Introduction to Point Set Topology, (Part - I)
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Module - 33
Lecture - 33
Connectivity

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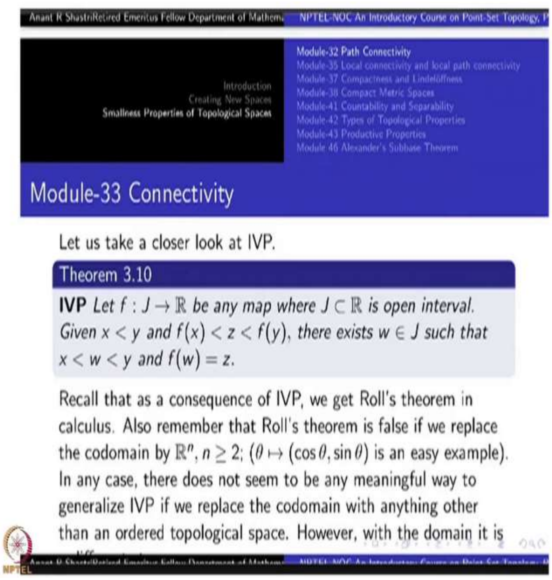

Welcome to module 33 of Point Set Topology part 1. Last time I mentioned some property of real numbers namely, if you remove a point from an arc or from $\mathbb{R}$, then it gets disconnected. How does one prove that? Intermediate value theorem from real analysis. That will give you automatically that $\mathbb{R}$ minus any point is not path connected. So, intermediate value theorem is something which is built-in in the definition of or in the construction of or in the creation of real numbers.

Let us take a closer look at it. Because it is very very relevant to the concept that we are trying to develop here. So, I am just recalling it here, I am not going to prove intermediate value theorem. Let f from J to $\mathbb{R}$ be any map, where J is some open interval. Given $x < y$ and $f(x) < z < f(y)$, there exists a w inside J such that w is between x and y and $f(w) = z$. So, that is the intermediate value theorem.

As a consequence of intermediate value theorem you must know, you must recall that we get Rolle's theorem in calculus about functions which are differentiable in an open interval and blah blah blah right? Also remember that Rolle's theorem is false if we replace the codomain by $\mathbb{R}^n$, where $n \geq 2$. Simplest example I would like to recall is θ going to $(\cos\theta, \sin\theta)$ which is differentiable everywhere ok? The derivative is never 0.

But several pairs of points may have same same value and so on. So, this is an easy example where you do not get Rolle's theorem. In any case there does not seem to be any meaningful way to generalize intermediate value theorem, if you replace the codomain with any thing other than some ordered topological space. Because you want to say intermediate value what is the meaning of an intermediate value given two values what is the meaning of intermediate value that does not make sense unless there is an order.

So, you must better take some order topology that is all. So, that is one way of making sense out of IVT. Otherwise, things do not work and so on. But you do not give up. So, try to keep the codomain as $\mathbb{R}$ itself and try to look at what is happening in the domain. Why do we need the domain to be an interval itself all the time? That is not obvious. So, here is an example wherein the domain need not be an interval ok.

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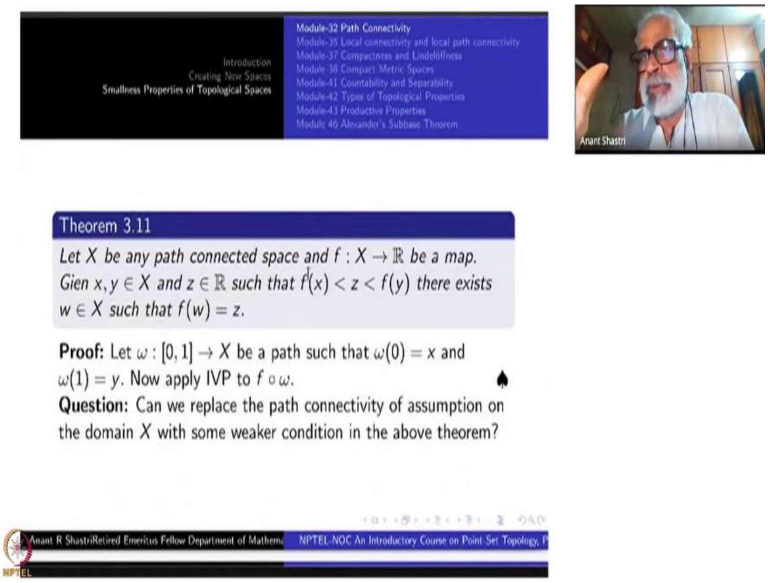

So, let X be any path connected space, we have defined what is a path connected space right? Recall that a path connected space means any two points can be joined by a path. So, take such a space X. Let f from X to $\mathbb{R}$ be any map, map means of course, continuous function. Now given x and y in X there is no way of saying that $x < y$ or $y < x$ and so on ok. But in the codomain, z is between $f(x)$ and $f(y)$, makes sense, these are elements of $\mathbb{R}$. So, then I say that there exists a point w inside X such that $f(w)$ is equal to z.

So, it is at least like a part of the intermediate value theorem. Intermediate value has been obtained though the point wherein it is attained that point maybe you do not know you cannot compare it with x or y, because there is no order in X. This is quite a satisfactory generalization of intermediate value theorem ok? So, how does one prove it? One line proof, you have two points x and y inside X. Join them by a path ω from $[0,1]$ to X such that $\omega(0) = x$ and $\omega(1) = y$.

Now, we apply intermediate value theorem to $f \circ \omega$, ω is continuous and f is continuous the composite map is continuous. This will start from $[0,1]$ and ends into $\mathbb{R}$. So, IVP is applicable ok. So, you get a t between 0 and 1 such that $f \circ \omega(t)$ will be equal to z which is between $f(x)$ and $f(y)$. This $\omega(t)$ is precisely the point some w here ok. So, put $\omega(t)$ equal to w, then $f(w)$ is equal to z. Over.

So, now comes the next question. I have done something fine, but I am not satisfied, can I replace the path connectivity assumption on X ok? assumption on this domain here with some weaker condition in the above theorem ok. Finally I do not want to get into this paths at all, is there some way of telling that? Why I am asking that question? You may ask why this question at all and you know you can question the question itself.

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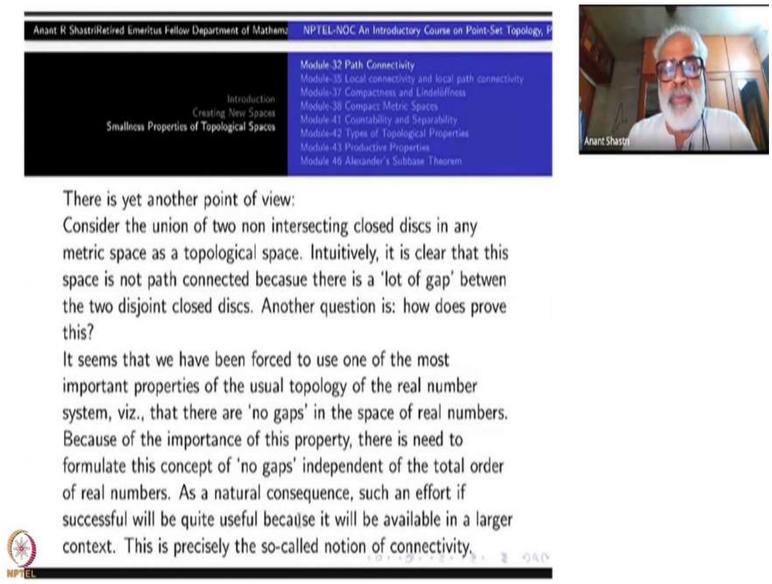

So, there is another point of view. Instead of answering that I will take note of it and look at the thing in another way. Consider the union of two non-intersecting closed discs inside a metric space. If you do not want too much of that you can do it in $\mathbb{R}^2$ or $\mathbb{R}^3$ and so on any $\mathbb{R}^n$, but there must be non intersecting non empty closed discs let us say ok? Immediately your intuition tells you that there is no path from one to the other ok? So, that space the union of two non-intersecting disjoint closed discs is not path connected why?

Have you proved it? Now you cannot use intermediate value theorem. Intermediate value theorem needs that the codomain to be an interval in $\mathbb{R}$.

If you are doing it inside $\mathbb{R}^n$, there may be a way to convert it like we did the conversion here, but I am taking arbitrary metric space ok? But you still feel that this is true right? Why? There is no path from one point in the closed disc this to a point in the other closed disc right. Now how to make this idea rigorous? So, this is the question, how does you prove this one? That is all ok.

So, it seems that we have been forced to use one of the most important properties of the usual topology of the real numbers here that in the real the numbers there are no gaps. So, this 'no gaps' concept, it may be difficult to understand in the case of arbitrary metric spaces, but we seem to understand it in the case of real numbers. In fact, this filling up the gaps was the motivation of construction of real number, the gaps were there in between the rational numbers or any algebraic numbers and so on ok.

But, so we have been able to convince ourselves that the real number system has no gaps right? So, that property we come back again and again we have to use that one. So, because of the importance of this property, there is a need to formulated this `no gaps' you know this concept, independent of the order of real numbers. It is a possibility that as a natural consequence of such an effort, if this effort is successful, we will be able to use it meaningfully. This can quite useful because it will be available in a larger context.

So, this is precisely the so called notion of connectivity. So, all this I am talking to is to motivate why we need the nonintuitive notion of connectivity as compared to very very intuitive notion of path connectivity which is very easy to understand ok?

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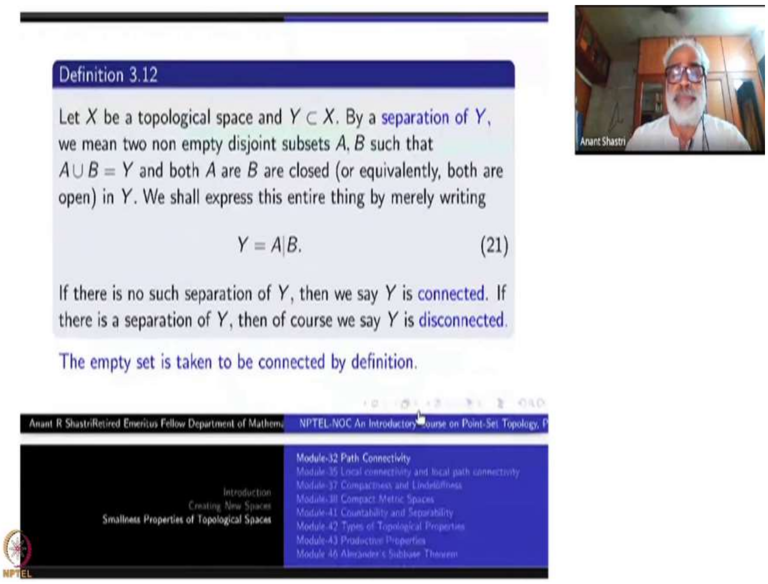

So, let us see how connectivity answers these two questions that we have raised right now. Let X be a topological space and Y be a subspace. So, I am making some definitions now. By a separation of Y, we mean two non-empty disjoint subset A and B such that the union is equal to Y, both A and B are closed, OR equivalently, both are open in Y.

So, all these things I express by just writing very cryptically Y equal to $A|B$, Y equal to A separate B. You read it as A separate B. So, this vertical line is like, you know, a cardboard partition. So, remember what are the conditions: both of them must be non-empty; they must be disjoint. Both of them closed or both of them open, there are two different ways of looking at them. The union must be Y. So, these are the things which you have to remember all the time just by this symbol ok.

If there is no such separation, i.e., the negation of this is very important. Even if one of these condition is not satisfied it is a negation. So, you have to understand that way ok. If there is no such separation of Y then we say Y is connected.

So, connectivity which is which is a very nice concept finally, is defined in a negative way here ok. So, but I have tried to put it in a positive tone --- if there is no separation. So if there is a separation, then we say Y is disconnected.

I would like to include the space empty set also in the definition. It does not come all that easily? Because I am assuming A and B are non empty here? So, an empty set cannot be union in this fashion. Therefore empty set is connected by definition ok? So, this is you know this actually forced by this definition. In any case, this is the convention.

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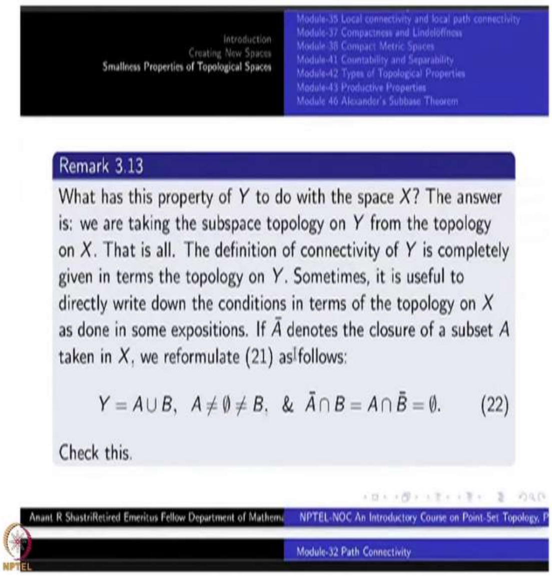

What has this property of Y to do with the space X? See I started with Y inside X ok? X is topological space and Y is subset ok.

See the phrase closed in Y; that is the phrase that will tell you what is that relation, what is the meaning of closed in Y? In the subspace Y, so I have to take the subspace topology from X to Y ok? That is the cliche here. Once you take the subspace topology you can forget about X. Then you can talk about closed in Y to mean closed subspace of Y, that is all ok? So, only to get the subspace topology this X is there that is all. So, what is what has this property to do why to do with is the space X? The answer is we are taking the subspace topology on Y from the topology that is all.

The definition of connectivity of Y is completely given in terms of the topology on Y. Sometimes it is useful to directly write down the condition in terms of the topology on X. As done in some expositions they give you a different version of the same definition. So, that definition will become a consequence of this definition and vice versa. So, let me give you that, what is that? Let us denote $\bar{A}$ denote the closure of a subset A taken inside X, the closure is taken inside X.

For every subset A which may be subset of Y does not matter ok, then instead of writing this one I can write this set of conditions namely Y is the union of A and B; A and B are non-empty; finally, instead of saying A and B are closed and so on, all that I say is $\bar{A} \cap B$ is empty and $A \cap \bar{B}$ is empty ok? In particular, you will see that since A is contained in $\bar{A}$, so, A and B are disjoint. That will come automatically, ok. So, the only new thing is, instead of saying that A and B are closed inside Y ok? The topology of Y is not bothered about it everything is referred to now to the topology of X.

So, this is another definition. You can verify very easily that this definition is the same thing as, I mean it is equivalent to this definition (21) here.

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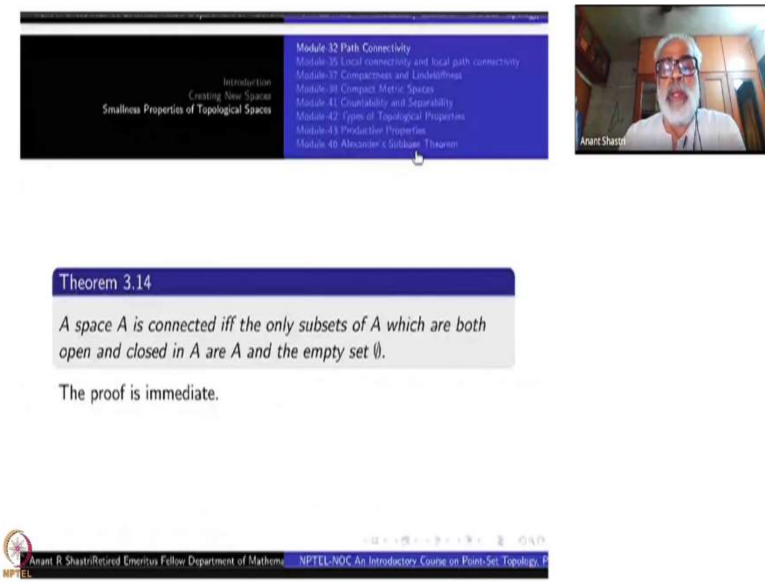

Let us carry on with this definition and do something. A space is A connected if and only if the only subsets of A, (subsets of A; I am taking A as the space now instead of Y deliberately) which are both open and closed in A are empty set and A itself ok?

So, these two sets are improper, proper subsets means what? Non-empty and not equal to the whole space. There is no proper subset of A which is both open and closed. So, that is the connectivity. This is very immediate because suppose you have say B is a subset of A which is both open and close and neither empty nor A then you can write A as $B|B^c$. Over right?

Conversely also, if you use (21) of course notation is different. If $Y = A|B, A$ and B nonempty and both are closed, or both are open, it follows that A is a proper subset of Y which is both open and closed. So, here I have not written down. So, that part at least you should do on your own.

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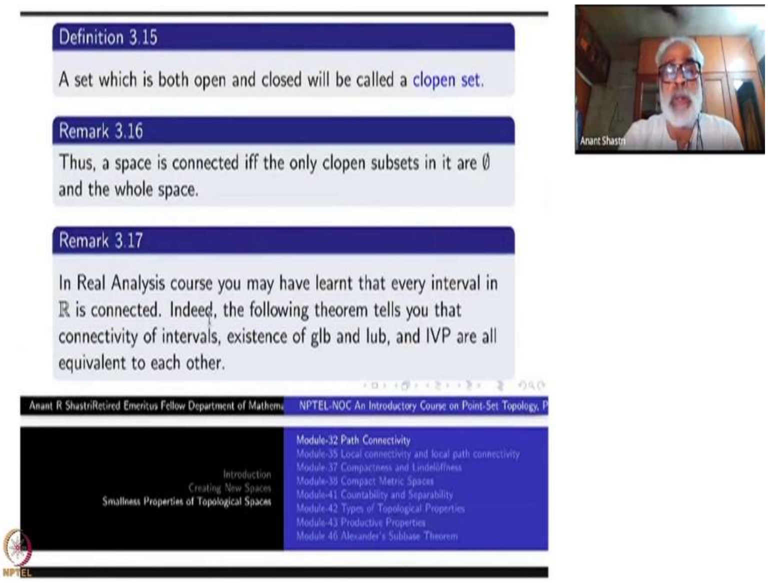

A set which is both open and close will be called a clopen set. So, this is a short for saying both open and closed that is all. So, I may or may not use this at all, but some authors use this one quite often. Thus a space is connected if it only if the only clopen subsets in it are or empty set and the whole space, the improper sets. This is just rewording this theorem you know, just reformulating this theorem that is all.

In your real analysis course you may have learned that every interval in R is connected, (I will come back to this one now), with this definition. You may have come across with this definition then what I am content, but if you have not, we will do it here do not worry. Indeed the following theorem tells you that connectivity of intervals, existence of greatest lower bound, existence of least upper bound (greatest lower bound least upper bound have go hand in hand) and the intermediate value property, these are all equivalent to each other.

Therefore, the notion of connectivity that we have introduced is pure topological. Now you see that it could have been used instead of glb and lub in the construction of real numbers ok? At least we will see that these things are equivalent right. So, I am redefining this one here the Intermediate value property.

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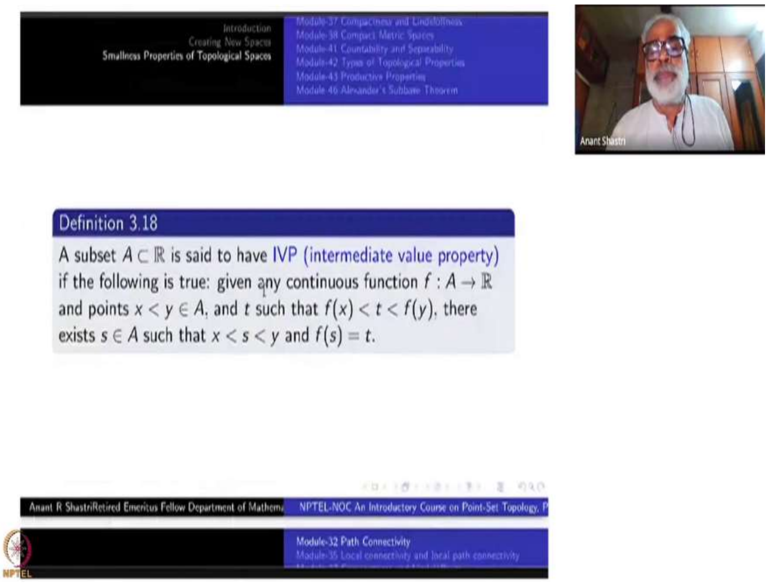
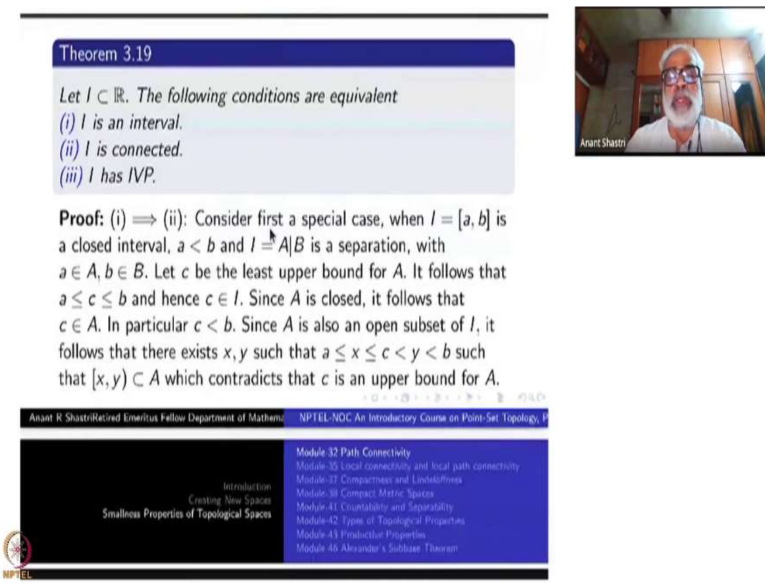


A subset A of $\mathbb{R}$ is said to have intermediate value property(IVP) if the following is true. Given any continuous function f from A to $\mathbb{R}$ and points $x, y \in A$, say $x < y$, (x and y are points of A where I have some order and so that either $x < y$ or $y < x$ that is all) and t such that t lies between $f(x)$ and $f(y)$ there must exist s belonging to A such that $x < s < y$ and $f(s) = t$. So, whatever the statement, you know, of intermediate value theorem I made it to a property. The theorem says that if A is an interval then this has this property right? So, we are going to prove that one and whatever, so that is the part of the game here now. (Refer Slide Time: 22:00)



So, let I be any subset of $\mathbb{R}$. So, I have used this notation I for usual, interval here I am not assuming that I is an interval ok? Then the following conditions are equivalent. (i) I is in interval. (ii) I is connected. (iii) I has IVP. So, I think I want to prove these things are equivalent that is the whole idea here ok. So, in particular, this will prove intermediate value property for intervals alright. So, now we are bringing the connectivity in between that.

Consider first a special case ok, where I is a closed interval ok. To prove (i) implies (ii) means what? If it is an interval it is connected is what I have to show. Suppose it is not connected. Then I must get a contradiction that it is an interval. Everybody knows the definition of an interval I hope ok? So, suppose there is a separation $[a, b] = A|B$, with a belonging to A, and b belonging to B. This is the special case. Because once it is separation both A and B are non-empty, you choose these points accordingly. Let c be the least upper bound for A.

Now I am using the least upper bound property here ok? For the real numbers. A, B are subsets of the closed interval they are bounded. Take the least upper bound for A. Closed and bounded subsets always have a least upper bound. It follows that this c must be between a and b right? Because I have assumed $I = [a, b]$ is a closed interval.

So, c is inside I ok. No botheration. Since A is closed it follows that c is inside A. So, this is another property of least upper bound I am using, least upper bounds are limit points of the corresponding set. So, in particular c is less than b, Why?

Student: sir, A and B they are disjoint, we know.

Right, they are disjoint that is all yeah you are right. So, c is less than b ok. First I said c is less than equal to b you know, since A is also an open subset of I, you see we have assumed that it is a separation. So, both A, B are open and closed ok. So, it is an open subset of I, ok? It follows that there exist x and y such that $a \leq x \leq c < y < b$. Since c is a point of A ok and A is open of course c is strictly less than b. Therefore, I can always choose some y here ok?

Now, c is less than y and that y is also less than b, ok right. This elementary property of real numbers and open subsets of real numbers. But this on side I do not know c may be equal to a

also. So, I do not know that. So, I have may have to take $x \leq a$. I have to include equality sign as well, ok? But what is the conclusion? That the whole $[xy)$ is contained inside A, why?

Because A is open in the closed interval $[a, b]$ and A is open a closed interval $[a, b]$ means for every point there is a neighborhood of this nature inside A that neighborhood can be chosen. So, that this y is smaller than b is an extra thing that is all ok. In any case if you have chosen it inside A automatically it will be less than b there is no problem, but this contradicts the fact that c is an upper bound for A because there is y also in A now bigger than c, a contradiction.

So, in other words what I have used here is: if you have an open set ok of real numbers, bounded, the upper bound or even the lower bound ok cannot be a point of the open interval. So, that is all I have used here alright. So, what we have proved? We have proved that every interval, so only closed interval is connected.

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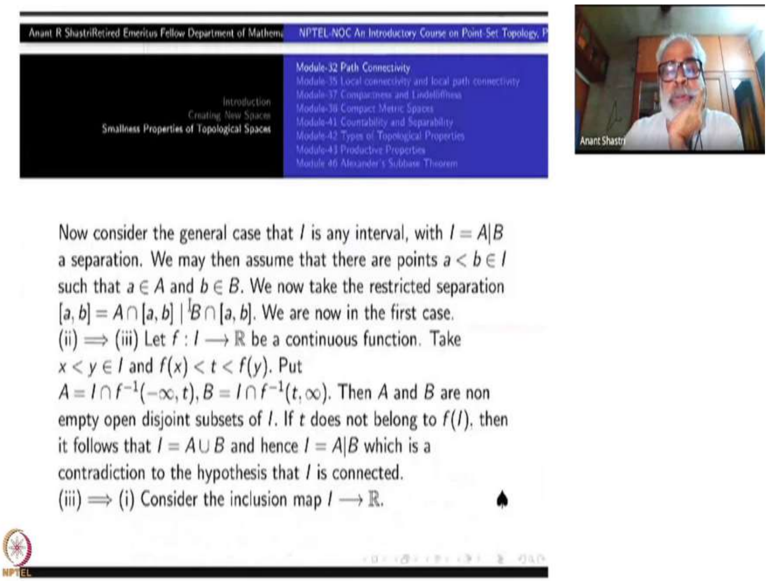

Now, let us prove the full thing, now consider the general case wherein I is any interval having a separation. We may assume that there are points a less than b belonging to I such that a is in A and b is in B ok. By interchanging A and B if necessary, one point here, one point here. Why this a should be less than b? That one I do not know. But you interchange them that is all, then you can assume that a is less than b.

We now take the restricted separation: look at this closed interval $[a, b]$ ok once a is inside A and b is inside B both of them are inside I the entire interval $[a, b]$ is inside I. I is an interval that is what is assumed right. So, the entire interval $[a, b]$ is contained in I, that is the definition of an interval. So, look at the two sets, $A \cap [a, b], B \cap [a, b]$. They will form a separation for this $[a, b]$ now. Only thing you must see that both of them are non-empty other things will be automatic.

They will be disjoint, union will be the whole thing and both of them are closed. Why they are non empty? Because a is here this is non empty and b is there fine? So, we are now back into the first case. So, the proof is over. Now let us go to (ii) implies (iii). So, what is (ii)? (ii) says I is connected ok? (iii) says it has intermediate value property ok. So, (i) implies (iii) will come automatically every interval has intermediate value property that is a theorem that you know in analysis, but here we are going to prove it.

So, first I have to connectivity implies the intermediate value property ok. So, let f from I to $\mathbb{R}$ be a continuous function. Take x less than y belonging to I, $f(x) < t < f(y)$ these are the hypothesis for intermediate value theorem ok. Then you have to find out some element here between x and y such that element goes to this t here. So, that is what you have to do right.

To prove the intermediate value property that is what you have to do. So, once these things are given to you put A equal to $[x, y] \cap f^{-1}(=, t)$. All the all the points mapped below t, all points which go to less than t are inside inside is A. All those which are mapped to bigger than t are inside B ok? Then A and B are non-empty because x is there in one and another one y is there open disjoint subsets ok.

Suppose t does not belong to $f(I)$. See we want to show that there is some $z \in I$ such that $f(z) = t$ that is the same thing as this t belongs to $f(I)$, right. If t does not belong to $f(I)$ that is the only point which is missing from here, then it follows that these two sets will cover I right? Therefore, this $A|B$ becomes a separation, $I = A|B$. That is a contradiction because we have assumed I is a connected set here. Be careful here I have never used that I is an interval, the last part I am using that I is a connected set ok.

Now, (iii) implies (i) is a very straightforward thing. If I has intermediate value property ok then we consider the inclusion map here I to $\mathbb{R}$. What is the meaning of I is an interval? Given any two points $x < y$ in I, everything between them must be there right? The inclusion map has intermediate value property means what? Because it is continuous, it must have intermediate value property any point between x and y must be there. That is all.

So, this (ii) implies (iii) is more or less topology and (iii) implies (i) is completely trivial. Just apply the property (iii) to the inlcusion map.

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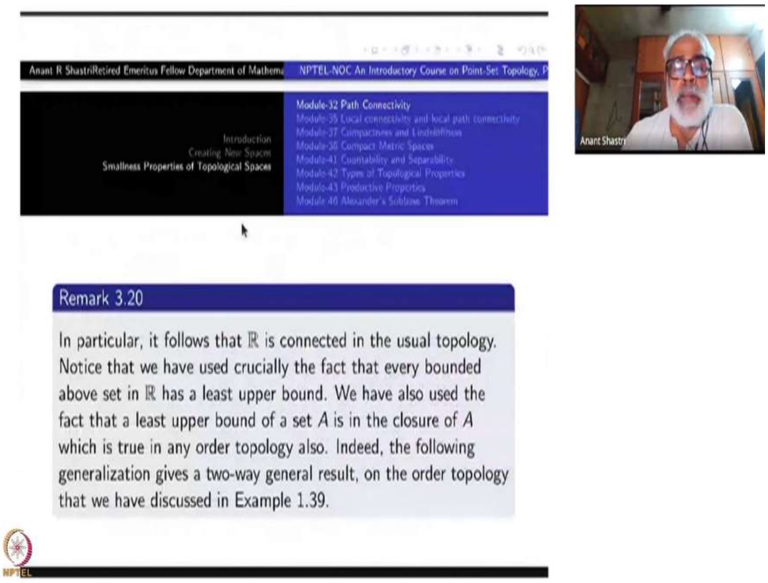

In particular it follows that $\mathbb{R}$ itself is connected, because $\mathbb{R}$ is an interval. Notice that we have used crucially the fact that every bounded above set in $\mathbb{R}$ has a least upper bound that is the only thing which you have used which is equivalent to saying that every bounded below set is has a greatest lower bound.

(REMARK: THE PROOF OF (ii) IMPLIES (iii) GIVEN HERE IS INCOMPLETE. REFER TO THE LIVE SESSION 3).

We have also used the fact that a least upper bound of a set A is in the closure of A which is true in any order topology. It is nothing special about real numbers. It is true in every order topology. Indeed the following generalization gives a two-way general result on the order topology that we have discussed in example 1.39 in the first chapter ok. So, you see I wanted to show you that these three things here are all same: connectivity, being an interval, existence and IVP. So, first I took only this one now I am attacking this glb ulb itself ok.

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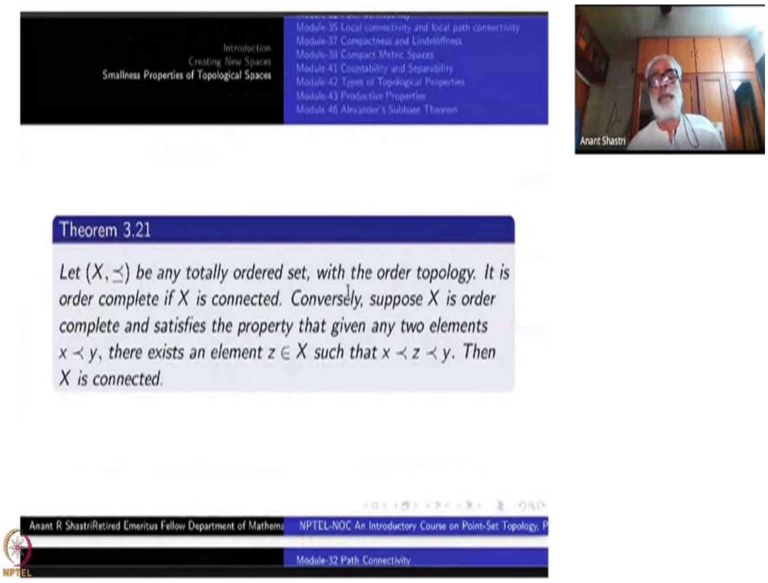

So, here is a theorem to understand this one I have to go to arbitrary totally ordered sets with a certain property I do not want to impose myself that I am looking at real numbers. That will be like in a vicious circle ok. So, start with a totally ordered set and give the order topology on it. It is order complete if X is connected. Order complete means what? Every bounded above set has a least upper bound that is what you have to prove ok?

And every bounded below set has a greatest lower bound that is the meaning of order complete. If it is connected I want to say that it is order complete. For the converse you have to assume one more condition. Conversely suppose X is order complete and satisfies the property that given any two elements x less than y, there exists a third element between them ok. This is like what this is similar to or equivalent to the Archimedean property of real numbers ok.

Then X is connected. So, connectivity is equivalent to this order completeness. Only thing you have to assume that totally ordered set which has this Archimedean property. See here there are no integers, no rational numbers, no additive structure etc. So, you have to reformulate the Archimedean property in some way this is the way it has been done here. So, all these goes back to Cantor it is not my invention or anything. So, I am doing this because many standard books do not have these things that is all ok.

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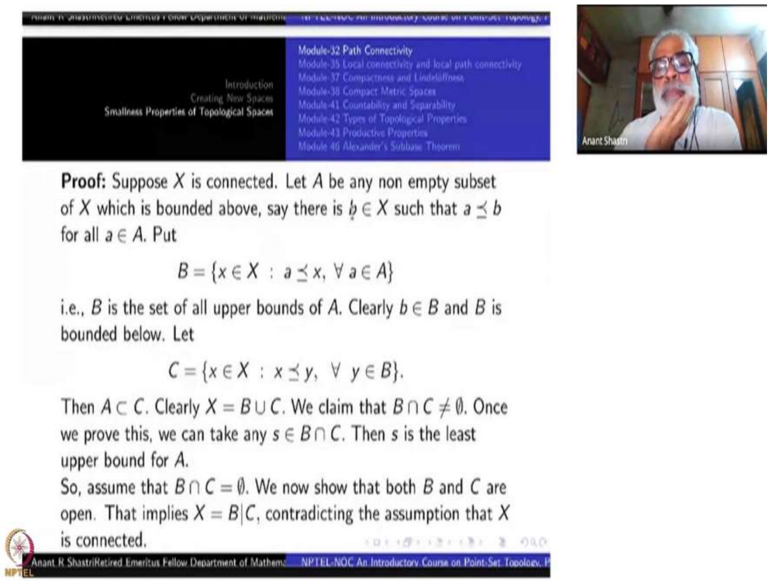

Suppose X is connected. Let A be any non-empty subset of X which is bounded above, say there is b in X such that a is less than or equal to b for all a in A ok? That is the meaning of an upper bound. Now take B to be all $x \in X$ such that a is less than or equal to x for every $a \in A$, ok.

This B is non-empty because this b is there. You start with a subset which is bounded above now you take all possible upper bounds x of A, $a \leq x$ for every $a \in A$. Then x will be there in that set ok that is B the set of all upper bounds of A. Clearly b is in B and so ok.

Now B is bounded below why? A is non empty. All elements of A are such that they are all less than or equal to elements of B, each element here. So, it is bounded below.

Let C equal to the set of all $x \in X$ such that $x \leq y$, for every $y \in B$. What I am doing here now? I am looking at all the lower bounds of B, take all of them that is C ok. Then A is contained inside C because all elements of A are already lower bounds for B.

But this set C may be very larger much larger than the set A, ok. So, A is contained inside C. Now, clearly X itself is the union of B and C. So, elements of X are either inside B or they must be inside C ok? That is all I am claiming here. What is the meaning of that x is inside B ? Inside B means what? Every element of A must be smaller than that element x ok. If that is not true, then there is some $z \in A$ such that $x < z$. But then x is less than all elements of B as well and hence is in C.

So, now the claim is that $B \cap C$ is nonempty. Suppose we prove that $B \cap C$ is non-empty then you can take any $s \in B \cap C$ then s is the least upper bound for A. Elements of B are upper bounds for A that is clear. If s is also inside C then s has to be smaller than all the upper bounds. That is, it must be the least one ok.

So, how do you prove $B \cap C$ is non-empty? If $B \cap C$ is empty, we get both B and C are first of all open subsets ok? That we have not shown. We will show that. Then it will follow that $X = B|C$. So, this will be a separation of X. But we started with the assumption that X is connected ok. So, we have to prove that $B \cap C$ is non-empty. So, if it empty then there will be a contradiction ok. So, I assuming that it is empty and then we are getting a contradiction. I will show that both B and C are open. So, that is what I have to do that now.

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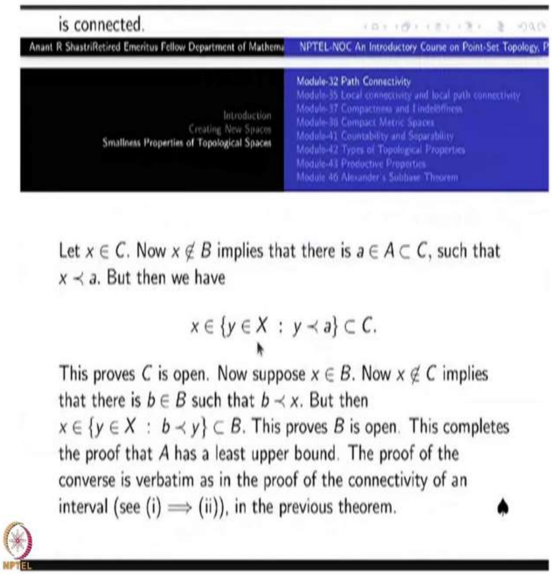

So, let x belong to C ok. Now x is not in B implies that there must be some $a \in A$ contained inside C of course, such that x is less than a, ok? Because B is by definition all the upper bounds of elements of A. So, at least one element of A must be bigger than x. So that is the meaning. But then x belongs to the set, namely all those $y \in X$ such that $y < a$, right.

So, this set is much smaller than C because C is the set of all elements less than or equal to y for all $y \in B$, ok. So, x belongs to this open set contained inside C. What I have done? Started with any element $x \in C$, I have got an open set. By definition this is an open set by the way, in the order topology on X. And this open set is contained inside C. Around every point inside C there is a neighborhood contained inside C; that means, C is open.

Same thing I want to prove for B also. Suppose x is inside B; that means, x is not inside C right. This implies that there is b in B such that $b < x$. That is the definition of C after all. If it is not in C then there is some b which is less than x. But then look at all $y \in X$ such that b is less than y. That will contain this x and is contained inside B. So, this prove B is open.

So, this completes the proof that A has a least upper bound ok.

The proof of the converse is verbatim as in the proof of connectivity of an interval (i) implies (ii). How did you prove that? Exactly same proof here. Instead of usual less than or equal to you replace it by this $\prec$ that is all this notation ok. So, this is something where I have taken you somewhat deeper. So, go through this one carefully because these things are purely logical here. that is all nothing else.

So, you can try to write down try to make a picture and so on, but when you make a picture you may be misled because then you may be already using the properties of real numbers. So, you make a picture, but throw it away and see that everything comes from pure logic. So, let me do a little more here.

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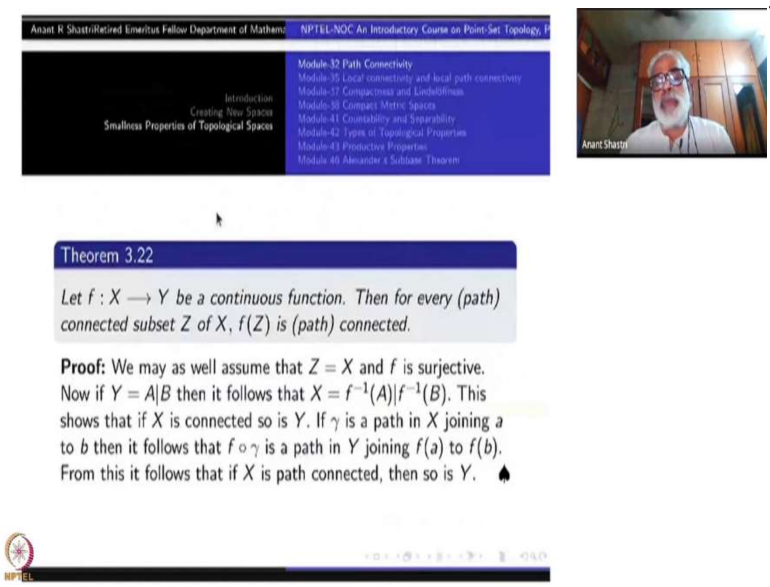

Let f from X to Y be a continuous function. Then every path connected subset Z of X, take a path connected subject here, the image of Z is path connected ok?

So, now, we are trying to prove something about properties of path connected spaces. In particular if X is path connected $f(X)$ will be path connected here. In particular if the map f is surjective and continuous and X is path connected will imply that $f(X)$ which is Y is path connected. So, other than continuity I not not assuming anything on f. If you have subset here which is path connected, it will remain path connected under f, this is the correct thing you have to do.

So, this is a one-line-proof. Assume that $Z = X$ ok, instead of now instead of taking subsets you restrict the whole function to Z, so that you can assume Z is equal to X itself and f is surjective. What do you mean by that? You replace Y also by $f(Z)$. Then it is as if you have taken a surjective map f ok? Now assume that X is connected. Suppose $A|B$ is a separation of Y, Y equal $A|B$. I must get a contradiction. Because I wanted to prove that Y is connected, First I am looking at connectivity. Later on path connectivity. You will see that is obvious anyway. So, A and B are forming a separation for Y. Then immediately it follows that $f^{-1}(A)|f^{-1}(B)$ the whole of X. Because because what? Because now f is surjective. So, $f^{-1}(A) \cup f^{-1}(B)$ will be whole of X. They are disjoint because A and B are disjoint.

They are closed because A and B are closed. They are non-empty because A and B are non-empty everything is from Y and is pulled back to X under f^{-1} everything comes back. So, this shows that if X is connected so is Y ok.

If X is path connected so is Y is easier. Because take two points inside Y, they are the image of some points here right like a and b; $f(a)$ and $f(b)$ are there inside Y. Now join a and b by a path γ in X, $f \circ \gamma$ will join $f(a)$ and $f(b)$ in Y.

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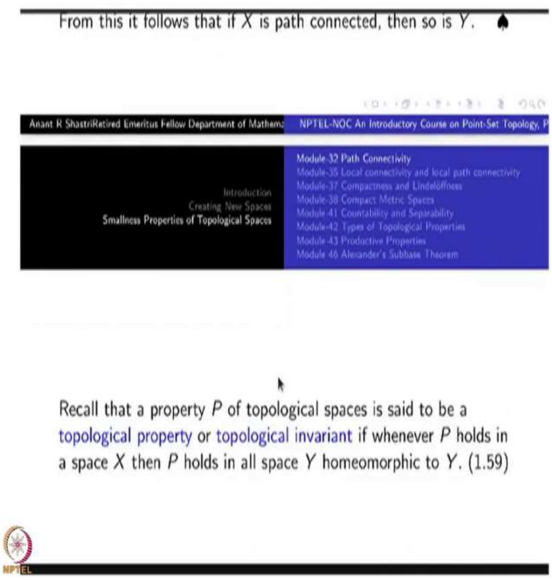

Maybe I should stop here now ok?