Introduction to Point Set Topology, (Part - I)
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Lecture - 32
Path Connectivity

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Welcome to chapter 2 of Points Set Topology course Part-1. In this chapter we discuss a number of topological properties which can be classified or which can be called smallness properties. So, what are they? I have listed them here: path connectivity, connectivity, compactness, Lindelof property, separability, 1st and 2nd countability. It is better to leave this phrase smallness properties undefined, it is not a part of mathematics ok, it is about mathematics.

So, there is no need to define this word rigorously. Let it stand as a you know whatever it implies in the English language there; however, as a temporary stop-gap, you can explain it as follows ok?

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Roughly speaking we just mean that the topology does not have too many open sets, but that is as vague as the words themselves. So, I will try to make a definition, but do not worry about that definition much. Please I am trying just to explain it. That is all.

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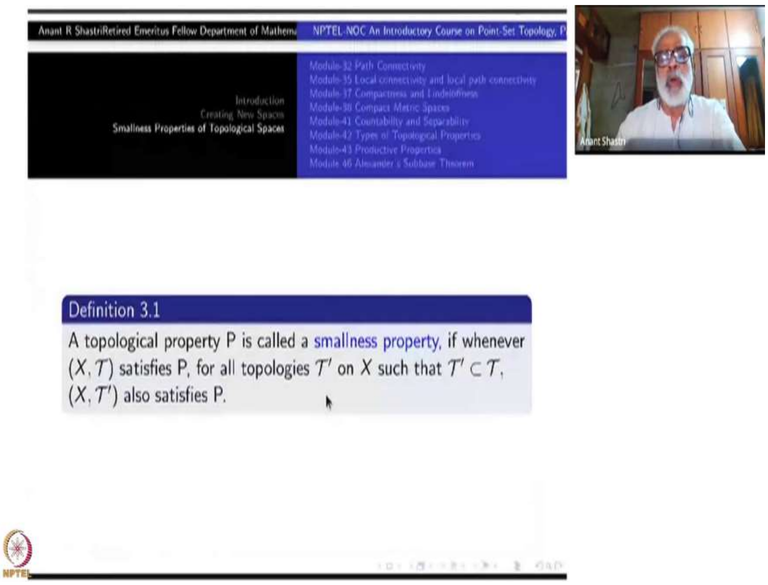

So, you can call a topological property P a smallness property, if whenever $(X, \mathcal{T})$ satisfies P, then for all topologies $\mathcal{T}'$ on X such that $\mathcal{T}'$ is smaller than $\mathcal{T}$ i.e., $\mathcal{T}'$ is contained in $\mathcal{T}$, $\mathcal{T}'$ also satisfies P. So, in that sense with respect to this P, $\mathcal{T}$ is small enough, after that everything which is smaller than that will definitely satisfy P. Bigger ones you do not know.

So, that is the meaning of this ok? So, with this definition you can keep track that many of these properties here do fit into this definition, I mean they satisfy this property. I caution you the last two in the list are not of that nature. Nevertheless they fit into the vague idea of smallness you will see why ok? So, that is why I am not very fussy about this definition ok?

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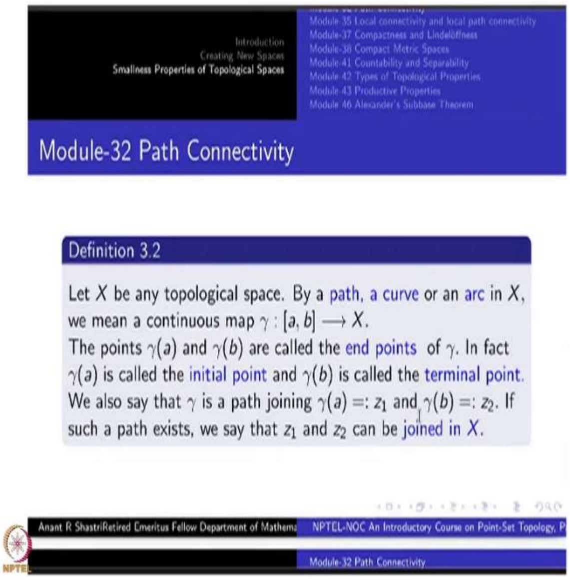

So, the first module here is module 32 Path Connectivity. This concept is directly from Layman's point of view or what may say elementary geometric concepts which are generalized into path connectivity. So, first of all let us make a rigorous definition what we mean by a path.

Start with any topological space. By a path sometimes called a curve or sometimes called an arc in X, (I will be using the word path only quite often), we just mean a continuous map this γ from a closed interval $[a, b]$ to X. The points $\gamma(a)$ and $\gamma(b)$ are called end points of γ. To be more specific you can call $\gamma(a)$ the initial point and $\gamma(b)$ the terminal point. We also say that gamma is a path joining $\gamma(a)$ which is z_1 and $\gamma(b)$ which is z_2.

So, all these terminologies which are very much just ordinary English words, they have been given some definite technical meaning here that is all.

If such a path exists we say z_1 and z_2 can be joined in X. Is that clear?

You may think that a curve should be something that it has some positive length and so on. With this definition there is no notion of length etc, first of all because X is only a topological space. Before you want to talk about length etc you must have a metric there. Secondly, even if you try to draw some picture of this path ok, picture means what? Then you have to assume X is $\mathbb{R}^2$ or something right? The first thing is that this is just a continuous function ok. So, in particular this may be a constant function. Then if you look at the image, it is just a single point. So, you may say that this is not a good definition of a path at all. Before throwing it away like that, you have to just wait why we have such a general definition for a path.

A point map is also treated as a path, it is called the constant path. That is all. So, let us hang on to that, there is no need to throw it away ok? though it is defying our common sense of a `path'. Alright? So, let us carry on with these definitions.

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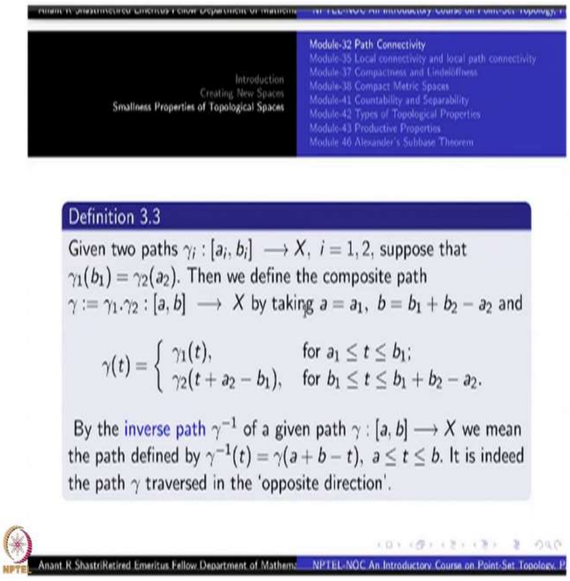

Given two paths; now paths are from some closed interval remember. So, there are two of them means γ_i from $[a_i, b_i]$ to X ok, i equal to 1 and 2. Suppose that $\gamma_1(b_1)$ is equal to $\gamma_2(a_2)$. In other words, terminal point of γ_1 is the initial point of γ_2. Terminal point of the first one is equal to the initial point of the second. Then we define the composite path ok this composite path is not a composition of maps, not a composition of functions. So, we have to pay attention to that. So, I am using this dot here $\gamma_1 \cdot \gamma_2$, from some interval $[a, b]$ which I am going to define, into X. Everything is inside X. By first taking a starting point a_1 here.

I go all the way to b_1 via γ_1 ok. So, γ_1 will be the map from $[a_1, b_1]$, but now the second point is some a_2 right? So, I have to shift a_2 to b_1 in the interval wherever it is and then trace the rest of the interval using γ_2. So, I take a equal to a_1, b equal to $b_1 + b_2 - a_2$ ok. So, $b_2 - a_2$ brings a_2 or $t - a_2$ whatever a_2 to 0 add b_1 so that it starts at b_1 that is the whole idea.

So, this is the shifting of the interval $[a_2, b_2]$ ok to start at b_1. After doing that the path γ is defined like this. $\gamma(t)$ in the first part is γ_1 between a_1 and b_1. From b_1 to $b_1 + b_2 - a_2$, this length is precisely equal to $b_2 - a_2$. So, it is $t + a_2 - b_1$. When t equal to b_1, what will be this? This will be a_2 right? t could be b_1 here.

So, this $\gamma_2(a_2)$ which is equal to $\gamma_1(b_1)$, where t equal to b_1 is $\gamma_1(b_1)$ and this is $\gamma_2(a_2)$ they are equal, therefore this is a well defined continuous function on $[a.b]$ ok? By the inverse path-- I am going to define what is the meaning of inverse path ok, of a given path--- we just mean the path defined by $\gamma^{-1}(t)$ equal to $\gamma(a+b-t)$. You just want to replace t by $-t$ so that you are just tracing the same path in the opposite direction.

But since you are not working on an interval around 0 but some $[a,b]$ you have to do this shifting, a bit of arithmetic here, $a+b-t$ is the correct thing. So, when t is equal to a. So, that is the end point now, $\gamma^{-1}(a)$ becomes end point $\gamma(b)$; when t equal to a, it is $\gamma(b)$ which is the which is the terminal point of γ which is starting point of γ^{-1}. So, the initial point and the terminal points are interchanged. So, that is the path, the inverse path of this one. Traverse in the opposite direction.

These definitions are borrowed from, what happens to when you do integration on paths. Unfortunately, on arbitrary paths you cannot do any integration, you have to have differentiable paths or piecewise differentiable paths. But we do not need that in topology. First of all we do not know what is the meaning of differentiability of a function taking values in arbitrary topological space. That will be too much.

We are doing topology here. Even in a metric space you will not know what to do with that. So we have come far away. From the Euclidean spaces to these topological spaces, wherein differentiability etc were valid. But the some basic idea still retained from that.

Often, by a curve one means the image of the path gamma, people always think of arc of a circle or an ellipse or a parabola.

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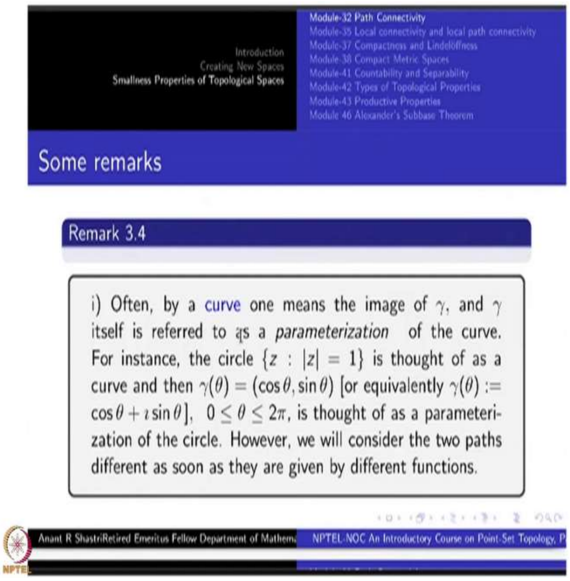

See in each of the cases whatever curve you mean, is described in a different way, the circle or an arc of a circle, the entire circle is described by a quadratic equation, parabola, by a quadratic equation. And we want to get rid of all that and that forces us to have this parameterization point of view ok? So, γ which is described in this curve now is called a parameterization, like t going to e^{it}, or $e^{2\pi it}$, whatever, is the description or parameterization of the circle ok?

So, our definition is called a parameterization of the curve, by those people who know what is a curve through a different definition ok? For instance the circle in the plane, $|z| = 1$, this is another way of describing the unit circle. This can be thought of as a curve given by $\gamma(\theta)$ maps to $(\cos(\theta), \sin(\theta))$ or equivalent to $e^{2\pi i\theta}$, or just $e^{i\theta}$ here ok?

So, those are all parameterizations of the same circle; however, we will consider two paths as different if they are given by different functions ok? The only condition on the function that we put is that it must be continuous. So, such things are called paths.

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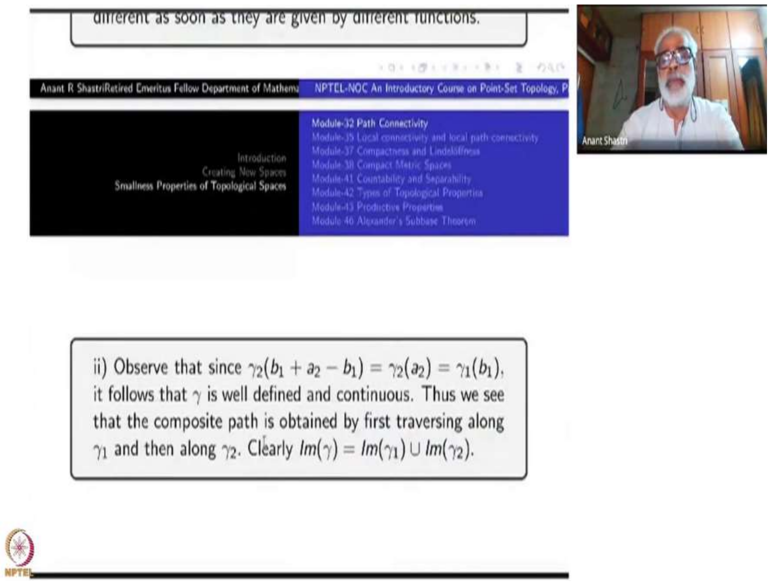
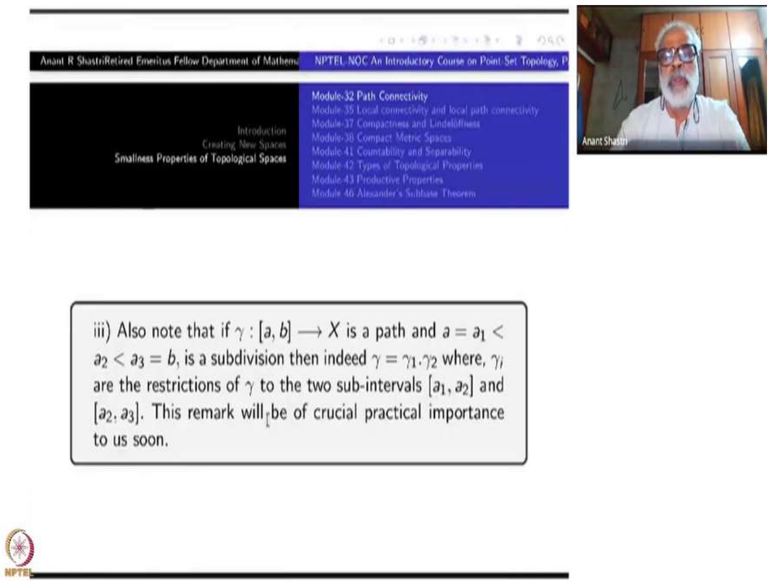


Observe that since $\gamma_2(b_1 + a_2 - b_1)$ it is $\gamma_2(a_2) = \gamma(b_1)$; it follows that the composite is well defined which I have already told you. Thus, we see that composite path is obtained by first traversing along γ_1 and then along γ_2. So, this is the geometric uh uh idea behind this definition. Moreover image of this γ is nothing but image of γ_1 union image of γ_2, they may overlap, they may whatever may have both of them may be constant at some point, then if that is the case that point will be the same for both of them because one point they are agree here. (Refer Slide Time: 15:01)



So, all those things are allowed because of the very generic nature of the definition here. Also note that if γ from $[a, b]$ to X is a path and a equal to a_1 less than a_2 less than a_3 etc. a_n equal to b. Suppose you take a division of the interval into two parts by taking a point between a and b namely a_2 ok. Then you can think of this γ as $\gamma_1 \cdot \gamma_2$ where γ_i's are the restriction of the original γ to the sub interval $[a, a_2]$ and $[a_2, b]$.

So, this remark will be crucial practical importance for us, this can be done for any subdivision I have put here. You can make finitely many cuts, finitely many divisions also ok.

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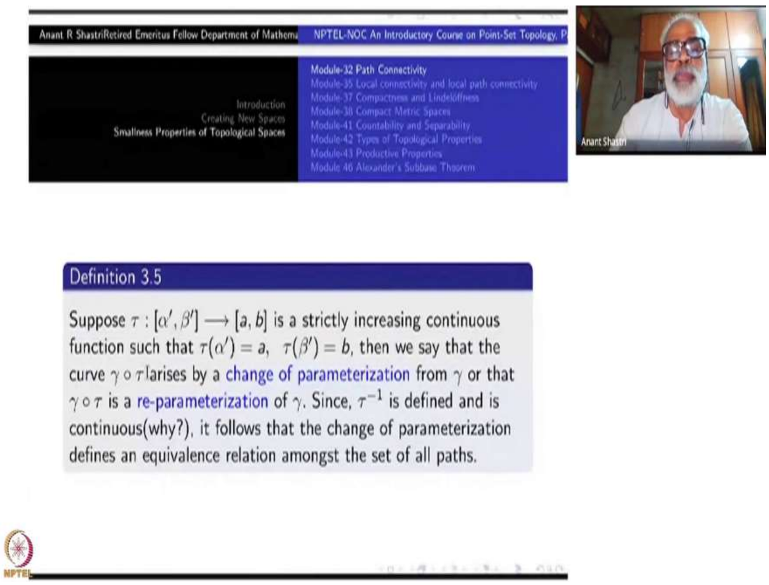

So, suppose you have a map τ from $[\alpha', \beta']$ to $[a, b]$, strictly increasing continuous function. So, that itself is a path actually because it is a continuous function it is a path where inside this space $[a, b]$. But it is strictly increasing and I am assuming that α' goes to a, β' goes to b ok. Then we say that the curve $\gamma \circ \tau$, see γ is from $[a, b]$ to X ok τ is from $[\alpha', \beta']$ to $[a, b]$ or just you can write alpha beta there is no need of putting primes. The composite curve arises by a change of parameterization from γ. This is the definition now. So, earlier we had gamma as a parameterization.

Now, $\gamma \circ \tau$ is a re-parameterization of γ. What is the condition for re-parameterization? This re-parameterizing factor must be strictly increasing continuous function. In particular, the direction with which you are tracing the curve is not changed. For example, the inverse path of a path is not a re-parameterization ok? Sorry, γ^{-1} which you have defined here, is not a reparameterization of γ. It is tracing the same curve in the opposite direction ok. So, here what I am saying is this τ is from here to here, but τ^{-1} from here to here, any strictly monotonically increasing function which is continuous automatically inverse is also continuous ok. τ^{-1} is defined and is continuous, and strictly monotonically increasing. It follows that change of parameterization is an equivalence relation among the set of all paths in a space X.

So, this is one of the reasons why the equivalence classes are considered as paths or curves ok. Not exactly, you are not exactly interested in the equivalence classes right so you can pick up one of the parameterizations whichever suits you. So, this will bring you more to the geometry of the curves rather than the parameterization itself ok.

But when you define such a thing you must see why you are using the parameterization and what is the intrinsic property, why it will not change under this change of parameterization all this you have to be worried about ok?

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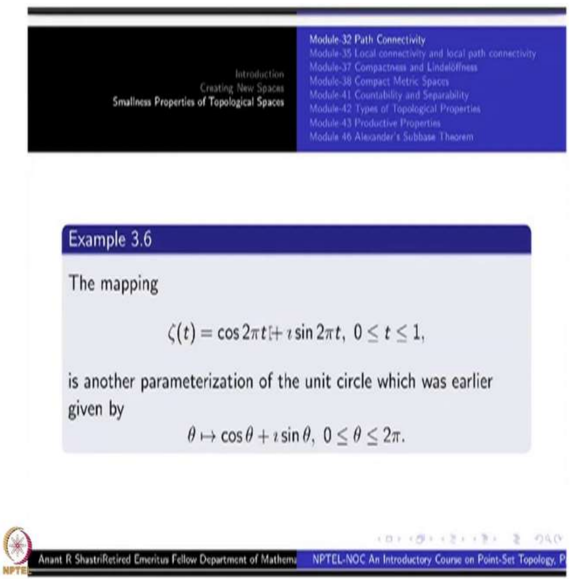

So, here are examples, the mapping $\zeta(t)$ which is $\cos(2\pi t) + i\sin(2\pi t)$ which I was telling you can write it as e^{it} or whatever, for $0 \leq t \leq 1$ is another parameterization of the unit circle which was earlier θ going to $\cos\theta + i\sin\theta$, but on the interval of $[0, 2\pi]$ ok. So, from here to here you take multiplication by 2π and follow it up then you get this one. So, that is the idea.

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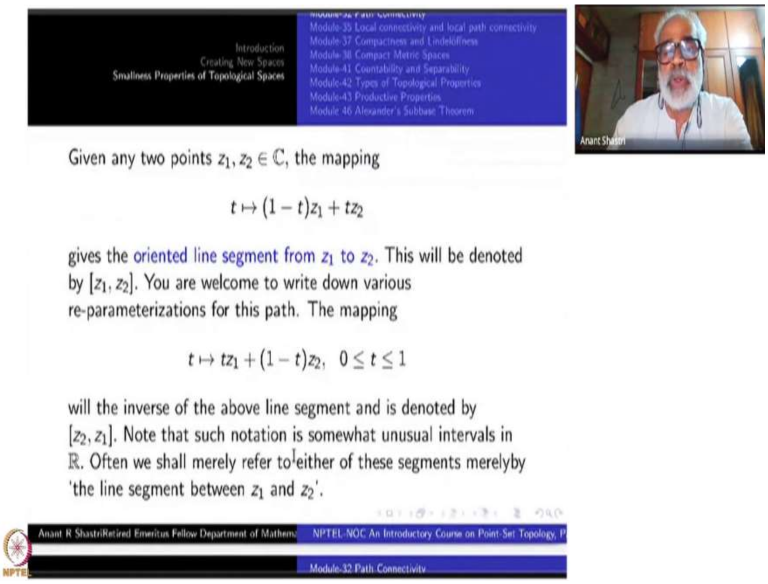

Given any 2 points z_1 and z_2 belonging to the plane $\mathbb{C}$, the mapping $(1-t)z_1 + tz_2$, this is called what? If t is restricted between 0 and 1, it is the line segment from z_1 to z_2 ok. So, this will be denoted by $[z_1, z_2]$. This $[z_1, z_2]$ will be called a line segment. This notation is borrowed from what we do in the real line, but need not be necessarily a line, it could be $\mathbb{R}^2, \mathbb{R}^3$ etc also. So, this is the notation, not only for the line segment, I am using this notation for the actual path z_1 to z_2, given by t going to $(1-t)z_1 + tz_2$. The domain must be $[0, 1]$ and z_1 and z_2 are inside a vector space over $\mathbb{R}$. You are welcome to write down various re-parameterization of the same path ok.

So, for example, t going to $tz_1 + (1-t)z_2$ again $0 \leq t \leq 1$ will be the inverse, because now when $t = 0$ it will be z_2 and $t = 1$ it will be z_1. This will be denoted by $[z_2, z_1]$. So, this is the inverse path of that ok, t is replaced by $1 - t$. Note that such notation is somewhat unusual. Other than intervals in $\mathbb{R}$ ok, intervals in $\mathbb{R}$ like that this you have to be now familiar with.

Often, we shall merely refer to either of these segments, merely by the line segment between z_1 and z_2.

This will make sense only if you are working inside a vector space ok, here I am talking about $\mathbb{C}$. So, why I have done just for $\mathbb{C}$ because these things are very useful in complex analysis when you are doing contour integration. That is the motivation for just quoting these things that is all. After all whatever topology you study you would like to use them elsewhere also.

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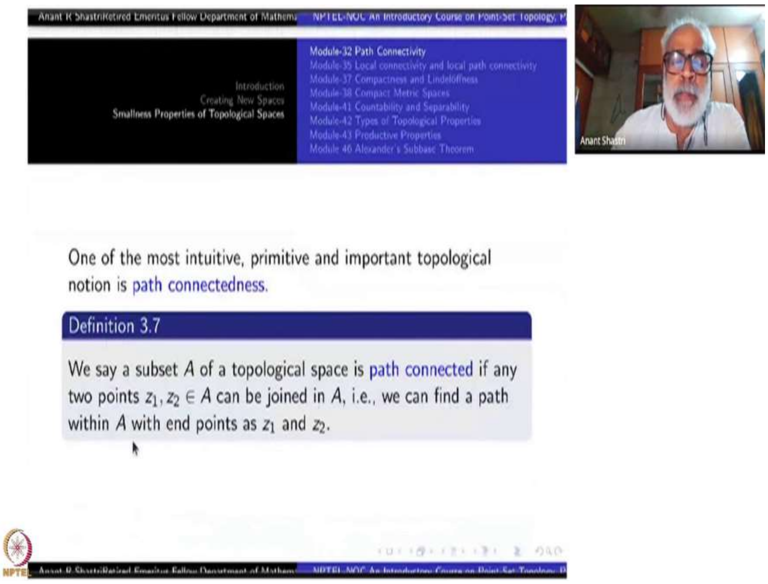

One of the most intuitive and primitive and important topological property is path connectedness, ok. The time we just say no connectivity just means that there is no road, there is no vehicle from here to there, we are all isolated and so on. So, the word path there, what do you mean by we have a path? So, that we can drive a vehicle we can try riding a bicycle or whatever ok, there must be a path.

We say a subset A of a topological space X is path connected if any two points $z_1, z_2 \in A$ can be joined inside A. Remember what is the meaning of can be joined. We can find a path within A with end points as z_1 and z_2 ok. By the very definition if they can be joined means it is symmetric ok, I do not care whether z_1 is first one or z_2 is the initial and terminal points can be interchanged. it is a symmetric relation. Every point can be joint to itself by the constant path. So, it is reflexive right?

So, finally, if you can z_1 to z_2 and z_2 to z_3 all of them inside A what happens? You can take the composition that we have defined. So, that will take you from z_1 to z_3. Therefore, if you think of this as a relation viz., z_1 can be joined to z_2, that is an equivalence relation. Right now, I have just defined what is the meaning of path connectivity, that just means that there is only one equivalence class, every point can be joined to all other points ok.

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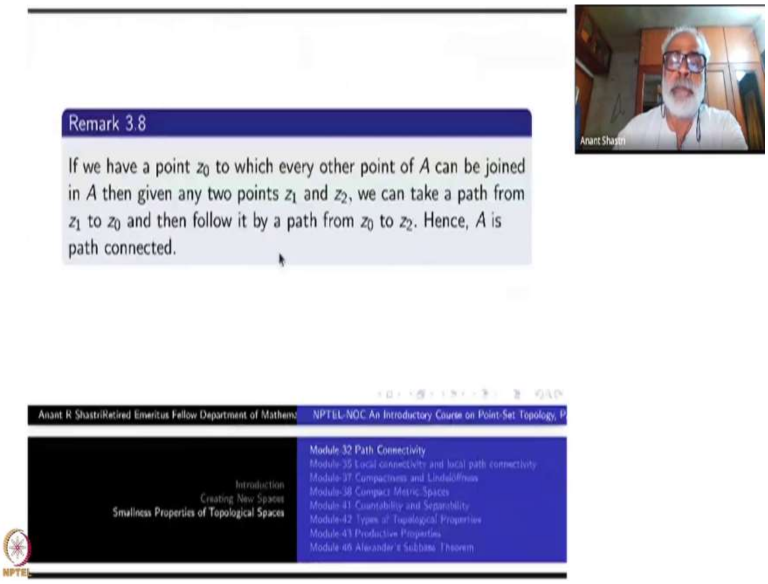

If we have one single point z_0 inside A to which every other point of A can be joined, then any two points of AA can also be joined with each other. What you have to do? First start with z_1 to z_0 and then z_0 to z_2. So, you take two of the paths like this and use the composition ok? One single point which we can be joined all other points inside A. That means A is path connected.

So, look at the union of x-axis and y-axis, $(0,0)$ can be joined to every point in the union of x-axis and y-axis right? So, that is the kind of situation I am in mind here ok.

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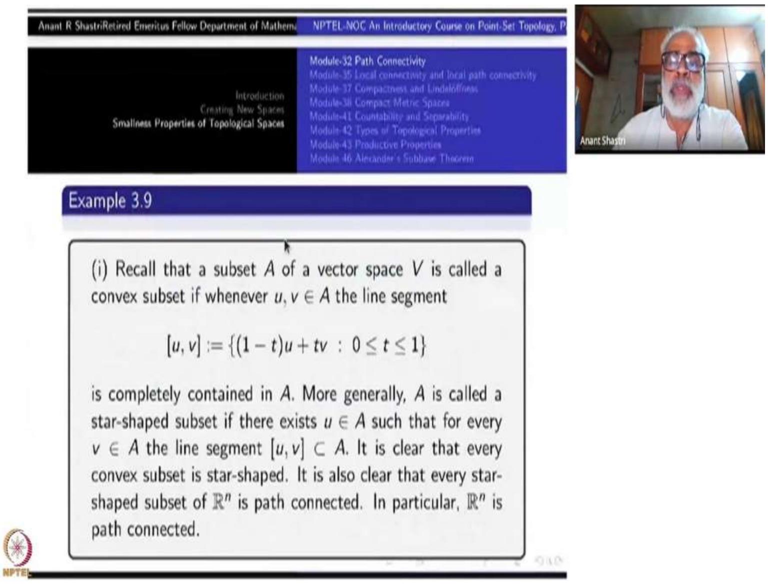

So, now just all the time we have examples inside $\mathbb{R}^2, \mathbb{R}^3$ and so on, the same thing can be done in any vector space V ok? Recall that a subset A of a vector space V is called a convex subset, whenever u and v are there then the entire line segment must be there right? So, that is the definition of convexity. Automatically what does it mean? Any two points can be joined by the line segment itself and therefore, in particular, every convex set is path connected.

And more generally what you can take is: A is called a star-shaped subset if there exists a point u belonging to A such that for every v inside A, the line segment $[u, v]$ is inside A. So, this is the case wherein you are taking union of say two lines, which are intersecting at a point or several lines which are intersecting at a single point, all of them. So, those things are star shaped ok, they are not convex ok. Yet they are path connected because of this property that we have discussed here.

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If we delete a single point from ℝ, it becomes disconnected. Well we have not defined what is the meaning of disconnected let alone prove it. But we immediately understand what it means. This is now just this English word ok?

So, right now let it hang like that. We will define what is the meaning of disconnected later and we shall actually prove rigorously that this happens or ℝ. Indeed, this is true of any interval also, if we delete a point ok that point should not be the end point. From the closed interval $[0, 1]$, if you delete 1 it will be still connected right?

So, here I meaning path connected ok because every point in the interval closed interval $[0, 1]$ say $[a, b]$., every any two point you can join them there the line segment is already there it is all the convex right. So, you can use the word path connected which you have defined. Then everything is clear here alright? So, if you delete one point from ℝ, why it is not path connected that is not obvious, you have to use something deeper about real numbers ok?

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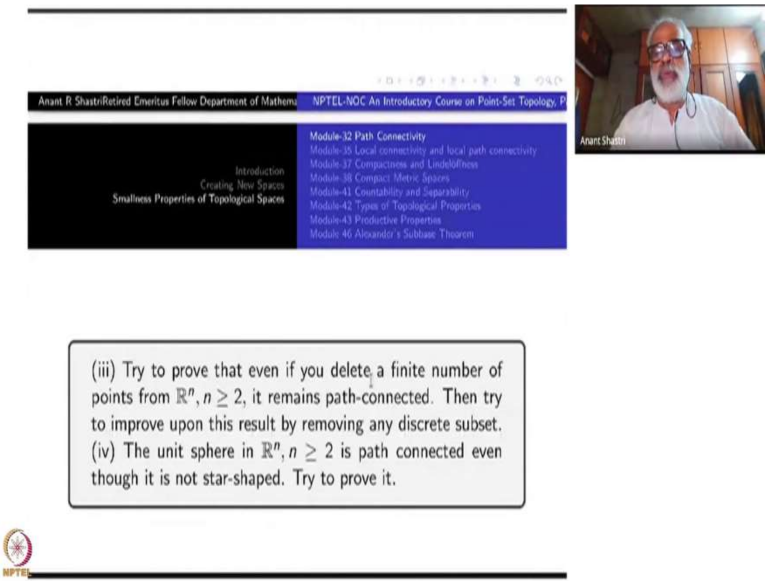

We will come to that one soon. Try to prove that even if you delete a finite number of points from $\mathbb{R}^n$ and $n \geq 2$ it remains path connected. So, this is immediate, this can be done immediately ok? See if you remove one point from $\mathbb{R}$ it gets disconnected, but if you remove finitely many points any number of finitely many points from $\mathbb{R}^2$, it is still path connected you can join them by path. How do you do that?

So, I would like to leave it to you as an exercise. If it is too difficult or you have not understood, you can contact us again ok. Similarly, the unit sphere in $\mathbb{R}^n$ for $n \geq 2$. For $n = 2$ it is the circle, $n = 3$ it is the 2 sphere and so on. They are all path connected even though they are not star shaped. How do you show that a circle is path connected? Given any two points there are two ways, you know, you can have two different arcs, you can use the restricted parameterization, done.

But when you go to the 2-sphere how do you do that? Think about these things, these are completely geometric and it is like an extension of your calculus, study of calculus ok.

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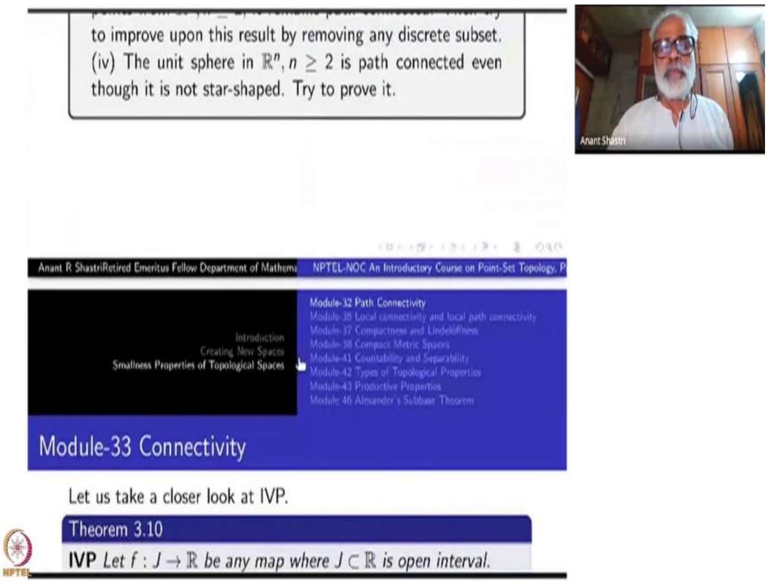

So, at this point we will take a break. So, we will continue it tomorrow, next time.

Thank you.