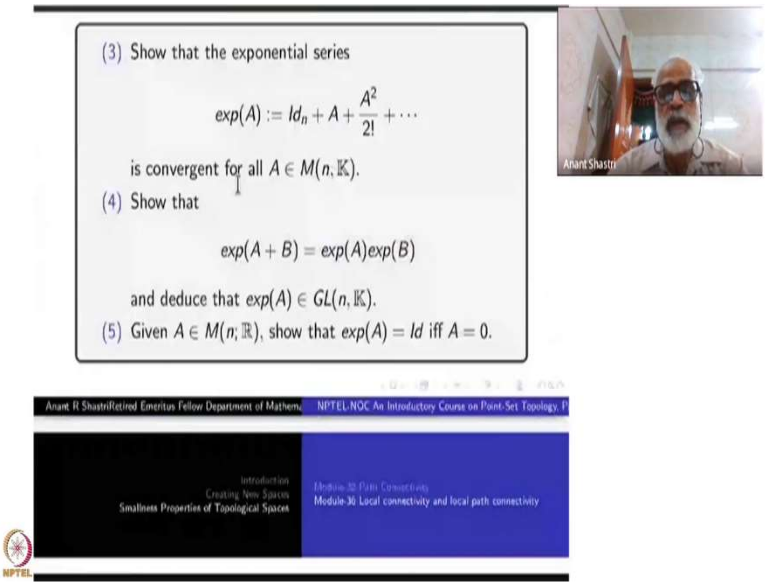

There are other things which you can have a look at ok.

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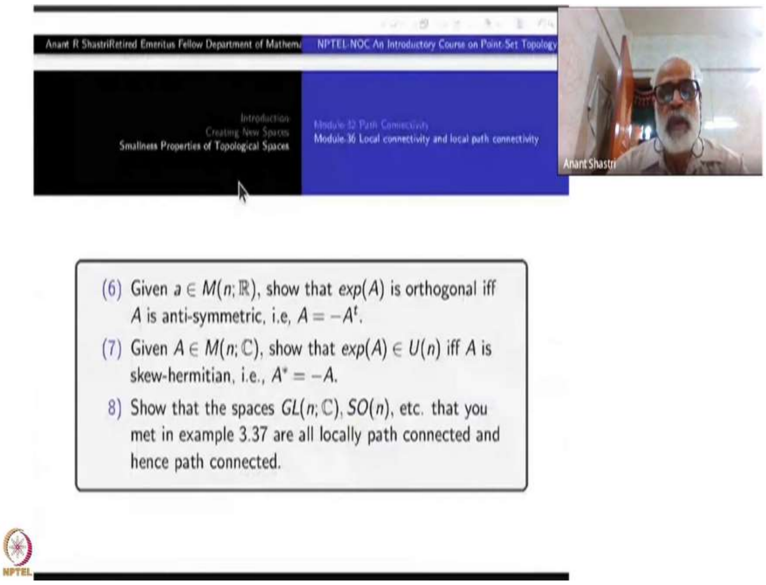

So, this exercise for example, is about metric spaces and part of it, most of them are from linear algebra and little bit of analysis. But last one is $GL(n), SO(n)$ etc, whatever you have met last time, they are all locally path connected spaces, the previous exercise told you that they are connected therefore, they will be path connected.

So, the exercise here is to show that they are locally path connected, the challenge. So, all these exercise, earlier exercises will help you to solve that.

Thank you. So, let us close it here.