Introduction to Point Set Topology, (Part I)
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Lecture - 31
Induced and Co-induced Topologies

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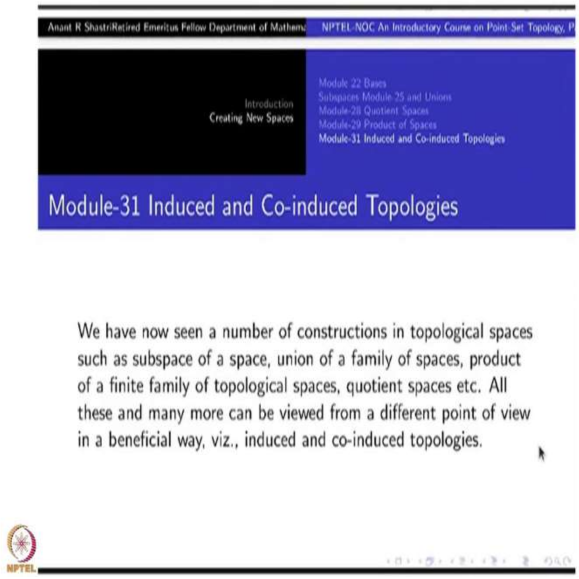

Welcome to module 31. Today, we will sum up the entire chapter of constructions of various topologies and a somewhat superficial study of these things so far as an introductory chapter. Though, all these constructions namely subspace, unions, quotient, product, right? They can all be put into two types of construction. One will be called induced topologies the other one will be called co-induced topologies. So, that is the topic for today.

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So, let us first consider induced topologies - Take a situation where we have a set X and a collection of topological spaces Y_i and functions f_i from X to Y_i. So, this is the situation we are facing with. We want somehow to relate the set X with the topological spaces Y_i through these functions f_i, in a topological way. Relating means what? Already there is a function you can say ok? Go ahead you know you want to bring in some topological aspect here.

So, in order to bring out the topological, or geometrical aspect of these relations it becomes necessary to put a topology on X, so that the functions f_i's are first of all continuous. You can say that it can be easily done because you can always take the discrete topology on X, then all the functions are continuous.

So, go ahead you may say. No, that would not do; because the discrete topology is too good in some sense and therefore it is useless also. Because no matter what f_i's are, what Y_i's are, what kind of topological spaces they are, the discrete topology is going to guarantee them that all these things are continuous.

So, it does not relate what is happening in Y_i at all ok? So, we want to have some topology which has something to do with these Y_i's and something with the f_i's as well. So, that is one thing why we reject taking the discrete topology. The second point is that it is too good it is too much to be expected, it is not economical to put so many open sets in X without being demanded.

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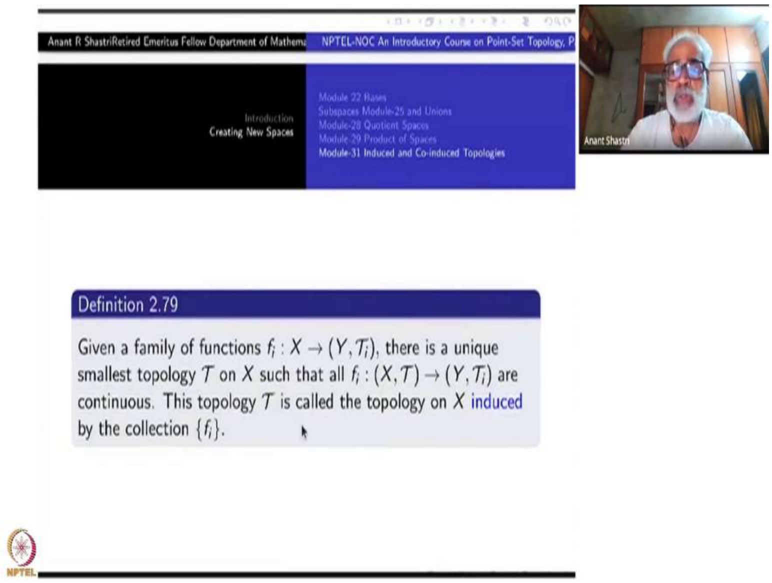

So, that leads us to the following definition we want to do it economically that is the key word here. Given a family of functions you want them to be continuous. So, take this definition. There is a unique smallest topology $\mathcal{T}$ on X, such that all the f_i's are continuous we have seen this ok. So, this topology tau is called the topology on X induced by the collection f_i, ok? alright.

The induced topology remember, now this is definition.

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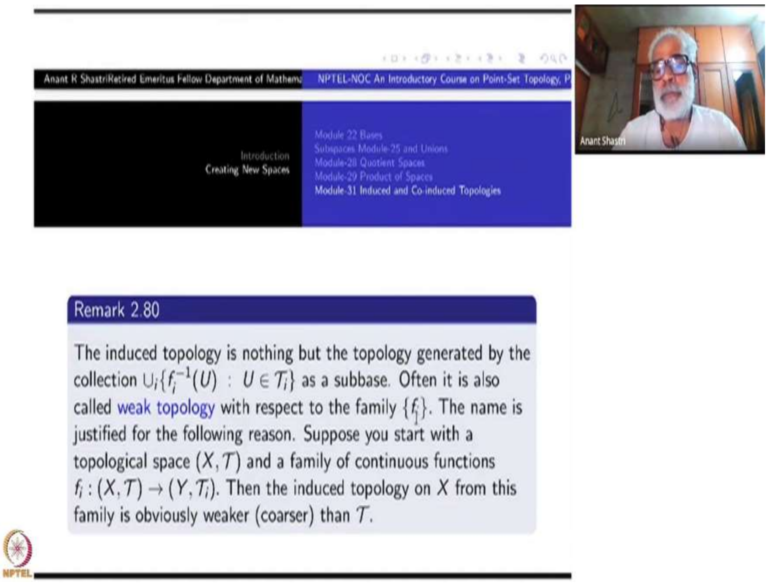

Why it is the smallest topology? What is it? It has the subbase $\mathcal{S}$ consisting of all $f_i^{-1}(U_i)$'s where U_i's range over $\mathcal{T}_i$. Take any element U_i of $\mathcal{T}_i$, take $f_i^{-1}(U_i)$, put all of them together in this collection $\mathcal{S}$. So, this is the topology with union of all $f_i^{-1}(U_i)$ where U_i belong to $\mathcal{T}_i$, as a subbase ok?

Often it is also called weak topology with respect to the family f_i. This name weak topology is justified for the following reason. Suppose, you start with a topological space $(X, \mathcal{T})$ and a family of continuous functions, right in the beginning. Then the induced topology on X from this collection will be, obviously, weaker because by definition because, it is the smallest one with this property. So, it will be smaller than this $\mathcal{T}$; anything which satisfies that all of f_i's are continuous will be containing the smallest topology right? So, that is why it is called weak topology. Weak means weaker means coarser smaller that is all.

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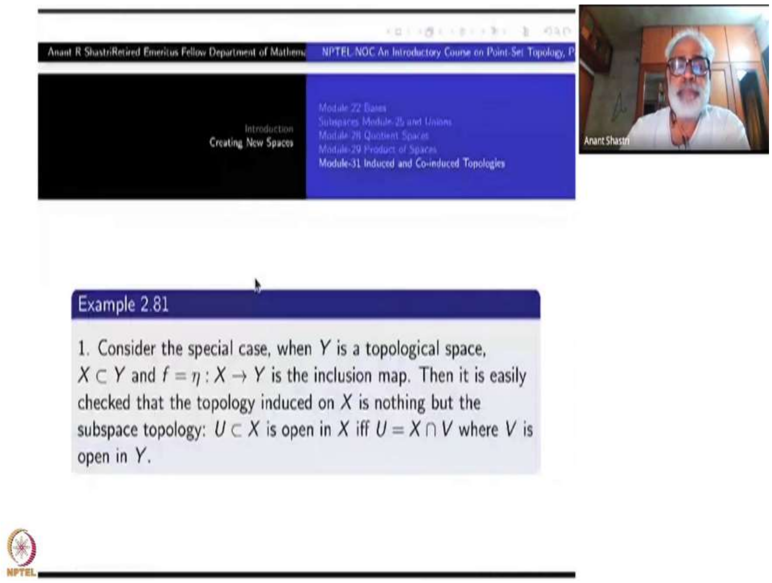

So, let us go back and examine whatever you have done already. So, it is not a new thing. I told you it is all summing up this chapter, nothing new. Consider a special case when Y is a topological space X is subset of Y. Now, you take $f = \eta$ equal to the inclusion map from X to Y ok? Now, what we have done? We have put the subspace topology on X here. Instead, just do this one namely. Take the inclusion map and put the induced topology whatever you have defined just now.

Then, what is this topology? I have to take an open set here take eta inverse of that open set which is nothing but open set U intersection with X ok? Therefore, this topology is easily seen to be the subspace topology; when you intersect U with X and take this collection that is a subbase. But it is already a topology, that is fine. If it is not a topology you better make one by taking the topology generated by that. But here is a case where it is already a topology ok? So, subspace topology is a special case of this, namely when there is only one member f_i is only one and that too is an inclusion map.

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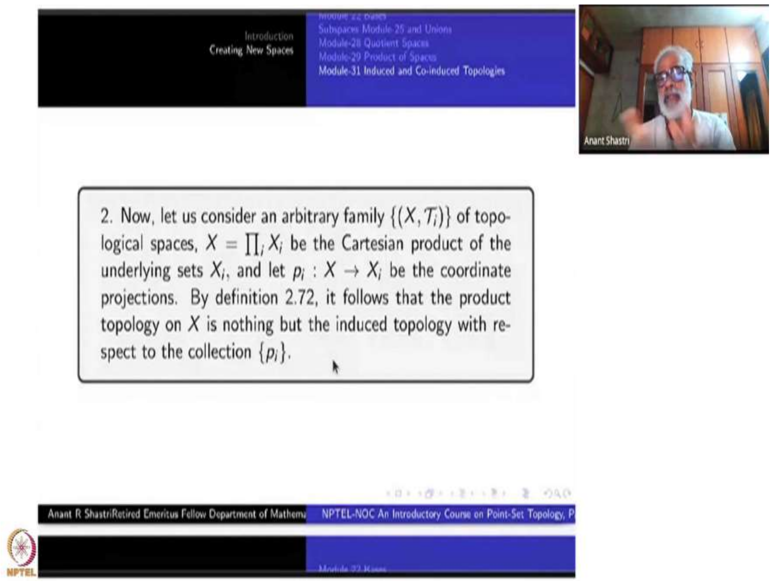

Take another example now which you have done, namely, the product - once again you have this product set X_i. Suppose each X_i has a topology, $(X_i, \mathcal{T}_i)$. Then each the underlying set X_i has a topology and I have taken Cartesian product of this family. And I have these projection maps p_i from X to X_i, coordinate projections.

By definition 2.72 whatever namely product topology, it follows that the product topology on X is nothing but the induced topology. It looks like the definition of product topology has been adopted here to define the induced topology that is all, it is the same definition ok. Only this is a special case that is all. Right? $p_i^{-1}(U_i)$ collection that collection is taken as the subbase. So, that is what it is ok?

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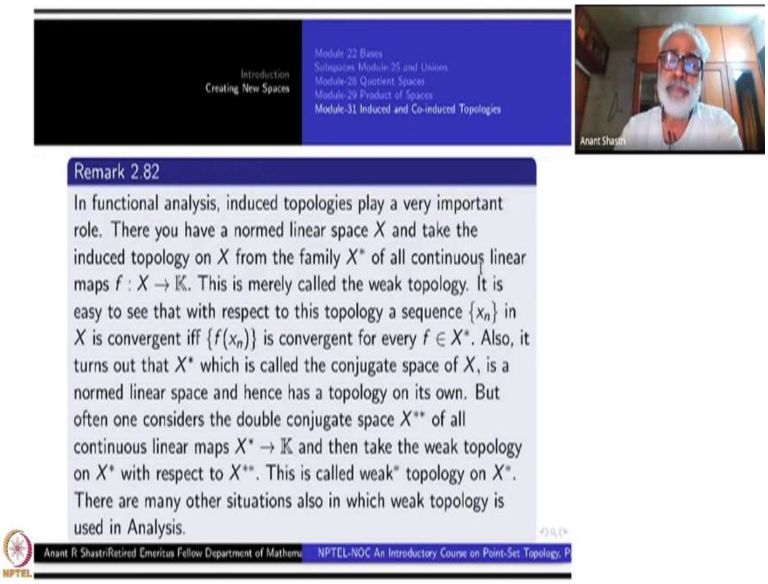

So, both these examples fit into one single generalization here. And this generalization can give you lots of other examples, unimagined examples, not very familiar ones also. Sometimes wonderful examples ok. If you develop some theory for just product spaces imitating that it may be available for any induced topology like this right; and then it will be available to those all strange kind of things also.

So, that is the idea of putting this one together. Right now, you cannot get much deeper into that one, right now. So, you have to be just aware of it that is all. So, here is a remark about the word weak topology; in functional analysis induced topologies play a very very important role. There you have a normed linear space X, one single normed linear space and take the induced topology on X from the family of all continuous linear maps f from X to $\mathbb{K}$, where $\mathbb{K}$ is the field either you may be real numbers or it may be complex numbers.

Why should I put continuous first of all? Why not just linear? When these spaces are large, namely infinite dimensional ok? Linear maps may not be continuous. I have already given you such an example in the previous chapter. In yesterday's lecture, in the previous module ok the n^{th} coordinate of the function was nt ok? That is a linear map, but it is not continuous ok?

So, so there are lots of linear maps which are not continuous linear maps are automatically continuous when you have finite dimensional normed linear spaces.

So, you take all continuous linear maps that is a smaller family linear maps into $\mathbb{K}$. They are called linear functionals. So, with this family now you put a topology on X ok. So, that topology is called weak topology. It is easy to see that with respect to this topology, a sequence $\{x_n\}$ in x is convergent if and only if $f(x_n)$ is convergent to $f(x)$ for every continuous linear map.

So, this was the motivating idea of putting this topology. It is convenient to have this one. So, that you know we can recognize the space by sequences.

Also, it turns out that X^* which is called the conjugate space of X. This is name by some author some people call dual space and so on dual. When you take all linear maps you take that is why they want to make it different this is called conjugate space continuous linear maps ok. So, conjugate space of X is a normed linear space, you can give it a norm also ok? and hence has a topology on its own. So, this norm is what is called the linear norm, supremum norm whatever ok.

But often one considers the double conjugate space of X. This X^{**} which is nothing but continuous linear maps you know $\mathbb{K}$ linear maps on X^*. So, all continuous linear maps from X^* to $\mathbb{K}$ take that. And then, use this family to give a weak topology on X^* ok. So, in functional analysis they have just a different name to distinguish this one, but this is also the weak topology only, this is called the weak star topology on X^*.

The weak topology is on X only that is the difference. but both of them are the induced topologies with respect to a certain family of functions. So, you can just call them weak topologies. There are many other situations also in which the word weak topology is used in analysis, we cannot go into them much deeper ok.

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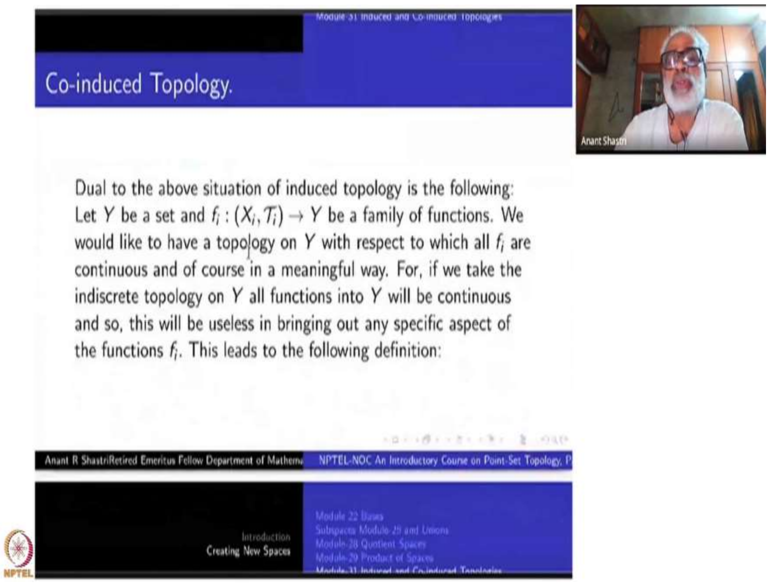

Now, let us come to the co-induced topology. In some sense it is a dual notion of induced topology ok. This time the one single Y is fixed, the codomain. A family of functions from X_i to Y are taken and each X_i is given a topology ok. So, in some sense arrows are reversed that is all that is why it is a dual situation.

Now, again we want to give some topology on Y, with consideration similar to the earlier case, a meaningful topology. Meaningful with respect to these functions as well as the topologies here. The first condition is that all the f_i's must be continuous. Once again you can just give the indiscrete topology on Y, the least one, the smallest one. Then automatically all functions will be continuous.

Once again this solution is useless. So, we reject this one outright for a similar reason as we have rejected discrete space in the case of induced topologies right. So, what we want to do? Automatically this brings us to the following definition.

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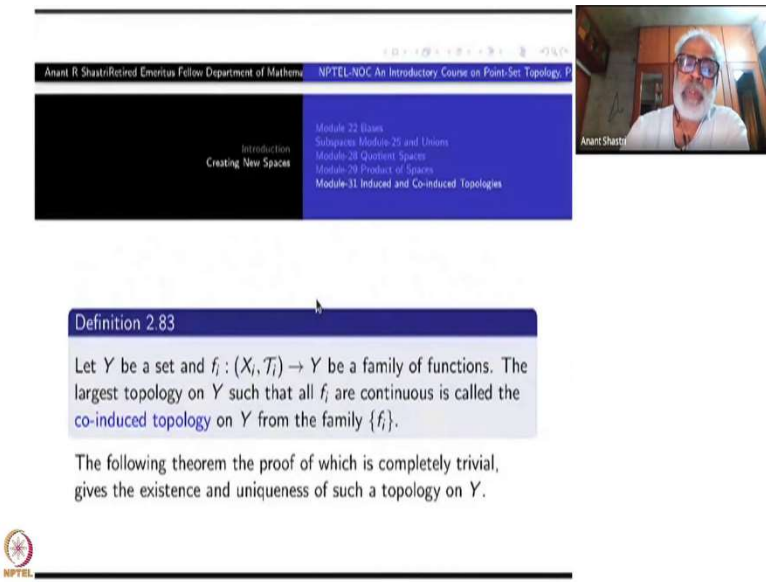
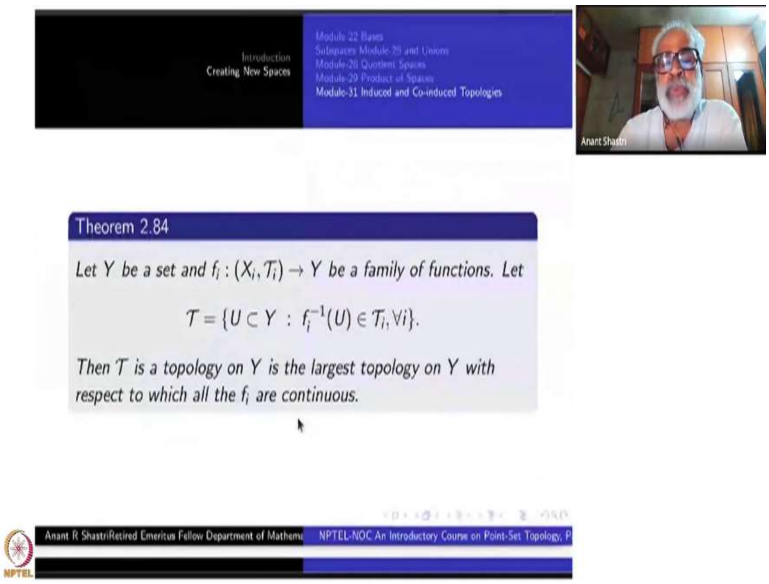


Now, let Y be a set and f_i from $(X_i, \mathcal{T}_i)$ to Y be a family of functions. The largest topology on Y such that all f_i's are continuous is called the co induced topology on Y from the family f_i or you may say with respect to f_i ok. The following theorem, the proof of which is completely trivial or similar to what we have considered in the case of quotient spaces or something like that gives the existence and uniqueness of such a topology on Y. What is theorem? (Refer Slide Time: 16:44)



Y is a set, f_i's are functions as before put tau equal to all U contained inside Y such that f_i inverse of U is inside $\mathcal{T}_i$, for every i ok. So, this is a very stringent condition you may say, but do not put anything else in here. Maybe it is too stringent, but do not make it smaller further, take only those U such that $f_i^{-1}(U)$ is inside $\mathcal{T}_i$ for every i. Automatically, this will be a topology, automatically whenever there is some $\mathcal{T}'$ here which is such that all the f_i's are continuous ok? $f_i^{-1}(U)$ will be inside $\mathcal{T}$; that means, $\mathcal{T}'$ is contained inside $\mathcal{T}$. So, this is the largest with respect to this property.

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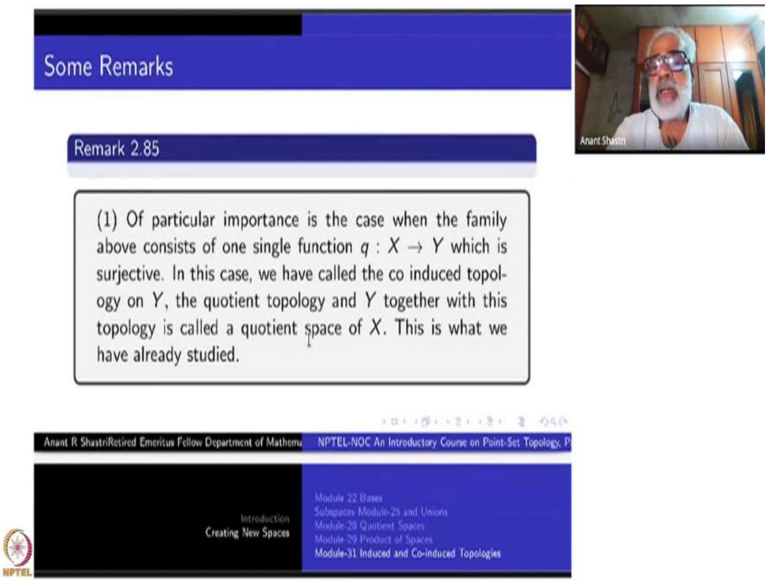

So, that is why this proof is completely trivial. That is what I have. Once again, let us have a few remarks here. One important special case is what we have studied quite thoroughly, but we will keep studying it again and again. What is it? It is one function q from X to Y and that is subjective function, just start with that.

As soon as you have a topology here on X, take the co-induced topology on Y, what is it by definition? It is the largest topology on Y such that q is continuous, what is the construction? Exactly the same as in the case of quotient space.

There is no indexing here. Just one q is there, f_1 equal to q. $q^{-1}(U)$ is open in X will mean that U is open in Y over. And that is the definition of the quotient topology there where q

from X to Y is a subjective function and X has a topology. So, I am just recalling that one right.

So, so this is a direct generalization you may say of the quotient space construction. If you have done this one before, then the quotient space is a special case, a very special case of coinduced topology --- only one function and also I am assuming that is surjective ok.

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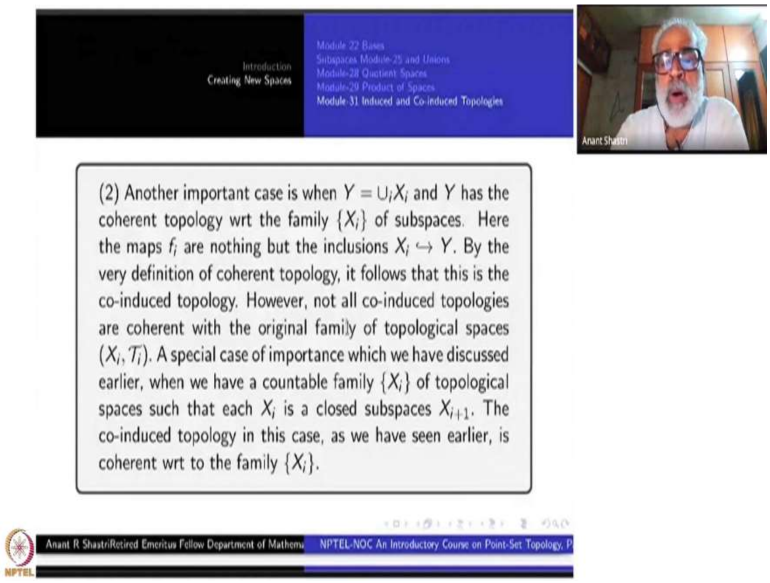

Another important case is that we studied you know before. So, this is a little more complicated situation now actually. So, you have to pay more attention to this one than to the quotient space construction where there is only one function. Take the case wherein Y itself is now union of X_i's and Y has the coherent topology with respect to X_i's which are subspaces.

Remember the what is definition of coherent topology. Y has a topology and in that topology, X_i's are subspaces and then it satisfies some condition. So, that is the meaning of coherent topology remember that ok.

You may say that this coherent topology is a special case of coinduced topology. Namely, you ignore the topology on Y with which you started right in the beginning. Now, give the co-induced topology on Y using the topologies on X_i's ok. And what is the map? Maps are from X_i to Y inclusion maps ok? X_i to Y inclusion maps. What you get is nothing but the original topology on Y. So, by the very definition of coherent topology, it follows that this is the co-induced topology.

Why, what is an open subset of Y? In the definition you see A is open in Y if $A \cap X_i$ is open in X if for every i, that is precisely the definition of coinduced topology here when each f_i is inclusion map. U contained inside Y will be put here if and only if $U \cap X_i$ is inside $\mathcal{T}_i$ for all i . Over.

So, it will give you the topology with which you started with. So, coherent topology is a special case of co-induced topology ok. However, not all co-induced topologies are coherent with the original family of topological spaces X_i to Y. I want to make it clear.

Suppose $(X_i, \mathcal{T}_i)$ are subspaces ok, and you have taken the inclusion maps and now you give the co-induced topology on Y that topology will be coherent maybe, but you started already with the topology there right. Some topology you started and took these subspaces, and then you took this co induced topology. That topology may not be equal to the original topology on Y. So, in that sense it may not be the coherent topology.

So, not all co induced topologies from the original subspaces may be coherent topology that is the meaning of this. A special case of importance which we have discussed earlier, when we have a countable family X_i's of topological spaces one contained in the other and each X_i is closed subspace of X_{i+1}. In that case the co-induced topology gives you the same topologies which you started with.

So; that means, it is coherent topology. The co induced topology in this case we have seen earlier is coherent with respect to X_i. There are special cases when this can happen like open subsets that is another case or locally finite closed sub families etc, ok. So, there are cases wherein this can occur, also cases wherein this does not occur ok.

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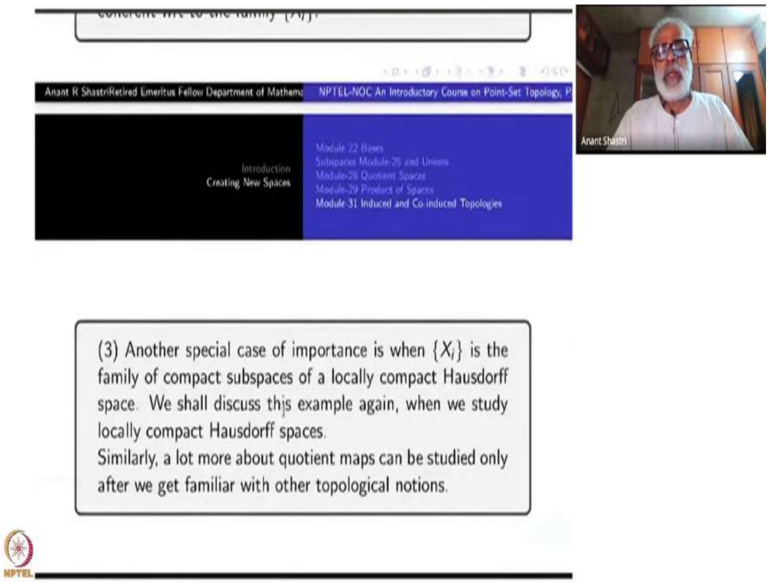

Another special case of importance is that X_i is the family of compact subspaces of a locally compact Hausdorff space. So, this I am mentioning that because it is so important. So, right now I just mentioned that we are going to study this one later on when we study compact spaces and locally compact Hausdorff spaces and so on ok. Right now, you do not know these terminologies let us say. So, do not worry about this.

Similarly, a lot more about quotient maps can be studied only after we get familiar with other topological notions ok. So, we have just begun the study of induced and co-induced topologies. So, we have explicitly five different cases of this ok; two for this two for the other five or four maybe. Subspace topology, product, quotient and this one is union right. So, and coherent topology and so on. Union under union coherent topologies under union itself anyway yeah?

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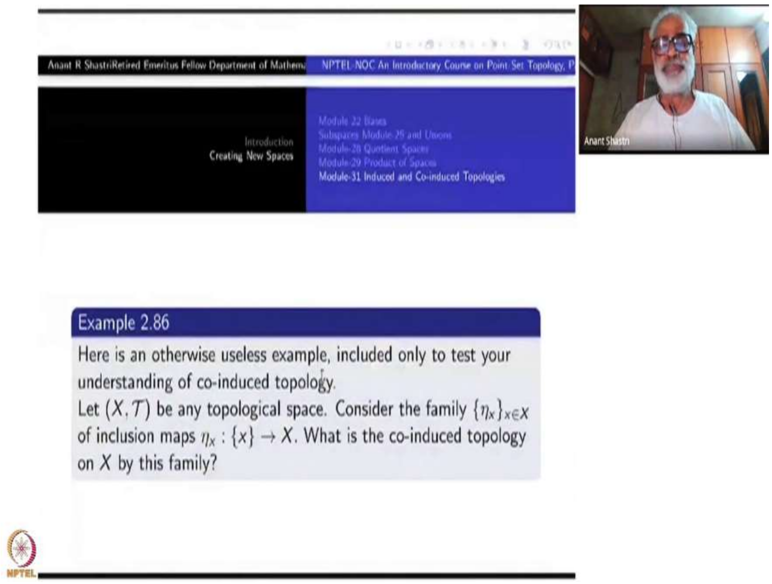

So, I will conclude it with a with an easy example here ok. But, this is only for testing the depth of our knowledge of co-induced topology alright. So, take a break that is all. This example is not of any use. Let $(X, \mathcal{T})$ be any topological space ok. Consider the family η_x, x belonging to X, what are η_x's? They are inclusion maps of the singletons inside X. Inclusion map from $\{x\}$ to X.

Now these singleton x are topological spaces right. What is the co-induced topology from this family on the set X? What is the co induced topology? That is what I am asking ok. By definition, something is open in X ok if and only if intersection with $\{x\}$ is open in $\{x\}$ for every X.

If this point is not in the set U, then the intersection is empty. So, it is open. If it is inside that, then the intersection is singleton x. Singleton x is an open subset of $\{x\}$. So, what I have concluded? I took an arbitrary subset of X, I have shown that it is open right? So, this will always give you the discrete topology on X right. This is a discrete topology on X ok.

For example, you know you can test some special case, you could have taken X to be $\mathbb{R}$, then all singletons will be actually closed subspaces. Yet, the co induced topology is discrete and not the original $\mathbb{R}$ at all ok.

So, just closedness is not enough. See, in this theorem we had everything is closed. In this one and X_i is closed inside X_{i+1} increasing sequences etc we have right; or we had what is called, locally finite family of closed sets also ok, neither of the condition is satisfied here.

This family is not locally finite ok, in general. Inside $\mathbb{R}$ for example, it is not locally finite. So, when it is locally finite it just means that the space is a discrete space you can prove that ok ah sorry, ah ah this is this is locally this is not locally finite, but it is point finite ok. So, this is also a nice example in a way, but let us leave it. I mean this is just for understanding what happens to the co-induced topology.

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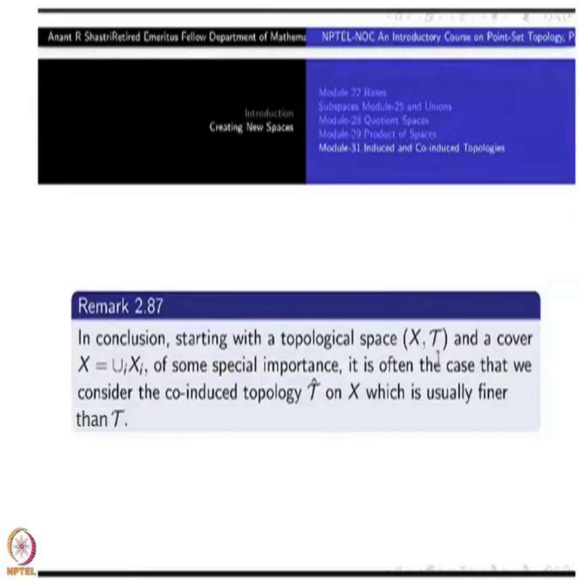

In conclusion, starting with a topological space $(X, \mathcal{T})$ and a cover. Just like when you had a function X to Y, a surjective map in the case of quotient space. You take all the points otherwise some points are left out. There will not be any structure on that part. So, X is union of X_i's of some special importance. These X_i's you have chosen nicely you know to represent the whole.

They cover the whole thing, like a representative subsets. It is often the case that we consider the coinduced topology $\widehat{\mathcal{T}}$, that is notation, which is usually finer than the given topology tau ok. So, it is a finer topology than tau. So, this is an important phenomenon. Many nice mathematical ideas have been developed from this one. This can be used by politicians also perhaps, in the construction of states and you know countries and so on.

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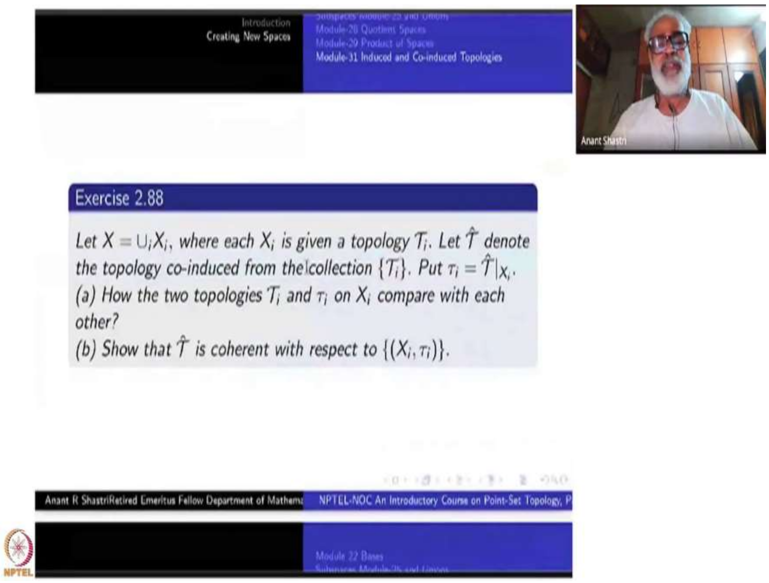

So, here is an exercise now which you can immediately answer because I have explained the things behind it ok. Let X equal the union of X_i where each X_i is given the topology $\mathcal{T}_i$. Let $\hat{\mathcal{T}}$ denote the topology co-induced from the collection. Put τ_i equal to $\hat{\mathcal{T}}$ restricted to X_i.

How the two topologies on X_i compare with each other? Next thing is show that this tau hat is coherent with respect to (X_i, τ_i), you started with arbitrary $\mathcal{T}_i$ on each X_i, it may not be coherent with respect to, but $\hat{\mathcal{T}}$ is coherent ok.

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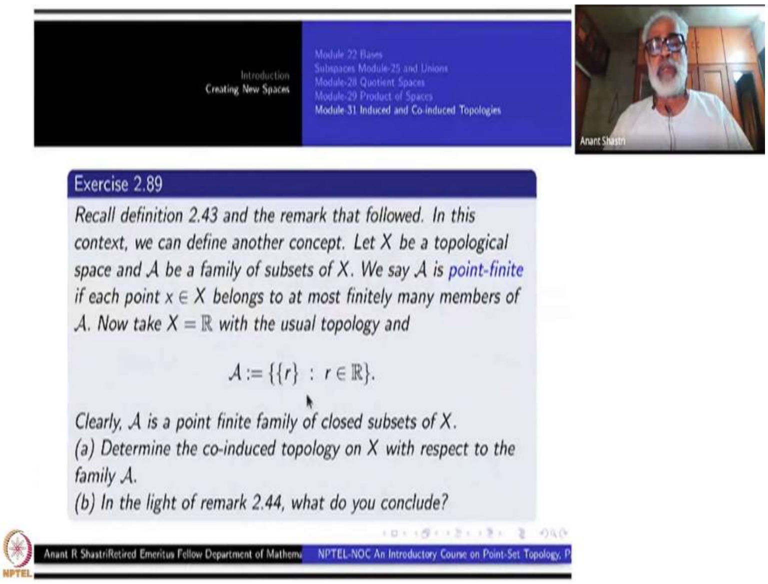

And this remark, this exercise all I have already explained it to you, X equal to $\mathbb{R}$, and I am taking $\mathcal{A}$ equal to this family $\mathbb{R}$, it is a point finite family. But what happens you can understand ok. So, this exercise is already explained to you. You can write down the details. So, that is all. Next time, we will meet we will meet with the chapter 3 ok.

Starting in earnest to study topological properties we have reasonably enough number of examples now other than the standard you know non-linear spaces metric spaces and so on ok. So, we can start studying some topological properties from next time.

Thank you.