Introduction to Point Set Topology, (Part - I)
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Module - 36
Lecture - 36
Local Connectivity

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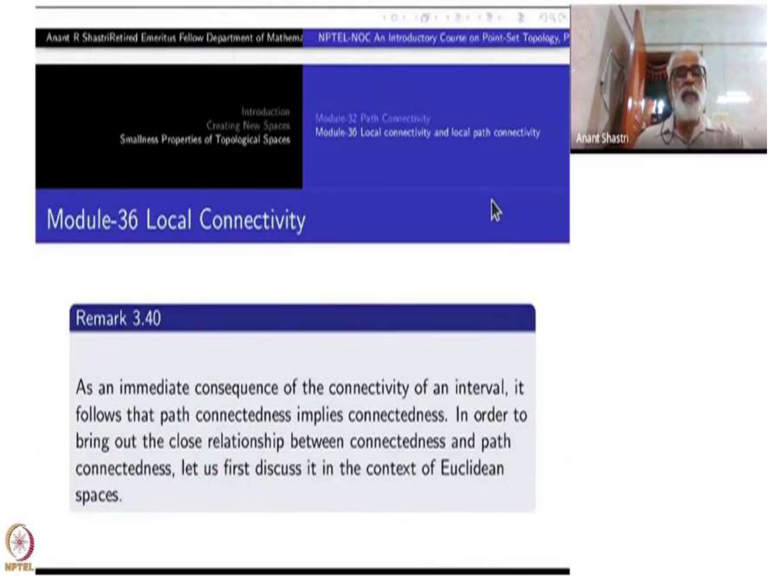

Welcome to module 36 Local Connectivity. Last time we studied path connectivity, connectivity, components and also studied one important example of topology, sine curve, which illustrates that connectivity need not imply path connectivity in particular. The intervals are connected. In fact, the connectivity was thought to explain the intermediate value theorem of an interval, alright.

And that is the key for path connectivity implying connectivity. So, though we showed that the other way around is not true there is very much close relation between them. So, now, let us look at what happens inside $\mathbb{R}^n$ instead of just $\mathbb{R}$, ok. $\mathbb{R}$ is too simplistic for further investigation for deeper analysis of the whole thing that is why we are going into R n and get some experience there and then study, you know, introduce more general concepts and so on. That is the whole idea.

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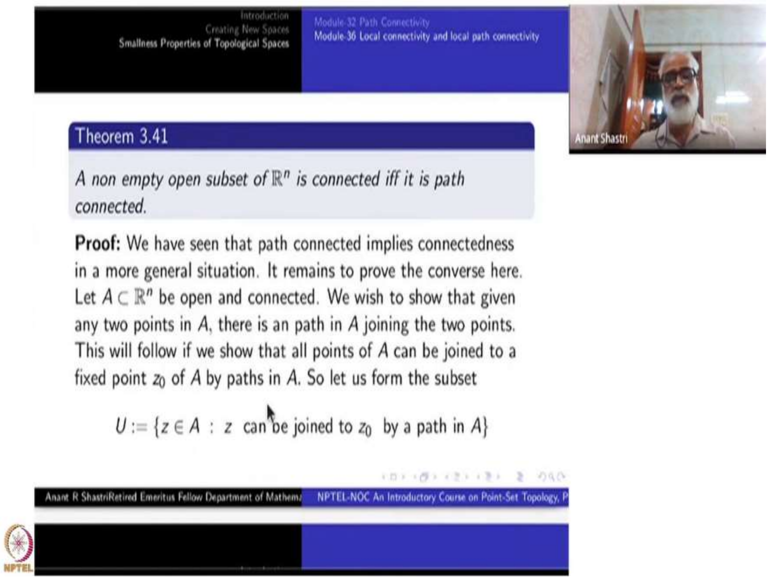

So, here is a result on $\mathbb{R}^n$. Take any nonempty open set you can take empty set, but that is not going to make any difference non empty open subset of $\mathbb{R}^n$, it is connected if and only if its path connected. So, here is a strong result connectivity need not imply path connectivity in general, but here for an open subsets of $\mathbb{R}^n$connectivity is the same thing as path connectivity, it is not just for intervals any open subset of $\mathbb{R}^n$. If it is connected it is path connected ok.

One way we have seen already path connectivity implies connectivity. So, you have to see only the other way around here ok. So, start with an open subset A we wish to show that given any two points of A, there is a path joining them completely inside A. Not just inside $\mathbb{R}^n$. It is no joke it is it is very straightforward because over $\mathbb{R}^n$ which is a vector space, so it is a convex.

Now, A is not given to be a convex set, A is only open set, that is given ok? So, this follows if you show that all points of A can be joined to one chosen point z_0. That z_0 is not very specific any z_0 you choose. First of all any two points can be joined means any point can be joined to z_0, but that is enough we have seen earlier. So, let us form a subset U of A, all z belonging to A such that z can be joined to z_0 by a path inside A.

What we wanted to show is that U is the whole of A ok. Starting with an open set and fixing a point z_0, we make this subset, but this subset which I have denoted by U what we want to show is this U is the whole of A. Now, the method of proof is going to be a like a principle here. How to use connectivity to show a lot of other results. So, this is the starting point of this one which you have not done so far. So, observe this method ok.

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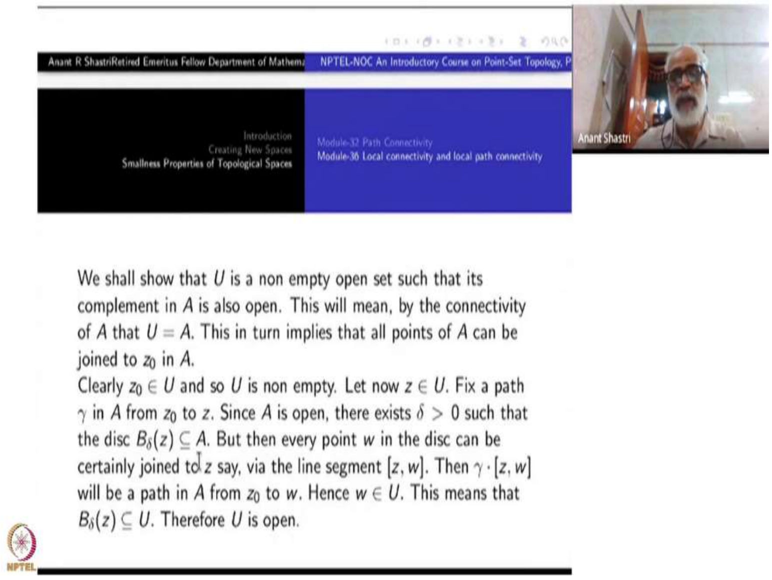

We shall show that U is a nonempty open set such that its complement is also open. Because you started with A connected then you want to show that this U is the whole of A, right? because A is connected it will imply that complement of U cannot be nonempty it has to be empty otherwise U and U^c will make a separation of A. So, U must be the whole of A.

So, what I have to show? First, this U is non empty, this is open and its complement is also open. Or equivalently you can show that U and its complements are both closed, either way. So, we will show that both of them are open ok. The first thing we observe is z_0 itself is inside U right? Because z_0 can be connected to z_0 by the constant path. Therefore, U is non empty that is easy.

Now, suppose z is inside U. Then I have to show a small open ball around z is contained inside U. That is enough to show that U is open ok. So, fix a path γ inside A from z_0 to z that is the meaning of z belongs to U, there must be a path. So, you fix it ok? So, there is a path from z_0 to z, completely inside A, but now A is an open set ok? z is inside U, but z is inside A also. By definition U is a subset of A. Therefore, you can find a positive δ such that the open disc of radius delta around z is completely contained inside A ok?

But then every point w inside this open ball can be certainly joined to z because all that you have to do is to take the line segment $[z, w]$. The line segment $[z, w]$ is completely contained inside $B_\delta(z)$. So, it is contained inside A also. So, γ comes from z_0 to z follow it with this line segment that will be a path from z_0 to w. Therefore, w is inside U. That just means that $B_\delta(z)$ the whole of $B_\delta(z)$ is inside U.

So, one part is over that U is open. Next part of the argument is very similar. This same argument will also show that U is closed or complement of U is open ok.

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So, let us just get through it. Suppose z is not a point of U. That means, it is in the complement ok? This $B_\delta(z)$ is the same choice namely it is contained inside A ok? Then no point w of $B_\delta(z)$ could have been joined to the point z_0 ok, inside A because if that is the case then from w to z, I can always take the line segment $[w, z]$.

So, if there is a path from z_0 to w, any point in the ball right? Then we could have joined z_0 to all of them ok; that means, that none of the points in $B_\delta(z)$ is inside U which is same thing as saying $B_\delta(z)$ is contained in the complement. So, the complement is also open ok? So, that completes the proof that U is equal to whole of A and therefore, A is path connected.

So, I repeat this process: suppose you want to show something holds for the whole of a subset A which you have chosen and that is connected. Then you make a subset of that set which consists of those points for which the property holds. Luckily if you can show that it is first of all nonempty and open and complement is also open then you have finished the proof that that subset must be the whole of A. So, this is a principle ok this is like a meta theorem it can be used several times, it has been used several times in topology ok.

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So, here what we have? What we have used? What was the key? The key is that the basic open subsets namely the balls in $\mathbb{R}^n$ they are path connected, any two points there can be joined. Indeed balls are actually convex. So, that is what we have used right? As usual what matters for us is that the path connectivity of the ball. So, now, you see if in a topological space basic open subsets are path connected, then the same proof will work.

Namely any connected open set will be path connected that is our next theorem. Before that perhaps, we should make this property as a you know, you can name it and that is what is called local path connectivity ok.

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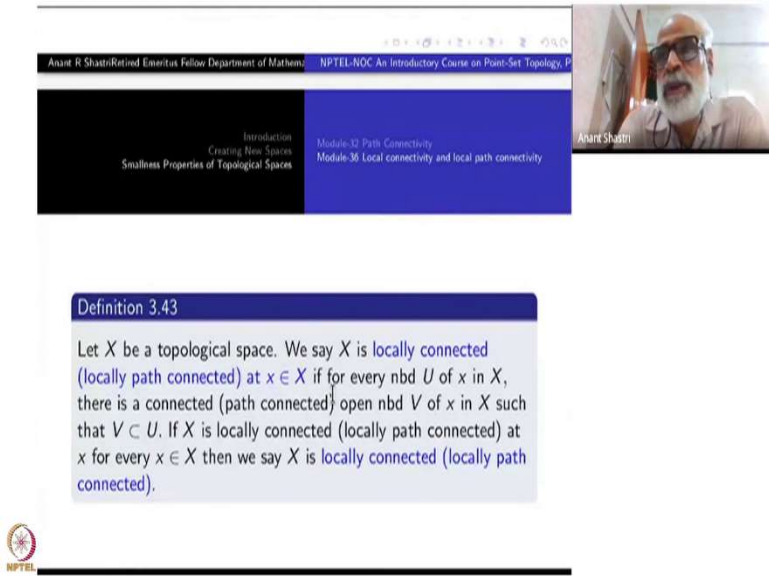

Since we are going to study both connectivity and path connectivity together maybe we should do the same thing for connectivity also.

So, we will have two different definitions here. Start with a topological space X. We say X is locally connected at a point (another pointwise definition), if for every neighborhood U of x there is a connected open neighborhood V of x inside U ok, such that that V is contained inside U. With any neighborhood you must be able to get a smaller open neighborhood of the point which is connected. That is called locally connected at the point x ok. For one open neighborhood if you find it that is not enough for every neighborhood you must be able to find such a thing a smaller smaller one which is connected.

So, that is the local connectivity property ok, no matter how small or how how big you choose the original neighborhood inside that you must be able to find an open connected neighborhoods. If you replace connectivity by path connectivity what you will call it locally path connected at that point. Just like other things if such a thing is true for all the points of

X, then we say the space is locally connected or locally path connected as the case may be, ok.

I repeat suppose the space X has the property that for every point x in X, it is locally connected, then X is called locally connected. Same thing if at every point it is locally path connected, then we say X itself is locally path connected ok.

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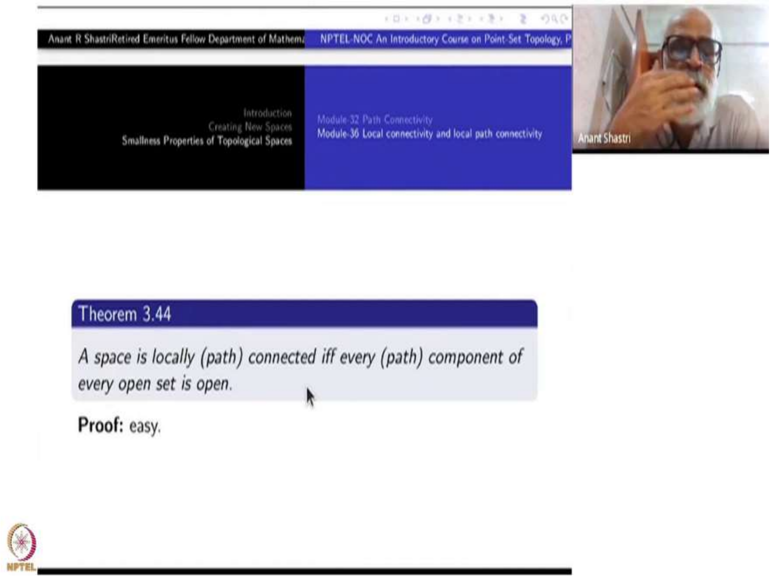

Now, as anticipated this was only a name we have given. What theorem we have immediately is this which is exactly a carbon copy of whatever you did for $\mathbb{R}^n$. A space is locally path connected; locally path connected first of all if and only if every path component of every open set is open ok.

So, I am going towards a characterization of locally path connectivity or locally connectivity. First let us do it for connectivity drop out this path locally connected part. Locally connected if and only if every component of every open set is open ok. What you have to do? Take an open set look at its connected components all of them are open.

How do you show that a connected component is open? Take a point there then you must produce an open subset contained inside that. But local connectivity says that at each point given any open set you started with an open set remember that, given any point, there is a connected open subset contained in the original open set. Therefore, that connected open subset is contained in the component because component is the largest one. So, every point has a connected onbd right. So, the union of all these open sets with the connected component.

So, the each connected component of an open set is also open. The converse is obvious because these connected components themselves you can take it as as neighborhoods ok. Start with any open set I want to find a neighborhood which is connected and contained in that open set take the connected component. Over ok.

Exactly same way the path connectivity also works. Because now the property that I have used is such that there is no difference between the two conditions ok, just put the word path connectivity wherever you are using connectivity that is all.

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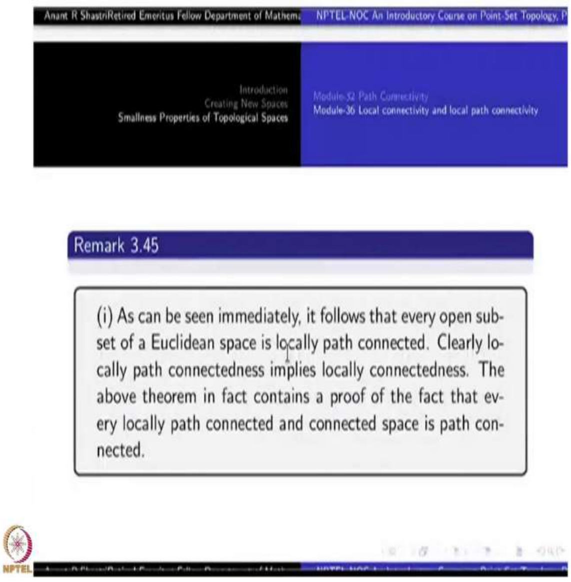

So, once again I repeat: it follows that every open subset of a Euclidean space is locally path connected that is the property we have used. Now you have given a name. Clearly locally path connected implies locally connected. Just like path connectivity implies connectivity.

The above theorem in fact, contains a proof of the fact that every locally path connected and connected space is path connected ok. Directly I could have proved it by using the carbon copy of the proof of the theorem that we proved for $\mathbb{R}^n$. But here is a shortcut. because I have already proved a stronger result here ok? So, let us see how this proves that. You start with a connected space which is locally path connected ok?

Therefore, its path components are all open. But its a connected space already ok? So, how do you use that to show that it is path connectred. If all path components are open all of them are closed also. In a partition if the partition members are open they are closed also. If they are close you cannot say they are open. Suppose you are taking one of the members look at all the other members each of them is open. So, union is open.

Therefore this set is closed. But A is connected. If there are more than one path component then that will give you a separation of A. So, that is the proof here, understand. So, that is why this result is stronger than just saying that locally path connected and connected implies path connected. So, this is stronger thing every open subset of a locally path connected space ok has the property that all the components are open.

Similarly, for connectivity. So, these two different statements, but use the fact that path connectivity local or otherwise implies connectivity local or otherwise. So, all that I have to do. So, this is a very strong thing. So, remember this one properly ok.

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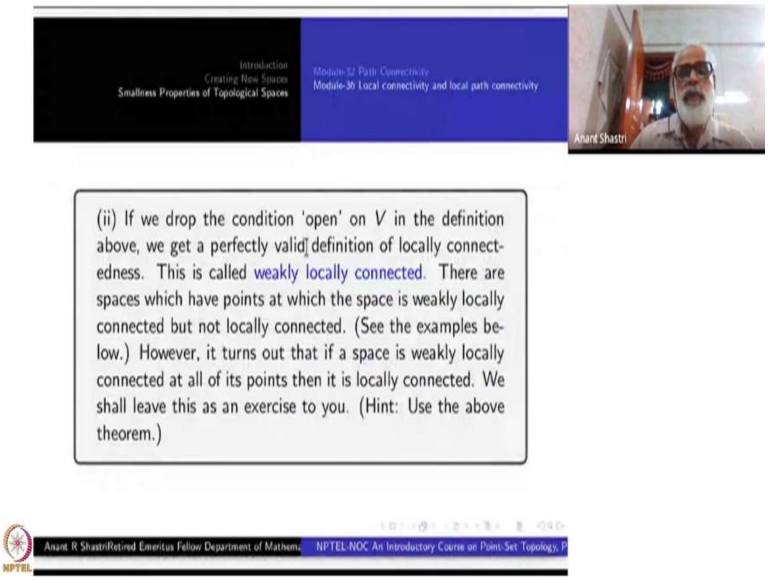

If we drop the condition open on V in the definition; in the definition of a local connectivity or local path connectivity I insisted let us go back again once again, every neighborhood of x there is a connected, (or path connected whatever) open neighborhood of V which is connected.

So, suppose I drop this openness here every neighbourhood has a smaller neighborhood which is connected that is; obviously, a weaker form of the local connectivity. So, we will call it weakly local connected. Similarly you can define weakly locally path connected. whether it is really weak or not that needs to be verified. Sometimes a weaker condition, weaker looking condition may be equivalent right. So, this is what I repeat here this is what I have done. You drop out the openness of V in the definition given we get perfectly valid definition of locally connectedness, which could be weaker.

Why I am giving this one is some authors may give you this as a definition, but then be careful. Our definition is stronger. So, this is called weakly locally connected. There are spaces which have points at which the space is weakly locally connected, but not locally connected. Once I give you such an example then you know that this concept is really weaker than the original local connectivity ok. So, we will come to those examples.

However, it turns out that if a space is weakly locally connected at all of its points, then it is locally connected in our sense. So, this is why many authors use this condition because they are not defining it pointwise they are only interested in local connectivity as a global thing. So, they define it everywhere. So, they are overlooking this one that is why they are all happy, they are not at all bothered about, that is all. So; however, it comes out that if a space is weakly locally connected at all of its points then it is locally connected, ok.

This theorem use it cleverly, it is not very difficult ok. To prove that if something is weakly locally connected at every point, then it is actually locally connected alright ok. You have to work out that when you think about how to prove that it is not difficult only then you understand the difference here. See you have a neighborhood open set you have a smaller neighborhood which is connected that neighborhood may not be open. So, to get an open set you may have to get take a smaller one, the smaller one may not be connected.

You see you take a connected space not all subspaces are connected otherwise the whole thing would have been you know just one global connectivity that is enough ok, when you go to smaller set or bigger set connectivity may fail, alright. So, that is why all these things have to be carefully understood, the definition should be carefully understood.

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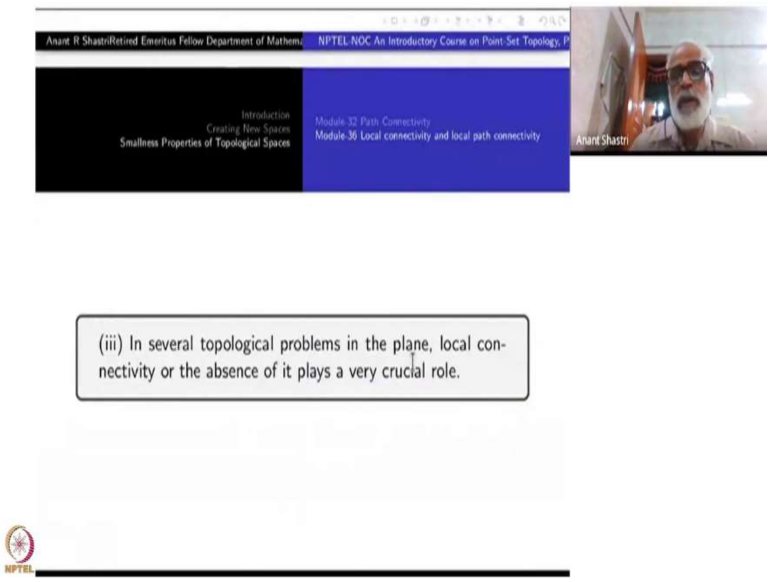

In several topological problems about the plane $\mathbb{R} \times \mathbb{R}$, local connectivity or absence of it plays a very crucial role. We won't have time to study all that especially in the dynamics of $\mathbb{C}$, you know there is a lot of what are called Julia set, Mandelbrot set and so on ok. In that study, this local connectivity and local path connectivity are important notions.

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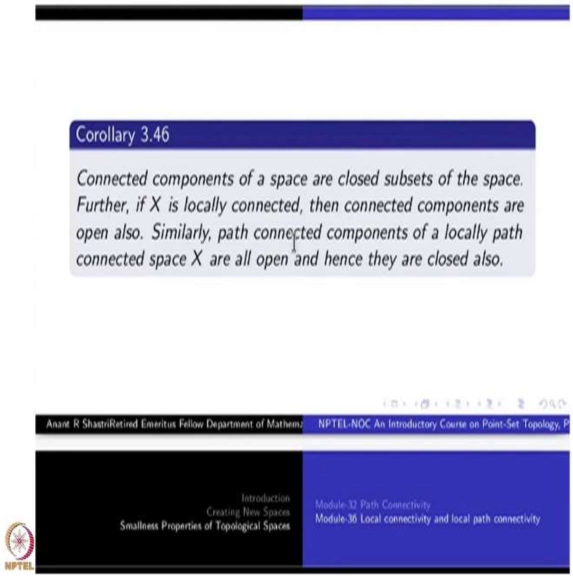

Connected components of a space are close subsets of the space which we have seen. Once you take a connected subset its closure is also connected. Therefore, components must be closed. Further if X is locally connected, then connected components are also open. This is what we have just seen.

Similarly, path connected components of a locally path connected space are all open and hence they are closed, but I am not saying that without local path connectivity, take any space and take a path connected component, it it may not be closed. So, you have to be careful about that. So, we have already seen an example of this in the topologist's sine curve.

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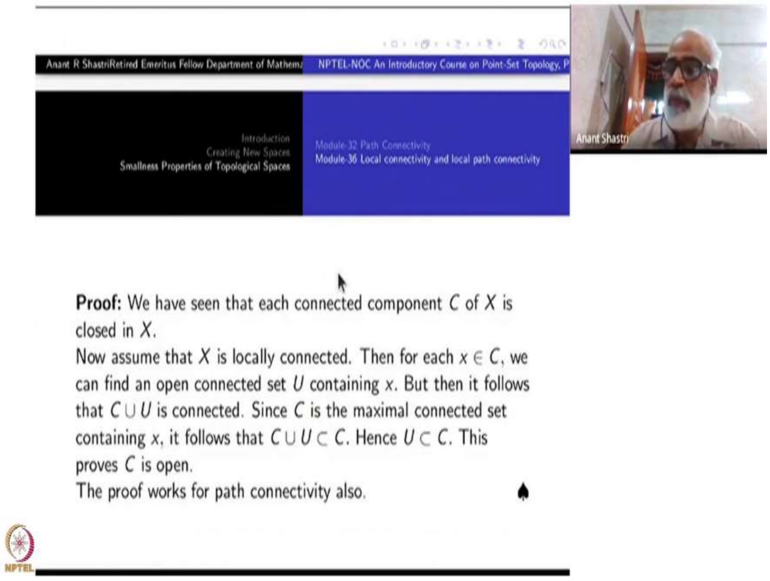

So, let us just go through the proof of this one. We have seen that each connected component C of X is closed right. Take connected component take the closure of that that is also connected, but it is larger no? it should not be larger because C is a connected component. So, therefore, C equal to $\bar{C}$.

Now, assume that X is locally connected. Then for each x belonging to C, we can find a neighborhood, open neighborhood which is connected. Therefore, every point has an open set contained inside C right. Actually why it should be contained inside C every point you have we have seen that I am proving it here again. $C \cup U$ will be connected right.

But C is contained in the union, but C is maximal, so they are equal. So, you have already proved it, but I have just put the proof here again the same thing works for path connectivity also, if C is path connected and U is path connected and the intersection has at least one point there then the whole thing is all the path connected. So, that that part is the same thing. Locally path connected or locally connected implies the components are open, ok.

So, next time you will see more and more examples especially to bring out the difference between local path connectivity and local connectivity. Thank you.