Introduction to Point Set Topology, (Part - I)
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Module - 35
Lecture - 35
Coneectedness- continued

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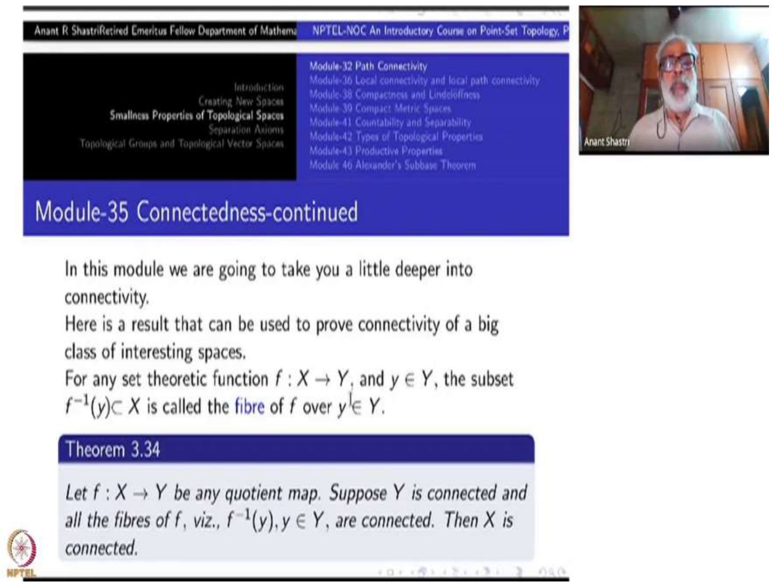

Welcome to module 35 of Point Set Topology Part I. We shall continue our study of Connectedness. So, far things were not all that difficult, but now we are trying to take you a little deeper into connectivity. Here is a result that can be used to prove connectivity of a big class of interesting spaces. So, let us formally make a definition here. Once again, this is for namesake only. Take any set theoretic function f from X to Y, for every y inside Y look at all points which are coming to y all points of X is coming to y.

Namely $f^{-1}(y)$ that will be called the fibre of f over y. What are the fibres of f? They are all inverse images of some point in y. For example, if you take a point y such that there is no point which is going to that point $f^{-1}(y)$ will be empty. So, then I will say fibre of f at that point y is empty. So, that is also allowed ok? It is purely set theoretic notion. There is no other condition on f. The word fibre is used in many higher mathematics, I am also used to that. That is why I am putting that terminology here ok.

So, you can just write $f^{-1}(y)$ for some y inside the codomain Y, then you do not have to use this fibre one alright. Let f from X to Y be any quotient map now, in particular, f is surjective ok. Suppose Y is connected and all the fibres of f are also connected. Namely, I am writing it to have just for definiteness sake, say $f^{-1}(y)$ for every y is connected, they are all connected. Then conclusion is that X is connected.

You see what we have seen earlier is that if X is connected, then $f(X)$ is connected. This f is a quotient map onto Y. So, $f(X)$ is all of Y. So, if X is connected $f(X)$ is connected and $f(X)$ equal to Y. So, Y is connected.

We are looking at the converse here. Y is connected. You do not immediately conclude that X is connected, but if the fibres are all connected, there is this extra condition, then you can do the other way around and conclude that X is connected ok.

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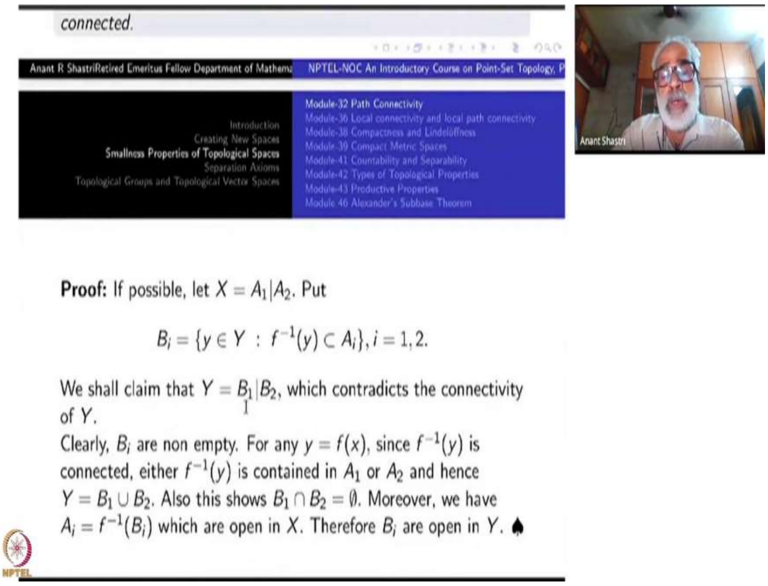

So, it is quite a profound result, but the proof is very easy here. All that I do is take a separation of X. Instead of $A|B$, I will call it $A_1|A_2$, does not matter. Take a separation of X, look at all points y below ok. So, there are points of y such that the fibre over that point is contained inside A_i ok.

Denote it by B_i, I am just defining B_i. This B_i is a subset of Y. So, this I do for both i equal to 1 and 2 ok. As such you know B_i's could be empty. there could be nothing right? But here I am using two things namely f is surjective, indeed, it is actually a quotient map and fibres are all connected.

So, together, these things will imply, first of all that these B_i's are nonempty. $B_1 \cup B_2$ the whole space X that is comes that comes from the surjectivity and sorry. Before that f^{-1} of each point namely fibres of f being connected, must be inside A_1 or A_2. Therefore $A_1 \cup A_2$ will be the whole of X ok? What we claim is that $B_1 \cup B_2$ is whole of X right? Because each $f^{-1}(y)$ is contained is either A_1 and A_2. So, union of all $f^{-1}(y)$ is the whole of X. Therefore, Y will be union of B_1 and B_2 ok.

What we want to show is this: we started with Y as a connected space, but this is a separation of Y. That will be contradiction. If A_1 is nonempty $f(A_1)$ will come to something it must be either B_1 or B_2 it has to be inside B_1. Similarly, f of a point in A_2 will have to be inside B_2. So, B_1 and B_2 are nonempty ok. Now, why B_1 is closed? $f^{-1}(B_1)$ is nothing but the entire of A_1. And to begin with we start A_1 and A_2 for closed subsets.

Inverse image of a closed set is closed implies the corresponding set is closed in the quotient topology, under this quotient map. Therefore, both B_1 and B_2 are closed.

Clearly, a fibre cannot be both B_1 and B_2. It has to be in one of them only right.? So, $B_1 \cap B_2$ has to be empty. So, all this show that $B_1|B_2$ is a separation for Y and that is a contradiction, why a contradiction? Because we started with a wrong assumption that X has a separation ok.

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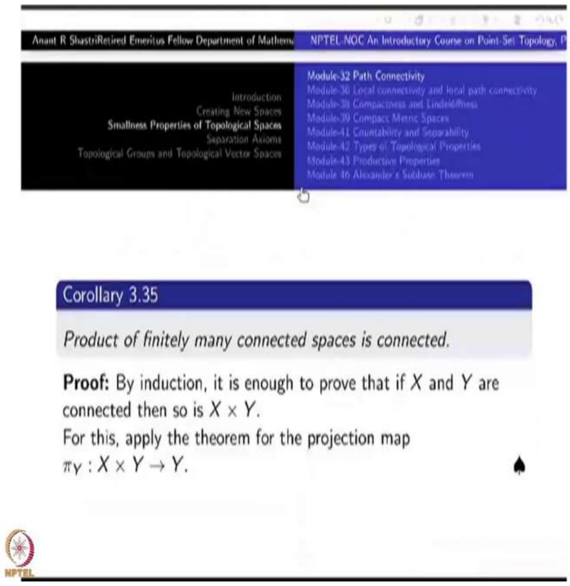

As I said, as an immediate corollary we get a big theorem here. Take any finitely many connected spaces and then take their product that is connected. How do we do this? For two at a time then by induction, it will follow for any finite product right. For two at a time how do you do? Look at X connected Y connected, look at $X \times Y$ to Y the projection map or you can take $X \times Y$ to X also no problem. That projection map is an open map. Any open surjective map is a quotient map.

So, you can apply the previous theorem to this quotient map. Y is connected, what are the fibres of the projection? Fix a $y \in Y$, all points x, y coming to y is nothing but $X \times \{y\}$, which is homeomorphic to X. Therefore, each fibre is connected because I started with the assumption that both X and Y are connected.

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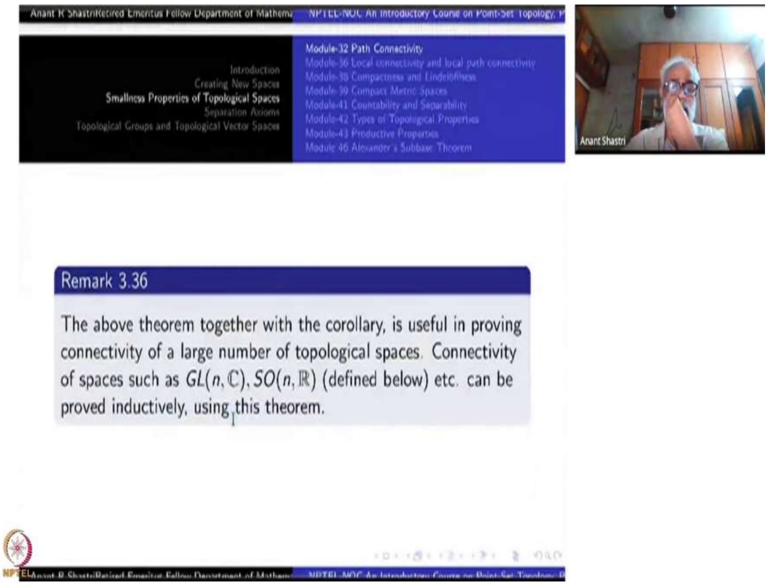
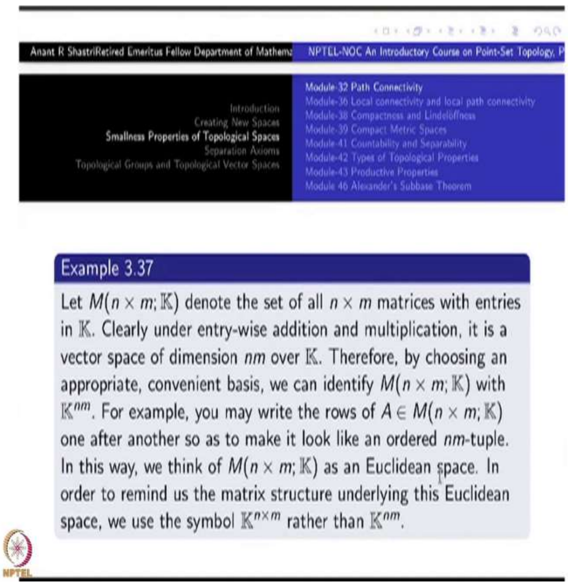


So, that is what we will do now namely what are called the general linear groups, the orthogonal groups, the unitary groups and so on they are all inside $n \times n$ matrices. So, let us make a formal introduction to these things which are very important in mathematics, in central mathematics. Of course, my $\mathbb{K}$ is any field to begin with, but finally, when you talk about topology and so on I am taking $\mathbb{K}$ as either real numbers or complex numbers, ok? Remember that. (Refer Slide Time: 09:32)



So, this notation $M(n \times m, \mathbb{K})$ denotes the set of all $n \times m$ matrices with entries in $\mathbb{K}$. Clearly under entrywise addition and scalar multiplication, it is a vector space of dimension $n \times m$ over $\mathbb{K}$. This is part of your linear algebra. Therefore, by choosing an appropriate and convenient basis we can identify this vector space with $\mathbb{K}^{nm}$, the Cartesian co-ordinate space because the dimension is nm, ok? There are many different ways of doing this.

For example, you may write the rows of A as follows: there are n rows and m columns ok? One single row is a vector inside $\mathbb{K}^m$. Write the next row, you know, on the right of the first one and so on instead of writing the one below the other. So, write them side by side, so that it will look like a nm ordered tuple. For example, if you have 2×2 matrix, then you will get a 4-vector ok? A 2-vector followed by another 2-vector, we get a 4-vector and so on.

So, that will give you an identification of this vector space $M(n \times m, \mathbb{K})$ with $\mathbb{K}^{nm}$, ok? It just depends upon which way you want to write. Any of them will be as good as any other one. So, you better choose according to your convenience appropriate to the context, but you should keep in mind that your original space has a matrix structure for many other purposes.

So, in order to remind you that instead of writing $\mathbb{K}^{nm}$ here, see mn once you write 5 into 3 as 15, the original data 3 and 5 are lost. 15 could be 1 into 15 also. So, you write it as $\mathbb{K}^{mn}$. So, this is just a reminder that it is $m \times n$ matrix ok. It is a very clever notation, it is not mine it is there in the literature. Instead of $\mathbb{K}^{mn}$, you write $\mathbb{K}^{mn}$. As a Euclidean space they will be the same, as vector space they are the same ok as matrices this should be different than $\mathbb{K}^{nm \times 1}$ or $\mathbb{K}^{1 \times nm}$ and many other possibilities.

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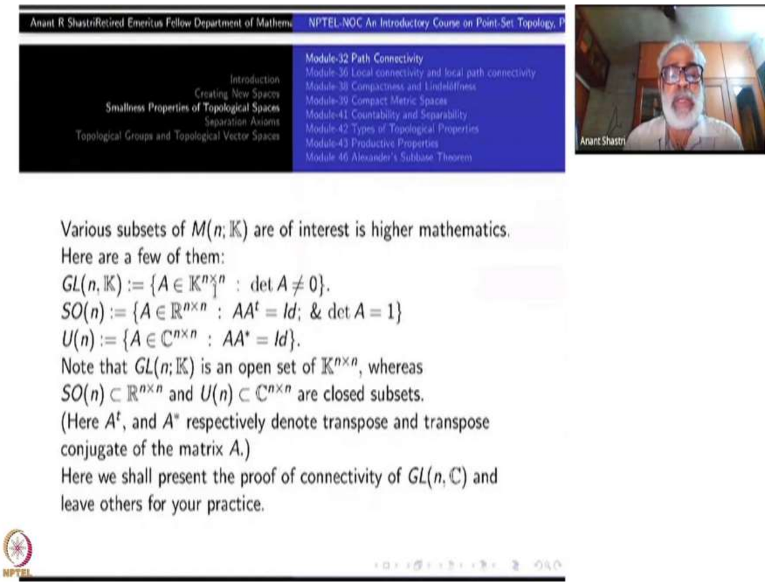

Various subsets of $M(n,\mathbb{K})$ are of interest in higher mathematics. Now, if this notation $M(n,\mathbb{K})$ stands for $n \times n$, ok. So, instead of writing two of them I am just writing n; that means, they are square matrices. Here are few of the subspaces. I am giving you the definitions. Look at all square matrices of size n such that their determinant is not equal to 0. It is the same thing as all those which are invertible. That is $GL(n,\mathbb{K})$.

Now $SO(n)$ is a subspace of $\mathbb{R}^{n\times n}$, i.e., with real entries such that AA^t is identity and determinant of A is 1. If you do not put this condition determinant of A equal to 1, then this is called $O(n)$ orthogonal group. So, this is a special orthogonal group ok.

$U(n)$ is all those complex n cross n matrices such that AA^* is identity ok. Here also you can have another subspace, $SU(n)$, wherein you put one extra condition viz., determinant of A equal to 1 ok? I am trying to give you only a few of them here, but I just told you two more also here.

Note that the first one $GL(n,\mathbb{K})$ is an open subset of $\mathbb{K}^{n\times n}$ of all $n \times n$ matrices. Why? It is given by the condition determinant of A not equal to 0. That just means that inverse image under the function determinant which is a polynomial function of all non zero scalars. You know determinant is a function into $\mathbb{K}$ right? $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{R}$. So, not equal to 0 means that is an open set. So, it is inverse image of that open set ok. So, this is an open set.

Whereas, for similar reasons viz., $SO(n)$ and $U(n)$ are given by equations, therefore $SO(n)$ and $U(n)$ are closed subsets of $M(n, \mathbb{K})$, where $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{R}$. This notation is A^t and this is A^* is conjugate transpose ok.

So, I claim that our previous theorem along with the corollary about finite products can be used to prove that all these three here at least and then many more are all connected spaces. So, let me present a proof of one of them the first one ok? When $\mathbb{K}$ is $\mathbb{C}$, I am taking the $GL(n, \mathbb{C})$ ok? If you take $GL(n, \mathbb{R})$ then you are in danger. Then you have to put determinant of A equal to 1 or -1 that will have two different components ok?

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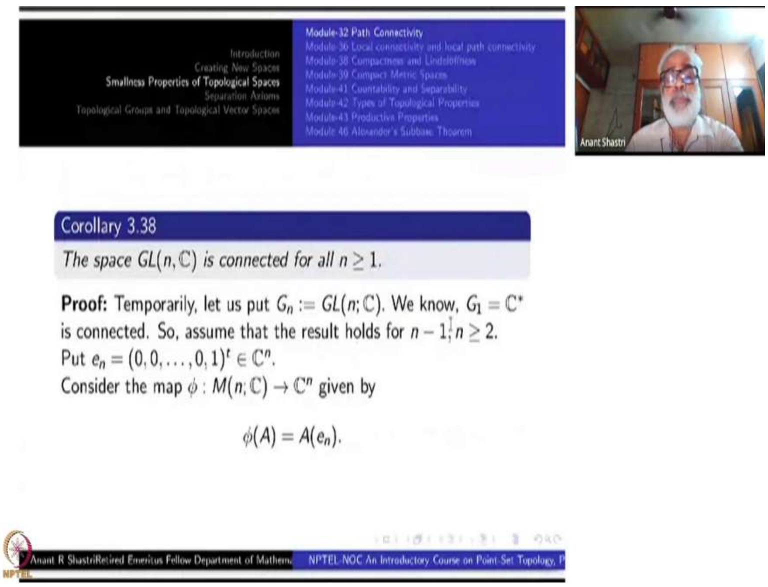

The space $GL(n, C)$, what is the space? It is an open subspace of the Euclidean space. This is connected for all $n \geq 1$. So, just a temporary short notation G_n equal to $GL(n, C)$, depends upon n right. So, I am going to prove this inductively. So, let us first prove this for $n = 1. G_1, G_1$ is just $\mathbb{C}^*, \mathbb{C} \setminus \{0\}$.

What is determinant of a 1×1 matrix? It is just the element itself and that should not be 0. So, it is just $\mathbb{C}^*$ which is $\mathbb{C} \setminus \{0\}$ ok. We know that $\mathbb{C} \setminus \{0\}$ is connected. Dropping out finitely many points from $\mathbb{R}^2, \mathbb{R}^3, \ldots, \mathbb{R}^n$ etc, they are all connected that we have seen already ok.

This formula is nothing but the last column of the matrix A. Writing the last column is what? You know, it is a bunch of coordinate functions its one of the coordinate functions. So, therefore, ϕ is open surjective mapping right? It is an open surjective mapping, but we are not interested in the whole space here. We are interested in only an open subset of that $GL(n, C)$ ok?

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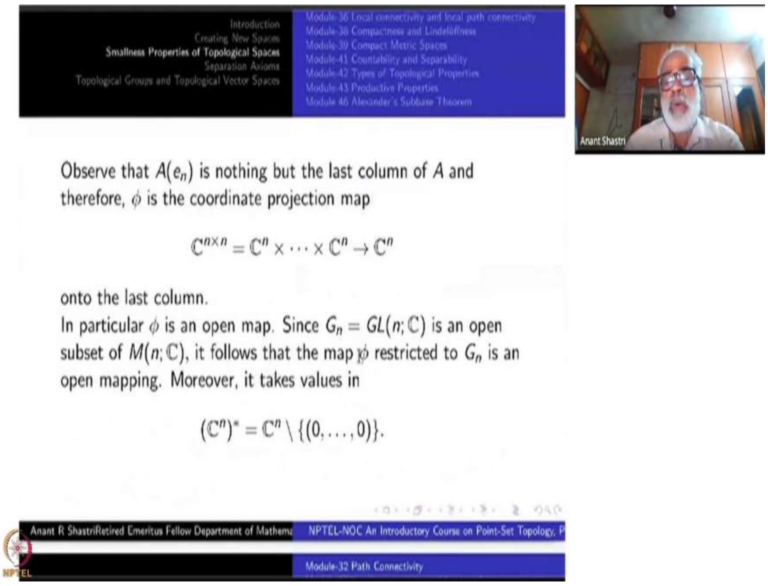

So, all this I have told you that it is just a quotient map here because it is a projection map to the last n columns ok.

In particular, ϕ is an open mapping now. $GL(n, C)$ is an open subset of matrices. So, it follows that phi restricted to G_n is also an open mapping, but what is the image? If you take an invertible matrix that is determinant of A is not equal to 0, each column is a nonzero vector that is a minimum condition. Therefore, the last column which is $A(e_n)$, it is a nonzero vector. Therefore, the function is taking value inside $\mathbb{C}^n \setminus \{(0, 0, \ldots, 0\}$. So, I will write it as $(\mathbb{C}^n)^*$.

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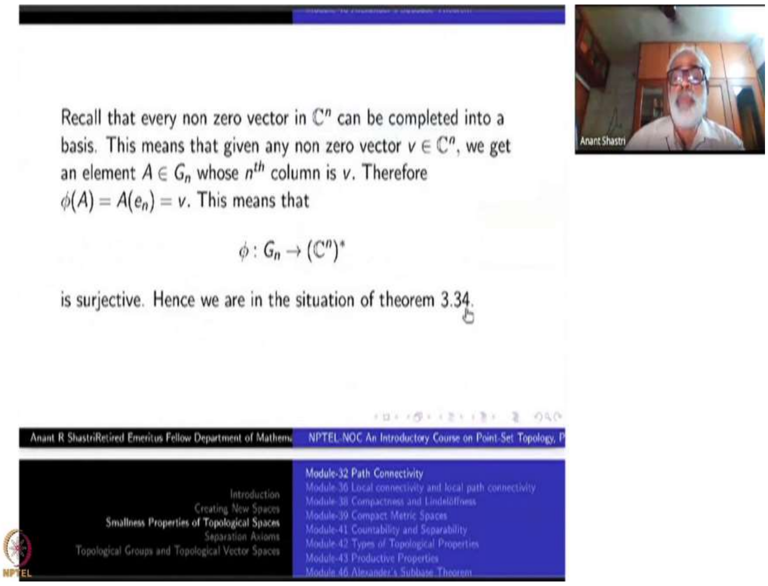

What we want to say is that this is also surjective namely from $GL(n, C)$ namely form G_n to $(\mathbb{C}^n)^*$. So, this is surjective is what I want to say. Why? Take a nonzero vector inside $\mathbb{C}^n$, use your linear algebra to complete it to a basis, how many elements will be there in the basis? Exactly n element.

The first vector you write it as the last one v is equal to the last vector then write the other $n-1$ vectors, all column vectors you treat them as column vectors you will get $n \times n$ matrix. That matrix, has columns which together span the whole of $\mathbb{C}^n$, because that is a basis. This is what you have chosen.

So, the matrix will be invertible. Determinant is not equal to 0, which is the same thing as a matrix invertible. So, what we have got is an element A inside G_n whose n^{th} column is v. It just means that if you take $\phi(A)$, it is v. Therefore, ϕ restricted to G_n to $(\mathbb{C}^n)^*$ is surjective. We are exactly in the situation of theorem 3.34 that we have just proved right. It is a surjective open mapping. So, it is a quotient map onto $(\mathbb{C}^n)^*$, what is it? $\mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C} \setminus \{(0, 0, \ldots, 0)\}$ which we know is connected.

What are the fibres of this ϕ? If we show they are connected then you are in a good shape you can immediately conclude that G_n is connected by this theorem ok?

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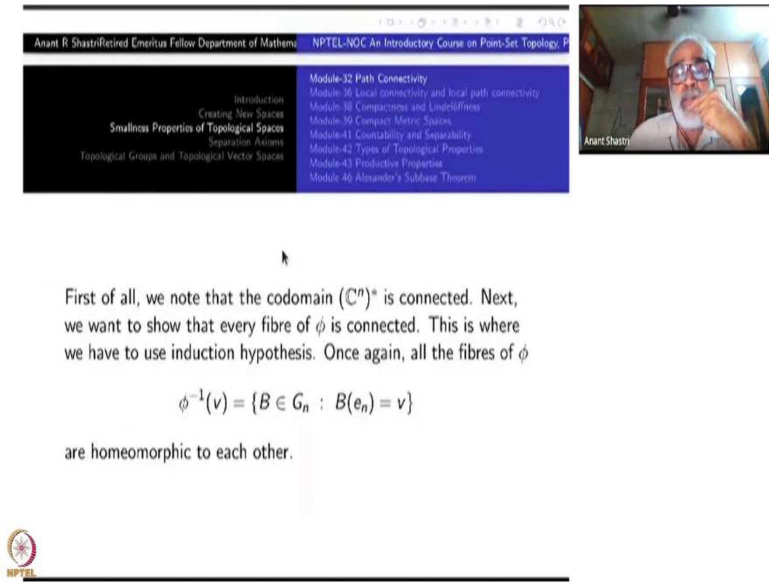

Therefore, what we need to show is that each fibre here, $\phi^{-1}(v)$ this is connected. By definition, what is $\phi^{-1}(v)$? All those $n \times n$ matrices inside G_n such that when evaluated on e_n, which is same thing as taking the last column, will be equal to v. This v is fixed right?

First thing I want to show is that all the fibres are homeomorphic to each other. What is the idea? Finally, I want to prove that they are connected right. So, if I prove one of them is connected all others will be connected. So, let me prove that they are homeomorphic to each other which is much easier to prove.

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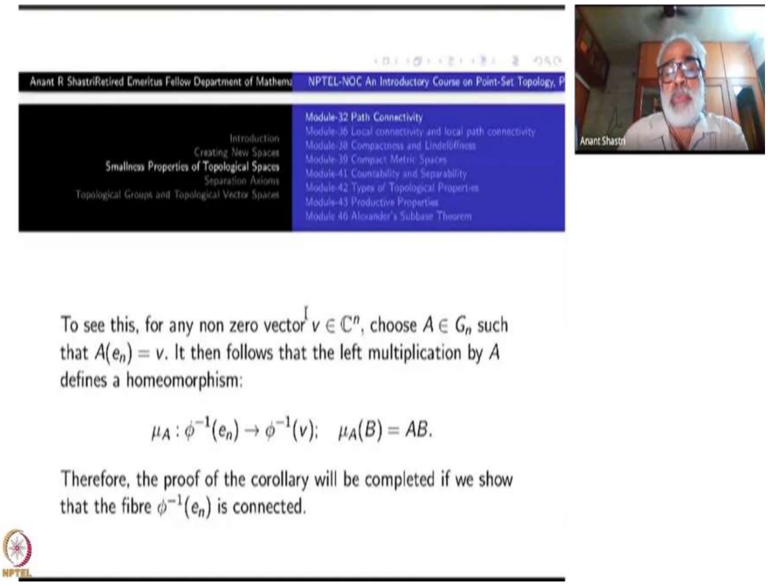

So, what you have to do? Start with any nonzero vector V choose A in G_n, this we have done earlier such that $A(e_n)$ is equal to v. Possible because we know ϕ surjective map ok?

It then follows that the left multiplication by A defines a homeomorphism of G_n onto G_n itself. Take an element here multiplied it by A on the left. What is the inverse map? Multiply by A^{-1}. So, this is a homeomorphism of the whole space to itself G_n to G_n. But I want to say that this fibre goes to that fibre. What is an element of this fibre? All those B such that $B(e_n) = e_n$. What is this one? All those say B' such that $B'(e_n)$ is equal to v right.

So, if you take an element B here, AB will have this property. So, homeomorphism is very easy to verify here.

Now comes the point. I have to just prove that one of the fibres namely $\phi^{-1}(e_n)$ this is connected. You can see why I have done this one in the next step; why the choice of e_n is prefered here, namely this allows us to apply the induction, the induction hypothesis ok.

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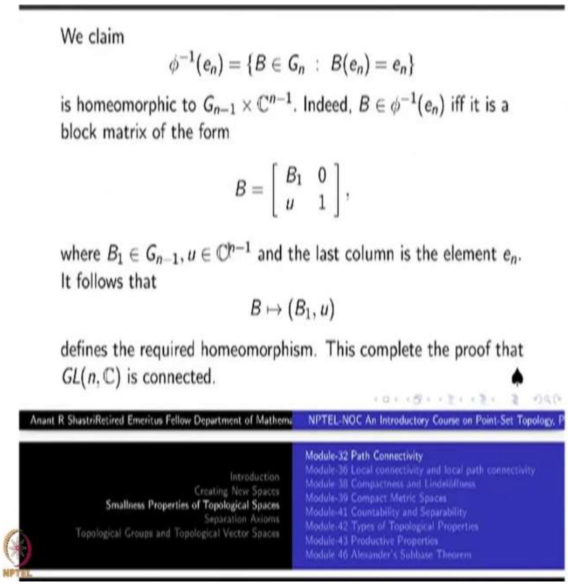

So, the claim is that $\phi^{-1}(e_n)$ is connected. So, but what is it? This is I am just repeating it here, what is it? All those B in $G_n, B(e_n) = e_n$ this is the definition. I want to say that this saying is homeomorphic to $G_{n-1} \times \mathbb{C}^{n-1}$. Suppose I have proved this one then the poof of the theorem is over ok. This was the only thing that was needed; once one fibre is connected, all the fibres are connected. So, I can apply the previous theorem along with the induction hypothesis it follows that G_n is connected.

So, all that I have to prove is that this set, this space is homeomorphic to this space. Look at what happens B is in $\phi^{-1}(e_n)$ means the last column is $(0, 0, \ldots, 0, 1)$ ok. So, this B has a last column this is a big 0, this is a block matrix by the way this is $(n-1) \times (n-1)$; this is $(n-1) \times 1$; this is $1 \times (n-1)$ and this is 1×1. It is a block matrix ok. So, this whole column is just $(0, 0, \ldots, 0, 1)$ that is the meaning that it is e_n right?

This is some block B_1 which is an $(n-1) \times (n-1)$ matrix and this is another just row here $(u_1, u_2, \ldots, u_{n-1})$. So, every B which has this form namely inside $\phi^{-1}(e_n)$ has this form, but determinant of this one is nothing but determinant of B_1 times 1. Therefore, this must be nonzero the moment this is nonzero it is an element of G_{n-1}, and what is this one this is just an element of $\mathbb{C}^n \setminus 1$.

Projection map here is to first $(n-1)$ coordinates here $(n-1)\times(n-1)$ coordinates that is continuous. Projection to these coordinates also continuous. So, breaking B into two parts like $[B_1, u]$ that is a continuous function this is what I am putting here. The inverse is also continuous from the whole things to this. Therefore, this becomes a homeomorphism, this completely proves that $G(n,\mathbb{C})$ is connected ok.

So, here little bit by little bit, many interesting topology is in involved here ok. So, the method employed here is educative not only the information that you have got now that $G(n,\mathbb{C})$ is connected ok. If you have learned this proof nicely it will help you in long way for many other things. Now, I will have to go to another important thing here now ok.

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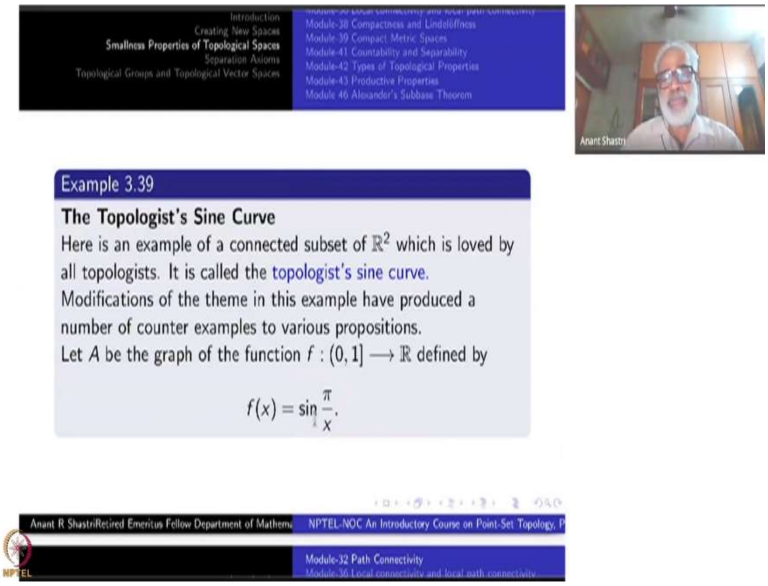

This time I may take a little more time. The topologist's sine curve. Remember that I promised that I will show you why the closure of a path connected space may not be path connected. That is what we are coming to here. This example will serve that purpose, but pay attention to this example because it will serve as counter example in many situations for us ok. For us as well as for other people also.

So, this is something you have to study properly. What is this sine curve? It is starting with the graph of a sine function only thing is instead of $sinx$ we are taking $\sin(\pi/x)$, the x is inverted here therefore, the domain should should exclude 0, at 0 it is not defined.

So, I do not want to take $(0,\infty)$, I just take $(0,1]$ that is enough for me, you can take $(0,\infty)$ for other purposes. 0 excluded 1 included. no problem. Take the function $\sin(\pi/x)$ and look at its graph, the graph is $(x, sin\pi/x)$, points like that which will be a subset of $\mathbb{R}^2$ ok. So, the whole topologist's sine curve is going to be a subset of $\mathbb{R}^2$ ok? you can just look at this curve first. Denote it by A.

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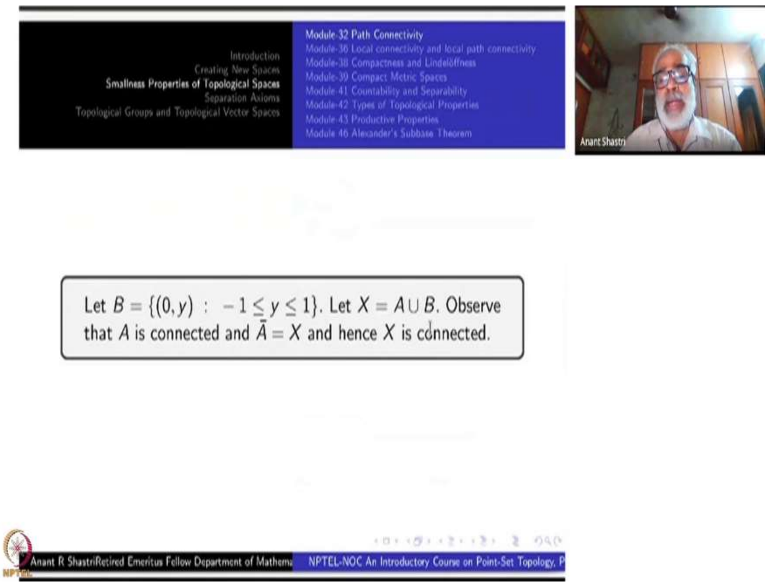

But now we want to do something more here, namely, we want to put the $y-$axis part between $-1 \leq y \leq 1$. So, $(0,y)$ where y is between -1 and 1. This a the closed line segment ok, lying on the $y-$axis; $(0,y)$. Denote it by B. So, you take the union of this sine curve namely the graph of $\sin(\pi/x)$, union with B that is my X ok. So, this is by definition the topology is a sine curve ok.

Being the graph of a continuous function defined on an open interval which is connected ok? this graph A is connected actually it is path connect this whole thing is a path it is also path connected alright. So, A is connected and $\bar{A}$ is the whole of X. So, this is what one has to see here. I will show you the diagram here then it will be very clear to you. So, X is connected A is actually path connected, but I am not going to use that just connectivity is enough to conclude that X is connected.

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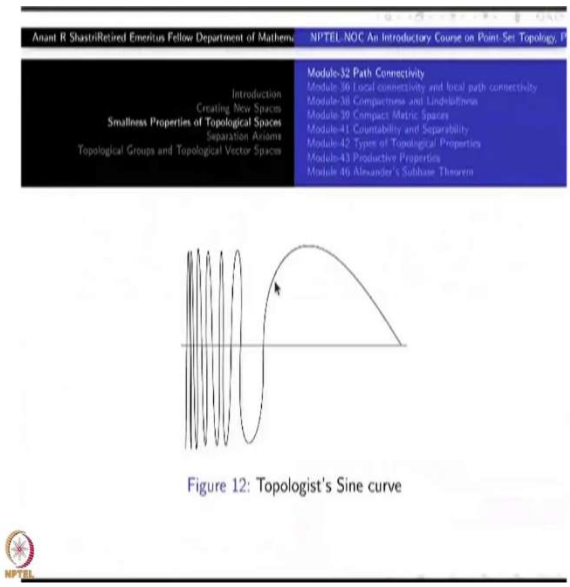

Therefore, our theorem says that this space is connected. A itself is path connected because it is actually its it is itself is a path ok, except that in the definition of path you must have closed interval. So, here the domain is not a closed interval.

So, what do you do? Suppose you take a point here from any other point here to any other point here you can take the restriction of the function $\sin \pi/x$ that will give you a path here. So, this is path connected alright? ok. So, since $\bar{A}$ is the whole of X, it follows that A is not closed ok? Of course, it is easily seen that B being a line segment that is closed ok? We claim that both A and B are path components of X that will show that X is not path connected ok?

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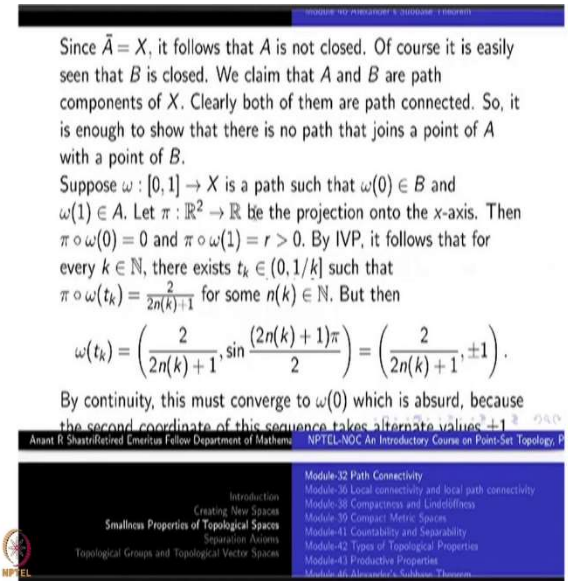

So, this is the final thing we will prove that A and B are both path components. They are path connected because, B is a closed interval homeomorphic a closed interval and the other one A is the image of a continuous function defined on an interval one side open and the other side closed does not matter, it is an interval ok. So, what we have to show is that take any point on B and any point on A, these two points you take and show that there is no path from one to the other ok? That will show that these two are, what? Path components.

See, if you have taken A what you have to do? bigger than A you have to take a point of B. So, you must be able to join a point of B to a point of A. So, I say there is no such path ok? So, if I prove this then the claim that A and B are path components follows therefore, X is not path connected, it has exactly two path components.

Suppose you have such a path. Path means now what? Some continuous function defined on a closed interval.

I can change the closed interval to any $[0, 1]$, there is no problem ok by a reparameterization. So, you can assume that there is a path ω from $[0, 1]$ to X such that $\omega(0)$ is inside B and $\omega(1)$ is inside A ok. So, remember X is a subspace of $\mathbb{R}^2$ ok. Look at the projection map π from $\mathbb{R}^2$

to $\mathbb{R}$ onto the $x-$axis, the x projection, the first projection (x, y) going to x. Then $\pi \circ \omega(0)$ this is what this is the $x-$coordinate of $\omega(0)$, $\omega(0)$ is inside B. So, the $x-$coordinate will be 0.

Because the whole of B is just $\{0\} \times [-1, 1]$ ok? So, $\pi \circ \omega(0)$ is 0. $\pi \circ \omega(1)$ will be something on the x-axis in the open interval $(0, 1)$. So, it is positive always. It is less than 1 less than or equal to 1, but it is not equal to 0. it is some positive number ok? By intermediate value theorem, see this map is from where from X to $\mathbb{R}$ ok. So, the codomain is $\mathbb{R}$. So, intermediate value theorem for this map $\pi \circ \omega$ from $[0, 1]$ to $\mathbb{R}$ ok? Not from X to $\mathbb{R}$.

But I have taken $[0, 1]$ to X and then composed with π. So, $[0, 1]$ to $\mathbb{R}$, it follows that for every integer k, there exist t_k belonging to the interval $(0, 1/k]$ (why I want to take this one? Because I want this t_k converge to 0 as k tends to infinity that is why I have made this choice) such that $\pi \circ \omega(t_k)$ is equal to $2/(2n(k) + 1)$.

So, what is the statement? Statement is look at $\pi \circ \omega(0)$ and $\pi \circ \omega(1/k)$ between these two there must be some point take that point conveniently t_k will go to that point that is the intermediate value theorem; $\pi \circ \omega(t_k)$ is equal to this point 2 divided by an odd integer, $2n(k) + 1$, ok.

Once you have chosen this way what happens is $\omega(t_k)$ will be what? Its x-coordinate is this point. So, I have to write the x-coordinate here once x-coordinate is written this $\omega(t_k)$ is some point inside the graph right? It will look like this ok. See, what I am try trying to do is if the points are on the y-axis, I ignore them, after all the curve starts from $\omega(0)$ and ends at $\omega(1)$. So, between them you have lots of parts of this path A this the graph of A. So, it must be somewhere in this graph that is what I want.

(SOME IRRELEVANT REPETITION In other words all these points see this point is the $\omega(1)$ right. Now, when you come here this must be because intermediate value theorem it must be ω of something, this must be omega of something, this must be another of something because they are all once I have chosen this one the point must be between these two, point must be between these two like this I keep choosing them closer and closer that is the whole.)

So, once you have chosen that way, the y-coordinate of this point will be $\sin(\pi 2n(k)+1/2)$ ok? So, what is $\sin$ of this one? It will be ± 1 ok and x-coordinate is this one. So, that is precisely what I have been pointing out. x-coordinate will be somewhere here or here or between these two points with that; x-coordinate will be somewhere here the y-coordinate will be that this that, this that does not matter. All of them may be this point ok.

But I will always choose something after that here one here which I can always choose them alternatively that is why I have taken its as odd numbers all these things must be there.

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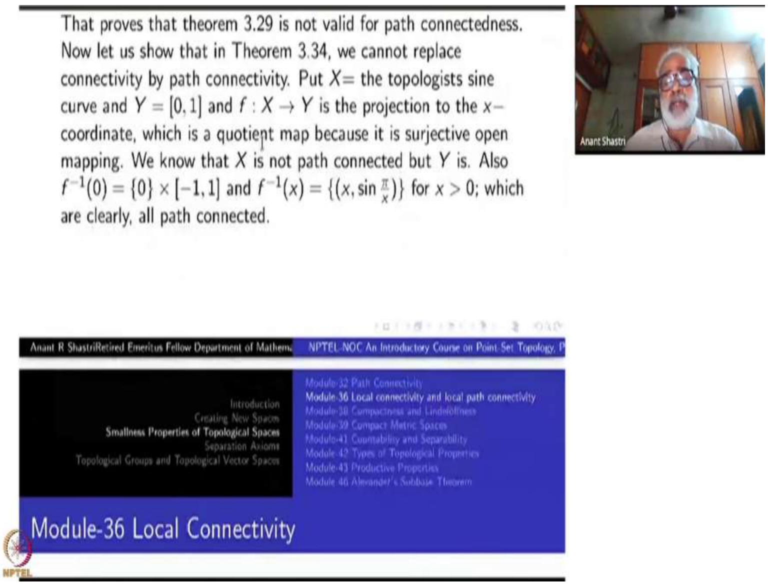

All that happens is by continuity if you take the limit of this one it must converge to some point in B what is that point? Actually it must be $\omega(0)$. This 0 it must be right. $\omega(0)$ is 0, it must come to that one because this goes to 0, but this point is plus minus 1, plus minus 1, this is not a convergent sequence.

So, this contradicts the continuity of omega. So, it proves that there is no path alright. So, that shows that theroem 3.30 is valid for path connected spaces; the closure of a path connected space need not be path connected.

Let us show that this same example gives you a counter example for non validity of theorem 3.34 for path connected spaces, ok. So, it does not take much time. So, I will show that one also. Now put X equal to the topologist's sine curve and Y equal to the closed interval $[0,1]$ and f from X to Y, the first projection, which is a quotient map. That we know because it is surjective open mapping ok?

We know that X is not path connected right, but Y is path connected Y is just $[0,1]$ ok. What happens to the fibres? The fibres of 0, take any point between $f^{-1}(0)$ is nothing but the component B 0 cross minus 1 plus 1. So, this is path connected; what is f^{-1} of any other point x where x is not equal to 0, look at this picture.

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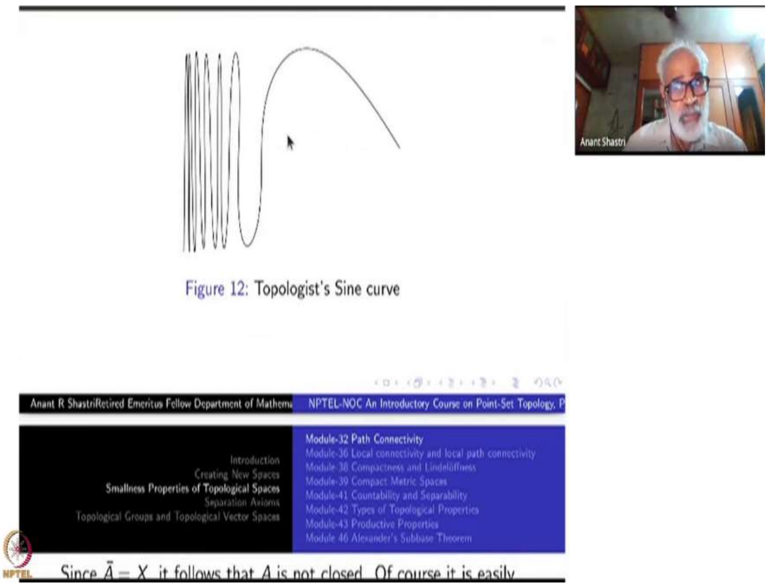

There the only the sine curve which is a graph. The graph means for each point x only one point here; each point gives only one point. What is it? $(x, \sin(\pi/x))$. Therefore, the inverse image under the projection is precisely one point here. The fibres of each point between 0 open to 1, are all singletons. So, they are also path connected.

So, the hypothesis is satisfied, everything is satisfied yet the conclusion is wrong ok. Conclusion says that X is path connected, but that is wrong. Therefore, the theorem is not valid for path connectivity ok?

So, this example, you know, has a very peculiar property. You take a small neighborhood of this point on the y-axis here, take a small neighborhood then look at the intersection of this entire thing there, it will have lots of segments of this sine curve infinitely many of them. They are all disjoint with each other.

Every point on the y-axis every neighborhood no matter how small the neighborhood is has this property. It is the union of all segments disjoint arcs, of course not exactly straight line segments. Except one, they are part of this sine curve right. So, this phenomena we want to study and make it into a property namely locally disconnectivity or locally connectivity that will be the next topic, we will do it next time.

Thank you.