Introduction to Point Set Topology, (Part - I)
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Module - 34
Lecture - 34
Connected components

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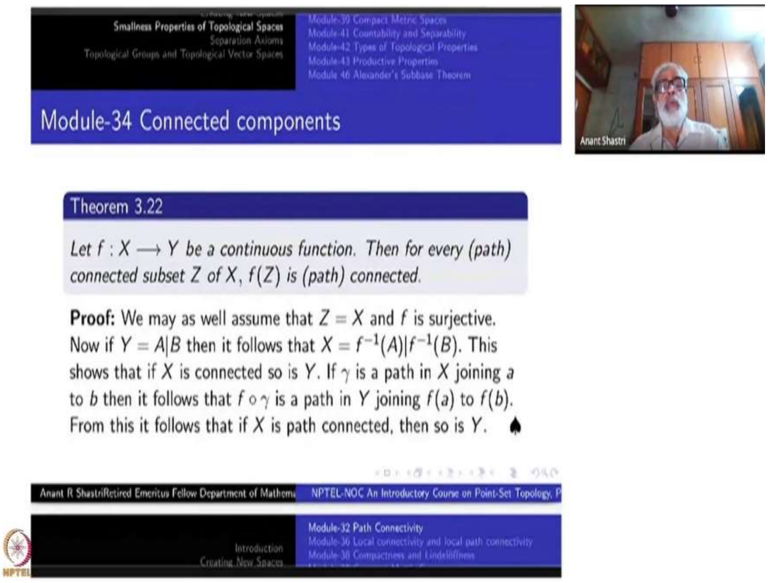

Welcome to module 34 of Point Set Topology Part-I. Last time, we introduced the notion of connectedness, motivating it from the notion of path connectedness and then, relating it to intermediate value theorem, least upper bound property etcetera. So, we will continue now the study of connectedness.

So, here is a simple theorem which says that if you have a continuous function from one space to another space, it will take a connected or a path connected subspace connected or a path connected space accordingly. So, it is actually two different statements here.

By taking a subset Z which is say connected or path connected and restricting the function to that and to the domain Z on the codomain $f(Z)$, we can as well assume that f itself is surjective and make the hypothesis that X itself is connected or path connected. So, let us first look at when X is connected, I want to show that Y is connected ok.

So, take a separation of Y, Y equal to $A|B$, it pulls back to a separation of X via f so, all that you have to do is $f^{-1}(A)|f^{-1}(B)$ that will be separation of X, pure set theoretic claim. Continuity of f is used only to see that these two sets, $f^{-1}(A)$ and $f^{-1}(B)$ are closed. Rest of the things, disjointness, union is the whole of X etc are are just pure set theoretic ok? So, that proves that image of a connected space is connected.

Now, let us assume X is path connected and we want to show that Y is path connected, but remember now, I have assumed f is surjective so, given any two points say here in Y, y_1 and y_2 whatever, they are $f(a)$ and $f(b)$ for some points a and b in X ok? But a and b inside X can be joined by a path $\gamma.\gamma$ is a path in X joining a and b. Then, $f \circ \gamma$ will be a path joining $f(a)$ and $f(b)$. that shows that Y is path connected.

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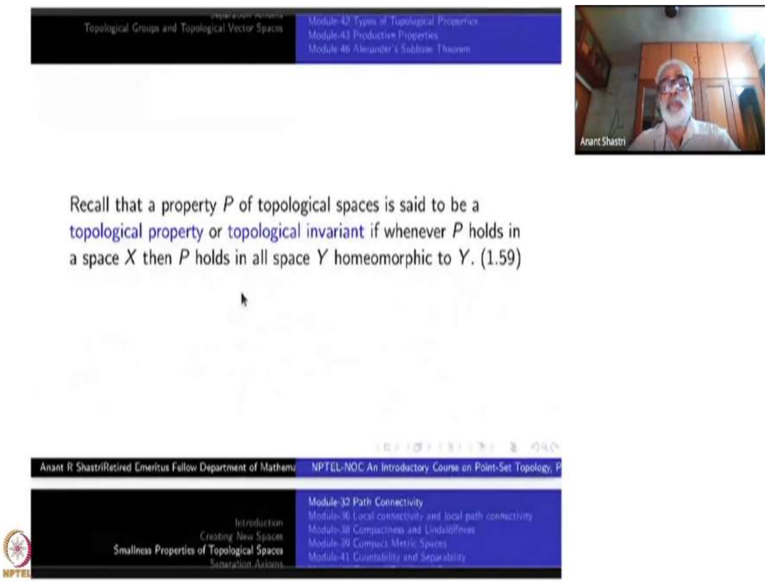

So, we had defined long back that a property of topological spaces is said to be topological property or a topological invariant if the following holds: whenever P is true for a space, it must be true for all spaces Y which are homeomorphic to X. So, that was the definition 1.59, I am just recalling it ok.

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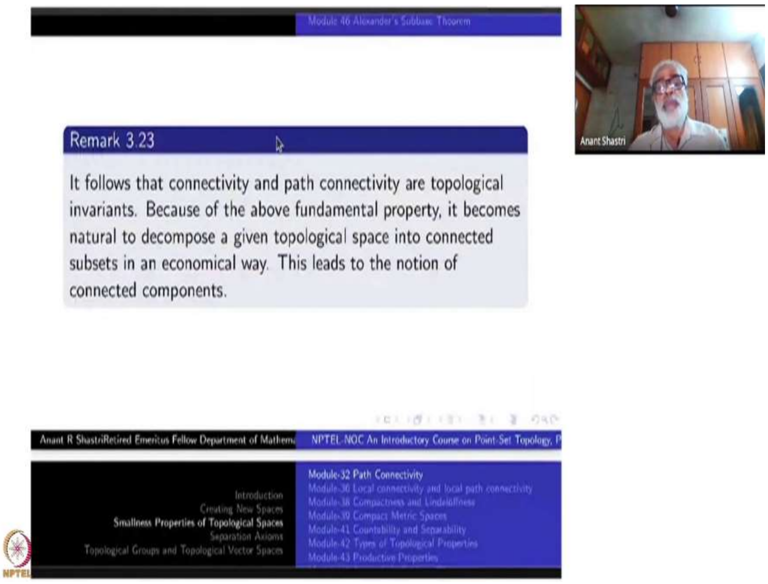

So, because image of a connected set is connected automatically under a homeomorphism, image of a connected set will be connected because homeomorphisms are are onto. So, homeomorphism preserves connectivity, similarly, it preserves path connectivity also. These two are topological invariants ok?

Because of this property, what happens is whenever you are studying some connected spaces and so on, you can actually assume the whole space is connected concentrate on a connected part of it ok and then discuss the whole thing because once we have that continuous functions from there, we will always be inside the same connected set, after all we are all the time studying continuous functions. So, lot of such discussions can be isolated just around a connected set. So, this leads us to the notion of what is called connected components.

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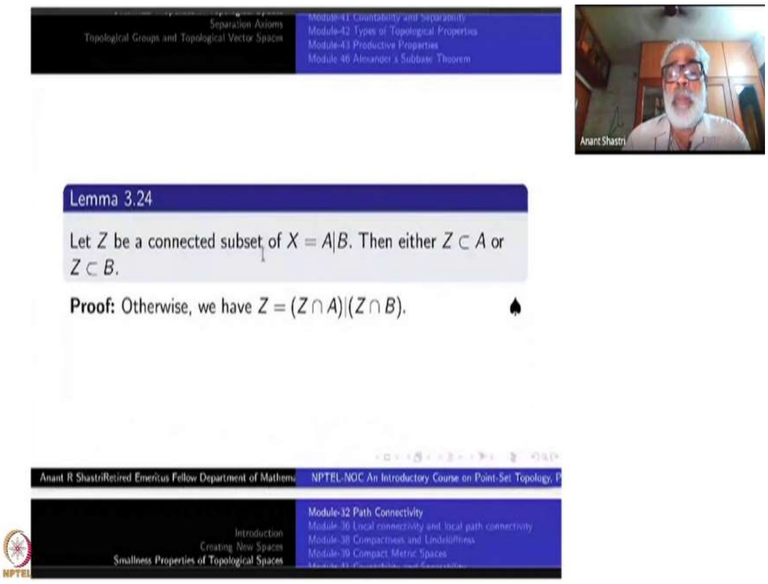

Suppose Z is a connected subset of X and X has a separation, then either Z is inside A or it is inside B. So, this is the property of a connected set otherwise what happens? You have to just take the restriction to Z of the separation. Z contained inside X, $Z \cap A$, $Z \cap B$ that is separation of Z. Only thing is you do not know whether each of these subsets are nonempty, $Z \cap A$ will be nonempty if Z is not contained inside B and vice versa if it is not contained inside A, then this will be nonempty.

So, if both of them are nonempty, that will be a contradiction to the fact that is Z is connected. So, one of them must be empty which means Z is contained inside the other one.

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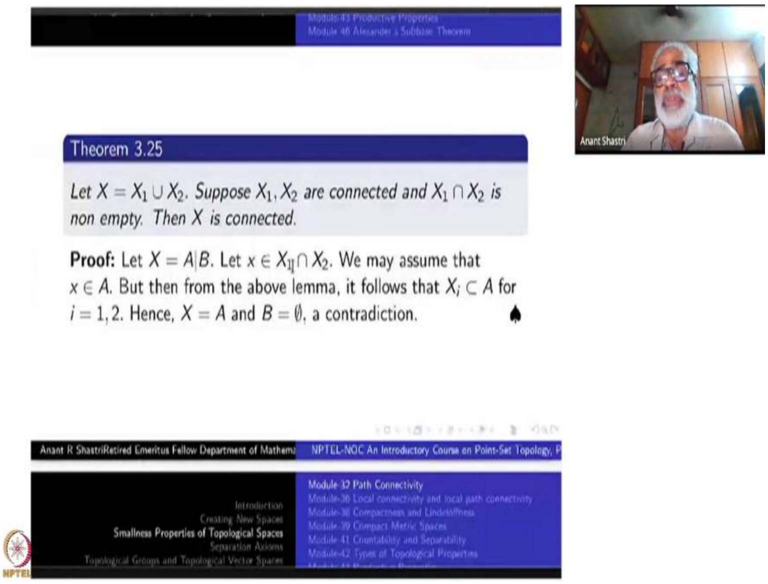

Let us understand one more thing about this connectivity. Suppose you have X as a union of two subspaces which are both connected and the intersection is nonempty, then X itself is connected. Later on we will generalize this one ok.

So, this is just a trick using the previous lemma. Suppose this union has a separation ok? I have assumed that $X_1 \cap X_2$ is nonempty so, take x to be a point in the intersection, then x being a single point, it must be either inside A or B and not both. You can assume that it is inside A, by changing the notation if you need.

But then, from the above lemma, it follows that since X_1 contains x right and X_1 is connected, ok? so, X_1 must be contained completely inside A. It is contained in A or B is the previous lemma, but since already x is inside A, it is a common point with X_1 so, it follows that X_1 is inside A. The same argument for X_2 also. So, both X_i's are inside A. That means, B is empty. So, that is a contradiction which just means that there is no separation of X and hence X is connected.

So, you can see that already this can be very easily to generalized for any family X_1, $X_2, \ldots, X_n$ etc not necessarily even finite ok. Here, I have not used any indexing set in the argument. This can be just arbitrary indexing set, it will be applicable for all. That is just an observation you can make once you have understood for X_1 and X_2.

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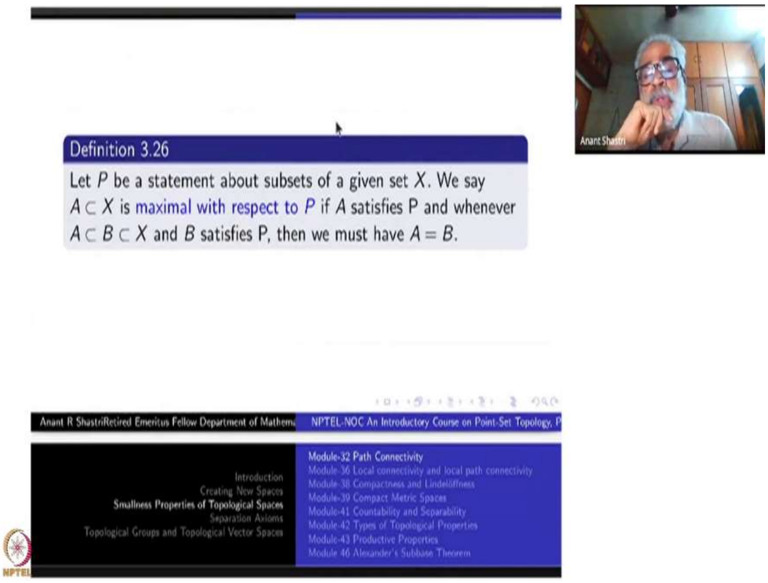

So, now we make a definition motivated by especially this lemma ok? And then, this theorem here ok. Let P be a statement about subsets of a given set X, this is general definition first of all just to tell you what is the meaning of this maximum.

We say a subset of X is maximal with respect to P, first of all it must satisfy that property P whatever. So, A satisfies P that is necessary. Then take any other subset which satisfy P, suppose B also satisfies P ok? then A cannot be properly contained inside B. A must be the biggest, A must be the maximal that is the meaning of this. So, if B also satisfies, and A subset of B then A must be equal to B.

A is contained inside B and B is also satisfy this, A must be equal to B this just means that if I take slightly larger subset, strictly larger subset, then it will not satisfy P ok. So, this property P must be for subsets of a topological space X that is the whole idea because it is a topological property. So, whereas, this definition is for all, this is just set theoretic definition, there is no topology here.

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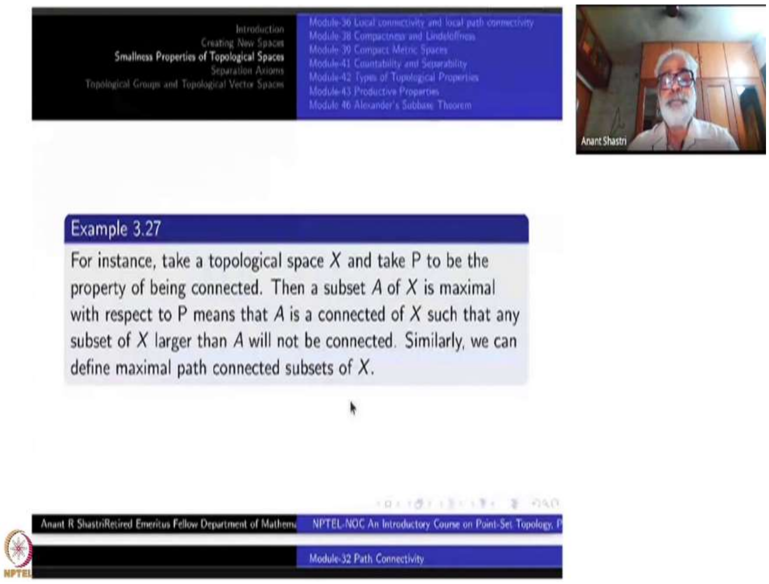

Now, we are going to make it a topological property especially take P to be the property of being connected. In other words, look at all subsets of X, collect all of them which are connected into a family of subsets of the set X.

Now, you take a maximal element A in this collection ok that means, anything every element here is connected first of all and anything slightly bigger than A will not be connected ok? That is the meaning of maximal element ok. So, we can make that as our definition now, let us see.

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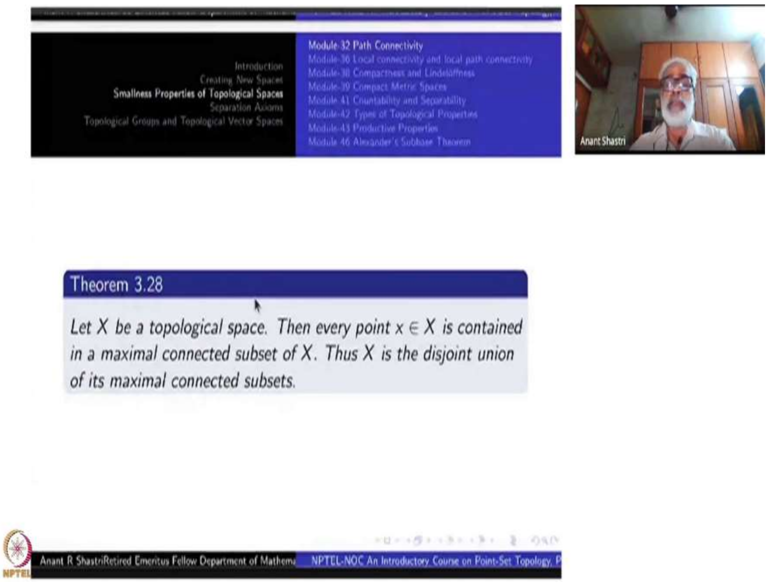

Let X be a topological space. Then every point x belonging to X is contained in a maximal connected subset. Thus, X is the disjoint union of its maximal connected subsets.

If I prove the first part, then the second part is purely set theory because every point is in a maximal set. So, union of maximal subsets will be the whole of X. So, we want to prove that each point is contained in a maximal connected subset. To begin with, singleton sets are connected right? So, they are the members of this set of all connected subsets of X alright. I want to show that it is in a maximal set, every x must be in a maximal set that is what I have to show ok.

So, now, use this property to enlarge this single point into a maximal set that is what we want to show, this property we keep using again and again so, here is the way how to do that.

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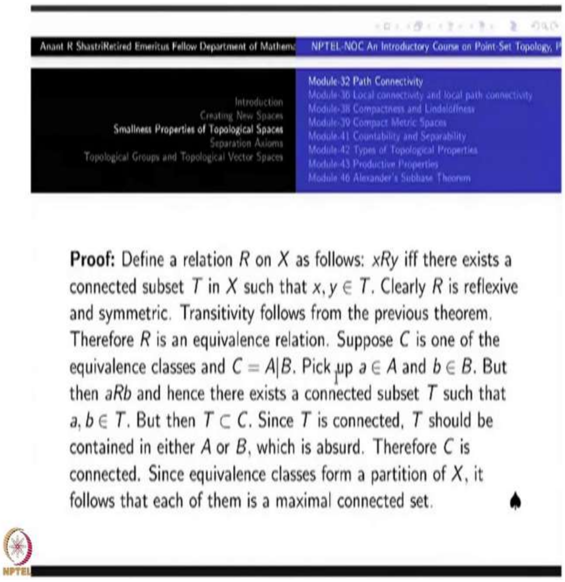

Define a relation I am saying here is a way that is not just this is the only way, there are many ways for example, you can use Zorn's lemma and so on, I do not want to use Zorn's lemma. I want to do this in a very elementary way.

Define a relation R on X as follows: x is related to y if and only if there exists a connected subset T inside X to which both x and y belong; both x and y belong to T.

Singleton sets are connected and so this relation is reflexive. Take T as singleton the singleton $\{x\}$ then x is related to x. If x is related to y, then x and y are both inside T. So, y is also related to x. Transitivity needs some proof but that is precisely what we saw in the previous theorem.

If T_1 is containing x and y and T_2 is containing y and z, then what happens $T_1 \cup T_2$ will be connected because both of them have a common point y. T_1 and T_2 are connected, they have a common point so, union is connected is a previous theorem. So, I have found a connected set which contains x and z ok? So, transitivity follows from the previous theorem.

Therefore, this R is an equivalence relation. As soon as you have an equivalence relation, you have a partition. Then, this is the second part was precisely a partition, what is partition here?

X is the disjoint union of maximal connected subsets. So, each member of a partition is a maximal connected subset. So, that part directly I am proving here ok? That will give you that every point is inside a maximal connected subset.

So, these equivalence classes I have to show, are connected, and they cannot be bigger. If you put any more point, it will be disconnected that is the maximality ok. To show that it is connected, take an equivalence class C ok? Suppose there is a partition, $C = A|B$. Pick up an element $a \in A$ and $b \in B$ ok, but they are in the same class which means a is related to b and hence, there is a common connected set T which contain both a and b right, but since T be connected it should be either in A or in B that is a contradiction, because one point is inside A another point is inside B ok?

(THIS IS IRRELEVANT AND A REPETITION).

Therefore, C is connected and there is no separation like this, there is no separation like this ok. Therefore, each equivalence class is connected right, equivalence classes are already define a partition of X. So, anything bigger than that cannot be connected, but a set which is bigger is not an equvalence class. It is a bigger class right so, these are actually maximal connected sets.

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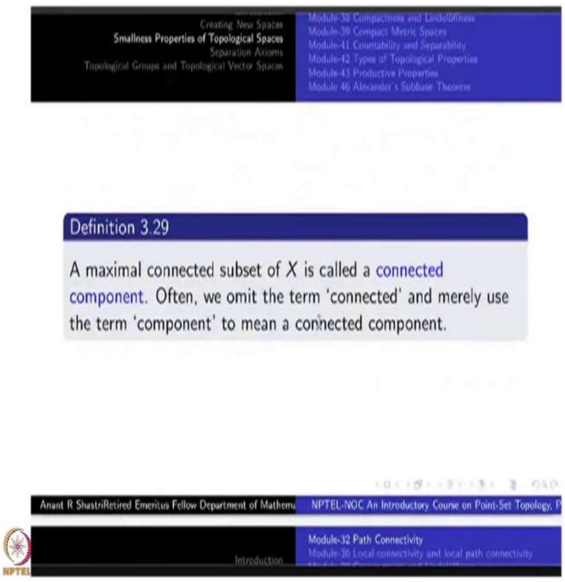

So, it is better to give a name for this maximal connected sets. After all, all the time you will have to keep on saying maximal connected, maximal connected. So, such things are called connected components and quite often when you are discussing connectivity, you will just say component that is the whole idea of naming this I mean this definition is just for the namesake precisely I mean literally it is for namesake ok.

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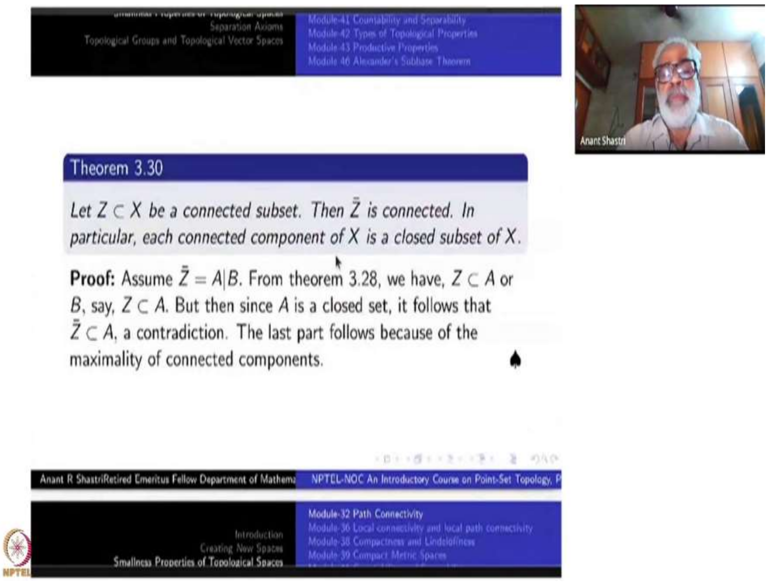

Let Z contained inside X be a connected subset. Then, the closure is connected. Now, why we are having such a thing you see? Now, we want to understand what happens to connected components, they are already giving you a partition of the whole space. Now, this theorem says that take a connected component, closure is automatically larger, but it cannot be larger, it has to be equal because the closure is also connected subset. Z being connected component, anything bigger cannot be connected therefore, there must be equality.

So, this theorem tells you immediately that connected components are closed inside the original space X ok? So, let us prove that $\bar{Z}$ is connected. Assume $\bar{Z}$ has a separation A, B ok separation, but then Z is connected so, Z is inside A or inside B. Remember if A is closed, Z is contained inside A, then $\bar{Z}$ is contained inside A, this was our old result about closures. Therefore, $\bar{Z}$ is either inside A or inside B, ok. So, there is no such separation.

So, you see several of these results we have been deducing by contradiction. Why because the connectivity itself is in that form, that is what I meant by it is some kind of a negative definition ok.

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Path connected components are also defined in the same fashion, what is it? Maximal path connected subsets of a given topological space ok. So, you can also say that they equivalence classes where the equivalenece relation is that x is related to y if and only if there is a path from x to y. No need to have that path connected subset and so on, the image of a path is already a path connected space. So, this is slightly easier to digest, this equivalence relation.

In fact, usually whatever happens to path connected space you can try to copy it for connected spaces if it works it works so, that is the way that perhaps this theorem has been used here, this definition has been obtained ok. So, path connected components also give you partition of the whole space ok. However, one has to be very careful when you keep saying it is same thing, same thing, same thing,.There is lot of close relationship, but but they are after all different notions so, somewhere they will be different. So, that is what you have to be careful about.

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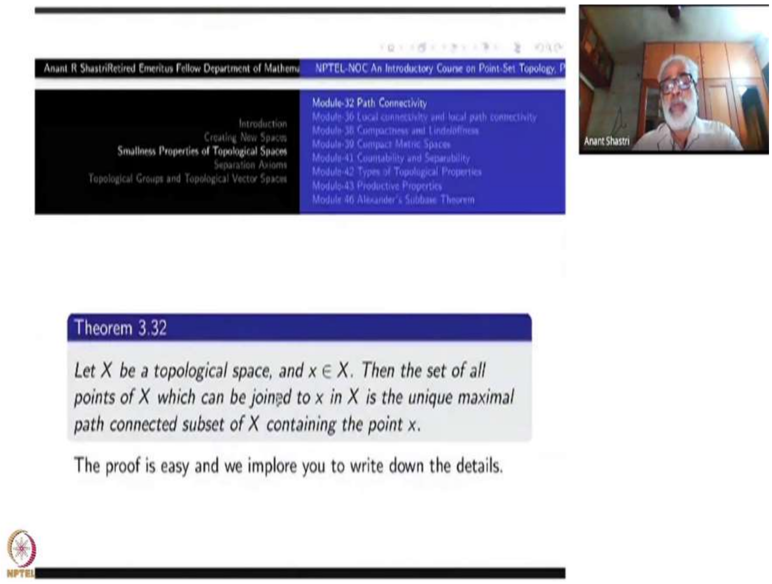

Let X be a topological space and x be any point in X. Then the set of all points of X which can be joined to x in X is the unique maximal path connected subset of X containing that point. So, this is another description of path connected components. Start at any point in a space, look at all those points which can be joined to that ok so, that has to be obviously, that is path connected, that has to be the component. If there is another point that can be also joined so, it is already there that is all, very very easy to look at this way ok. So, there is no need to write down formally the proofs of this one.

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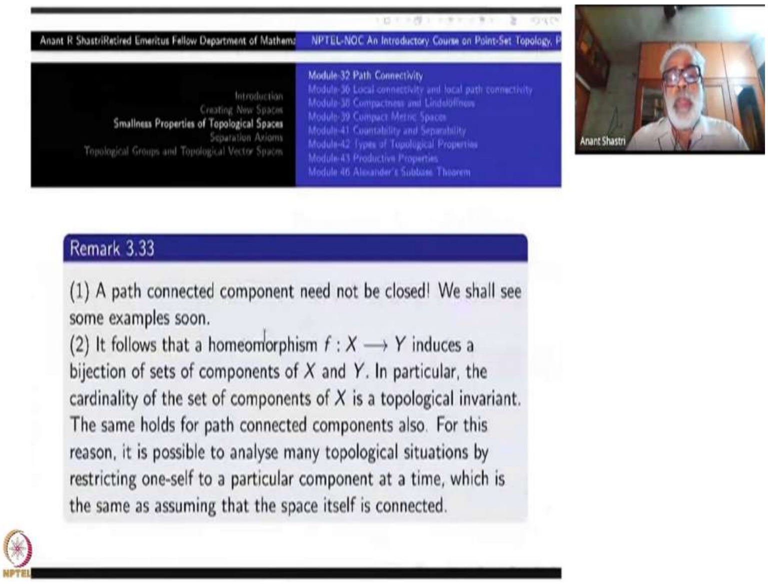

This is where I want to caution you that path connected component need not be closed. In other words, if Z is path connected, $\bar{Z}$ may not be path connected. If that was the case, then components would have been closed. So, we will see an example little later ok.

Secondly, if you have a homeomorphism from X to Y, then look at the path connected components of X ok, they will go in one-to-one fashion to path connected components of Y. Exactly, the components of X, connected components of X, they will go in one one fashion to connected components of Y ok.

In general, path connected components are connected also that we have seen right, but number of path connected components may be larger than the number of connected components, but this correspondence is true for both of them ok. Therefore, what happens is suppose you want to analyze a homeomorphism limit arbitrary X and Y and something, you can do that by component wise restrict f to one component here, when you go to the image, it will be another component there.

Therefore, right in the beginning you can assume that both X and Y are connected by restricting the whole thing to a connected component ok. So, this is how I already told this one, but I have repeated it now again alright. So, next time we will make it sure that you will be able to see a counter example also for this. In any case, we have a lot of work to do about connectivity alright. Simultaneously whenever such things are true for path connectivity, we will keep uh informing you or we will keep pointing out to you that is all ok.

So, until next time so, let us stop here.