

**Introduction to Point Set Topology, (Part I)**  
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**Lecture - 31**  
**Induced and Co-induced Topologies**

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Module-31 Induced and Co-induced Topologies

We have now seen a number of constructions in topological spaces such as subspace of a space, union of a family of spaces, product of a finite family of topological spaces, quotient spaces etc. All these and many more can be viewed from a different point of view in a beneficial way, viz., induced and co-induced topologies.

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Welcome to module 31. Today, we will sum up the entire chapter of constructions of various topologies and a somewhat superficial study of these things so far as an introductory chapter. Though, all these constructions namely subspace, unions, quotient, product, right? They can all be put into two types of construction. One will be called induced topologies the other one will be called co-induced topologies. So, that is the topic for today.

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Consider a situation where we have a set  $X$ , a collection of topological spaces  $Y_i$  and functions  $f_i: X \rightarrow Y_i$ , which somehow relate the set  $X$  with the topological spaces  $Y_i$ . In order to bring out the topological /geometric aspects of these relations, it becomes necessary to put a topology on  $X$  so that the functions  $f_i$  are all continuous. That is easily done, for, we could always take the discrete topology on  $X$  with respect to which every map is continuous. However, we would also want to do it in an economical way and that leads us to the following definition.



So, let us first consider induced topologies - Take a situation where we have a set  $X$  and a collection of topological spaces  $Y_i$  and functions  $f_i$  from  $X$  to  $Y_i$ . So, this is the situation we are facing with. We want somehow to relate the set  $X$  with the topological spaces  $Y_i$  through these functions  $f_i$ , in a topological way. Relating means what? Already there is a function you can say ok? Go ahead you know you want to bring in some topological aspect here.

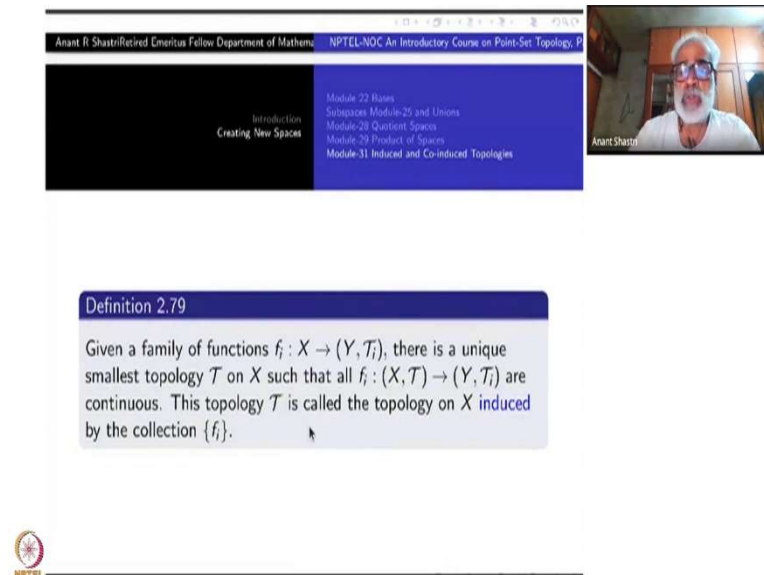
So, in order to bring out the topological, or geometrical aspect of these relations it becomes necessary to put a topology on  $X$ , so that the functions  $f_i$ 's are first of all continuous. You can say that it can be easily done because you can always take the discrete topology on  $X$ , then all the functions are continuous.

So, go ahead you may say. No, that would not do; because the discrete topology is too good in some sense and therefore it is useless also. Because no matter what  $f_i$ 's are, what  $Y_i$ 's are, what kind of topological spaces they are, the discrete topology is going to guarantee them that all these things are continuous.

So, it does not relate what is happening in  $Y_i$  at all ok? So, we want to have some topology which has something to do with these  $Y_i$ 's and something with the  $f_i$ 's as well. So, that is one thing why we reject taking the discrete topology. The second point is that it is too good it is

too much to be expected, it is not economical to put so many open sets in  $X$  without being demanded.

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The image shows a screenshot of a video lecture. At the top, there is a navigation bar with the text 'Anant R Shastri Retired Emeritus Fellow Department of Mathematics' and 'NPTEL-NOC An Introductory Course on Point-Set Topology, P'. Below this is a blue sidebar menu with the following items: 'Introduction', 'Creating New Spaces', 'Module 22: Bases', 'Subspaces, Module 25 and Unions', 'Module 28: Quotient Spaces', 'Module 29: Product of Spaces', and 'Module 31: Induced and Co-induced Topologies'. In the top right corner, there is a small video window showing a man with glasses and a white beard, identified as 'Anant Shastri'. The main content area of the slide is white and contains the following text:

**Definition 2.79**

Given a family of functions  $f_i : X \rightarrow (Y, \mathcal{T}_i)$ , there is a unique smallest topology  $\mathcal{T}$  on  $X$  such that all  $f_i : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}_i)$  are continuous. This topology  $\mathcal{T}$  is called the topology on  $X$  induced by the collection  $\{f_i\}$ .

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So, that leads us to the following definition we want to do it economically that is the key word here. Given a family of functions you want them to be continuous. So, take this definition. There is a unique smallest topology  $\mathcal{T}$  on  $X$ , such that all the  $f_i$ 's are continuous we have seen this ok. So, this topology tau is called the topology on  $X$  induced by the collection  $f_i$ , ok? alright.

The induced topology remember, now this is definition.

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Introduction  
Creating New Spaces

Remark 2.80

The induced topology is nothing but the topology generated by the collection  $\cup_i \{f_i^{-1}(U) : U \in \mathcal{T}_i\}$  as a subbase. Often it is also called **weak topology** with respect to the family  $\{f_i\}$ . The name is justified for the following reason. Suppose you start with a topological space  $(X, \mathcal{T})$  and a family of continuous functions  $f_i : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}_i)$ . Then the induced topology on  $X$  from this family is obviously weaker (coarser) than  $\mathcal{T}$ .

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Why it is the smallest topology? What is it? It has the subbase  $\mathcal{S}$  consisting of all  $f_i^{-1}(U_i)$ 's where  $U_i$ 's range over  $\mathcal{T}_i$ . Take any element  $U_i$  of  $\mathcal{T}_i$ , take  $f_i^{-1}(U_i)$ , put all of them together in this collection  $\mathcal{S}$ . So, this is the topology with union of all  $f_i^{-1}(U_i)$  where  $U_i$  belong to  $\mathcal{T}_i$ , as a subbase ok?

Often it is also called weak topology with respect to the family  $f_i$ . This name weak topology is justified for the following reason. Suppose, you start with a topological space  $(X, \mathcal{T})$  and a family of continuous functions, right in the beginning. Then the induced topology on  $X$  from this collection will be, obviously, weaker because by definition because, it is the smallest one with this property. So, it will be smaller than this  $\mathcal{T}$ ; anything which satisfies that all of  $f_i$ 's are continuous will be containing the smallest topology right? So, that is why it is called weak topology. Weak means weaker means coarser smaller that is all.

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**Example 2.81**

1. Consider the special case, when  $Y$  is a topological space,  $X \subset Y$  and  $f = \eta : X \rightarrow Y$  is the inclusion map. Then it is easily checked that the topology induced on  $X$  is nothing but the subspace topology:  $U \subset X$  is open in  $X$  iff  $U = X \cap V$  where  $V$  is open in  $Y$ .

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So, let us go back and examine whatever you have done already. So, it is not a new thing. I told you it is all summing up this chapter, nothing new. Consider a special case when  $Y$  is a topological space  $X$  is subset of  $Y$ . Now, you take  $f = \eta$  equal to the inclusion map from  $X$  to  $Y$  ok? Now, what we have done? We have put the subspace topology on  $X$  here. Instead, just do this one namely. Take the inclusion map and put the induced topology whatever you have defined just now.

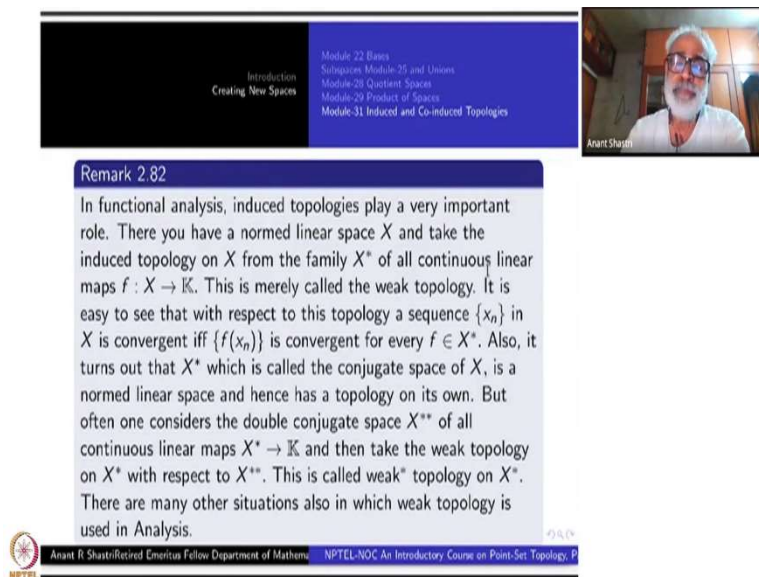
Then, what is this topology? I have to take an open set here take  $\eta$  inverse of that open set which is nothing but open set  $U$  intersection with  $X$  ok? Therefore, this topology is easily seen to be the subspace topology; when you intersect  $U$  with  $X$  and take this collection that is a subbase. But it is already a topology, that is fine. If it is not a topology you better make one by taking the topology generated by that. But here is a case where it is already a topology ok? So, subspace topology is a special case of this, namely when there is only one member  $f_i$  is only one and that too is an inclusion map.

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Take another example now which you have done, namely, the product - once again you have this product set  $X_i$ . Suppose each  $X_i$  has a topology,  $(X_i, \mathcal{T}_i)$ . Then each the underlying set  $X_i$  has a topology and I have taken Cartesian product of this family. And I have these projection maps  $p_i$  from  $X$  to  $X_i$ , coordinate projections.

By definition 2.72 whatever namely product topology, it follows that the product topology on  $X$  is nothing but the induced topology. It looks like the definition of product topology has been adopted here to define the induced topology that is all, it is the same definition ok. Only this is a special case that is all. Right?  $p_i^{-1}(U_i)$  collection that collection is taken as the subbase. So, that is what it is ok?

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The slide is titled "Remark 2.82" and contains the following text:

In functional analysis, induced topologies play a very important role. There you have a normed linear space  $X$  and take the induced topology on  $X$  from the family  $X^*$  of all continuous linear maps  $f : X \rightarrow \mathbb{K}$ . This is merely called the weak topology. It is easy to see that with respect to this topology a sequence  $\{x_n\}$  in  $X$  is convergent iff  $\{f(x_n)\}$  is convergent for every  $f \in X^*$ . Also, it turns out that  $X^*$  which is called the conjugate space of  $X$ , is a normed linear space and hence has a topology on its own. But often one considers the double conjugate space  $X^{**}$  of all continuous linear maps  $X^* \rightarrow \mathbb{K}$  and then take the weak topology on  $X^*$  with respect to  $X^{**}$ . This is called weak\* topology on  $X^*$ . There are many other situations also in which weak topology is used in Analysis.

The slide also features a video inset of a speaker, Anant R. Shastri, and a navigation bar at the bottom with the following text: "Anant R. Shastri/Retired Emeritus Fellow Department of Mathema NPTEL-NOG: An Introductory Course on Point-Set Topology, P".

So, both these examples fit into one single generalization here. And this generalization can give you lots of other examples, unimagined examples, not very familiar ones also. Sometimes wonderful examples ok. If you develop some theory for just product spaces imitating that it may be available for any induced topology like this right; and then it will be available to those all strange kind of things also.

So, that is the idea of putting this one together. Right now, you cannot get much deeper into that one, right now. So, you have to be just aware of it that is all. So, here is a remark about the word weak topology; in functional analysis induced topologies play a very very important role. There you have a normed linear space  $X$ , one single normed linear space and take the induced topology on  $X$  from the family of all continuous linear maps  $f$  from  $X$  to  $\mathbb{K}$ , where  $\mathbb{K}$  is the field either you may be real numbers or it may be complex numbers.

Why should I put continuous first of all? Why not just linear? When these spaces are large, namely infinite dimensional ok? Linear maps may not be continuous. I have already given you such an example in the previous chapter. In yesterday's lecture, in the previous module ok the  $n^{th}$  coordinate of the function was  $nt$  ok? That is a linear map, but it is not continuous ok?

So, so there are lots of linear maps which are not continuous linear maps are automatically continuous when you have finite dimensional normed linear spaces.

So, you take all continuous linear maps that is a smaller family linear maps into  $\mathbb{K}$ . They are called linear functionals. So, with this family now you put a topology on  $X$  ok. So, that topology is called weak topology. It is easy to see that with respect to this topology, a sequence  $\{x_n\}$  in  $x$  is convergent if and only if  $f(x_n)$  is convergent to  $f(x)$  for every continuous linear map.

So, this was the motivating idea of putting this topology. It is convenient to have this one. So, that you know we can recognize the space by sequences.

Also, it turns out that  $X^*$  which is called the conjugate space of  $X$ . This is name by some author some people call dual space and so on dual. When you take all linear maps you take that is why they want to make it different this is called conjugate space continuous linear maps ok. So, conjugate space of  $X$  is a normed linear space, you can give it a norm also ok? and hence has a topology on its own. So, this norm is what is called the linear norm, supremum norm whatever ok.

But often one considers the double conjugate space of  $X$ . This  $X^{**}$  which is nothing but continuous linear maps you know  $\mathbb{K}$  linear maps on  $X^*$ . So, all continuous linear maps from  $X^*$  to  $\mathbb{K}$  take that. And then, use this family to give a weak topology on  $X^*$  ok. So, in functional analysis they have just a different name to distinguish this one, but this is also the weak topology only, this is called the weak star topology on  $X^*$ .

The weak topology is on  $X$  only that is the difference. but both of them are the induced topologies with respect to a certain family of functions. So, you can just call them weak topologies. There are many other situations also in which the word weak topology is used in analysis, we cannot go into them much deeper ok.



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Module 31: Induced and Co-induced Topologies

## Co-induced Topology.

Dual to the above situation of induced topology is the following:  
Let  $Y$  be a set and  $f_i : (X_i, \mathcal{T}_i) \rightarrow Y$  be a family of functions. We would like to have a topology on  $Y$  with respect to which all  $f_i$  are continuous and of course in a meaningful way. For, if we take the indiscrete topology on  $Y$  all functions into  $Y$  will be continuous and so, this will be useless in bringing out any specific aspect of the functions  $f_i$ . This leads to the following definition:

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Now, let us come to the co-induced topology. In some sense it is a dual notion of induced topology ok. This time the one single  $Y$  is fixed, the codomain. A family of functions from  $X_i$  to  $Y$  are taken and each  $X_i$  is given a topology ok. So, in some sense arrows are reversed that is all that is why it is a dual situation.

Now, again we want to give some topology on  $Y$ , with consideration similar to the earlier case, a meaningful topology. Meaningful with respect to these functions as well as the topologies here. The first condition is that all the  $f_i$ 's must be continuous. Once again you can just give the indiscrete topology on  $Y$ , the least one, the smallest one. Then automatically all functions will be continuous.

Once again this solution is useless. So, we reject this one outright for a similar reason as we have rejected discrete space in the case of induced topologies right. So, what we want to do? Automatically this brings us to the following definition.

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**Definition 2.83**

Let  $Y$  be a set and  $f_i : (X_i, \mathcal{T}_i) \rightarrow Y$  be a family of functions. The largest topology on  $Y$  such that all  $f_i$  are continuous is called the **co-induced topology** on  $Y$  from the family  $\{f_i\}$ .

The following theorem the proof of which is completely trivial, gives the existence and uniqueness of such a topology on  $Y$ .

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Now, let  $Y$  be a set and  $f_i$  from  $(X_i, \mathcal{T}_i)$  to  $Y$  be a family of functions. The largest topology on  $Y$  such that all  $f_i$ 's are continuous is called the co induced topology on  $Y$  from the family  $f_i$  or you may say with respect to  $f_i$  ok. The following theorem, the proof of which is completely trivial or similar to what we have considered in the case of quotient spaces or something like that gives the existence and uniqueness of such a topology on  $Y$ . What is theorem? (Refer Slide Time: 16:44)

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**Theorem 2.84**

Let  $Y$  be a set and  $f_i : (X_i, \mathcal{T}_i) \rightarrow Y$  be a family of functions. Let

$$\mathcal{T} = \{U \subset Y : f_i^{-1}(U) \in \mathcal{T}_i, \forall i\}.$$

Then  $\mathcal{T}$  is a topology on  $Y$  is the largest topology on  $Y$  with respect to which all the  $f_i$  are continuous.

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$Y$  is a set,  $f_i$ 's are functions as before put tau equal to all  $U$  contained inside  $Y$  such that  $f_i$  inverse of  $U$  is inside  $\mathcal{T}_i$ , for every  $i$  ok. So, this is a very stringent condition you may say, but do not put anything else in here. Maybe it is too stringent, but do not make it smaller further, take only those  $U$  such that  $f_i^{-1}(U)$  is inside  $\mathcal{T}_i$  for every  $i$ . Automatically, this will be a topology, automatically whenever there is some  $\mathcal{T}'$  here which is such that all the  $f_i$ 's are continuous ok?  $f_i^{-1}(U)$  will be inside  $\mathcal{T}$ ; that means,  $\mathcal{T}'$  is contained inside  $\mathcal{T}$ . So, this is the largest with respect to this property.

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Some Remarks

Remark 2.85

(1) Of particular importance is the case when the family above consists of one single function  $q : X \rightarrow Y$  which is surjective. In this case, we have called the co induced topology on  $Y$ , the quotient topology and  $Y$  together with this topology is called a quotient space of  $X$ . This is what we have already studied.

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So, that is why this proof is completely trivial. That is what I have. Once again, let us have a few remarks here. One important special case is what we have studied quite thoroughly, but we will keep studying it again and again. What is it? It is one function  $q$  from  $X$  to  $Y$  and that is subjective function, just start with that.

As soon as you have a topology here on  $X$ , take the co-induced topology on  $Y$ , what is it by definition? It is the largest topology on  $Y$  such that  $q$  is continuous, what is the construction? Exactly the same as in the case of quotient space.

There is no indexing here. Just one  $q$  is there,  $f_1$  equal to  $q$ .  $q^{-1}(U)$  is open in  $X$  will mean that  $U$  is open in  $Y$  over. And that is the definition of the quotient topology there where  $q$

from  $X$  to  $Y$  is a surjective function and  $X$  has a topology. So, I am just recalling that one right.

So, so this is a direct generalization you may say of the quotient space construction. If you have done this one before, then the quotient space is a special case, a very special case of coinduced topology --- only one function and also I am assuming that is surjective ok.

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(2) Another important case is when  $Y = \cup_i X_i$  and  $Y$  has the coherent topology wrt the family  $\{X_i\}$  of subspaces. Here the maps  $f_i$  are nothing but the inclusions  $X_i \hookrightarrow Y$ . By the very definition of coherent topology, it follows that this is the co-induced topology. However, not all co-induced topologies are coherent with the original family of topological spaces  $(X_i, \mathcal{T}_i)$ . A special case of importance which we have discussed earlier, when we have a countable family  $\{X_i\}$  of topological spaces such that each  $X_i$  is a closed subspace  $X_{i+1}$ . The co-induced topology in this case, as we have seen earlier, is coherent wrt to the family  $\{X_i\}$ .

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Another important case is that we studied you know before. So, this is a little more complicated situation now actually. So, you have to pay more attention to this one than to the quotient space construction where there is only one function. Take the case wherein  $Y$  itself is now union of  $X_i$ 's and  $Y$  has the coherent topology with respect to  $X_i$ 's which are subspaces.

Remember the what is definition of coherent topology.  $Y$  has a topology and in that topology,  $X_i$ 's are subspaces and then it satisfies some condition. So, that is the meaning of coherent topology remember that ok.

You may say that this coherent topology is a special case of coinduced topology. Namely, you ignore the topology on  $Y$  with which you started right in the beginning. Now, give the

co-induced topology on  $Y$  using the topologies on  $X_i$ 's ok. And what is the map? Maps are from  $X_i$  to  $Y$  inclusion maps ok?  $X_i$  to  $Y$  inclusion maps. What you get is nothing but the original topology on  $Y$ . So, by the very definition of coherent topology, it follows that this is the co-induced topology.

Why, what is an open subset of  $Y$ ? In the definition you see  $A$  is open in  $Y$  if  $A \cap X_i$  is open in  $X_i$  for every  $i$ , that is precisely the definition of coinduced topology here when each  $f_i$  is inclusion map.  $U$  contained inside  $Y$  will be put here if and only if  $U \cap X_i$  is inside  $\mathcal{T}_i$  for all  $i$ . Over.

So, it will give you the topology with which you started with. So, coherent topology is a special case of co-induced topology ok. However, not all co-induced topologies are coherent with the original family of topological spaces  $X_i$  to  $Y$ . I want to make it clear.

Suppose  $(X_i, \mathcal{T}_i)$  are subspaces ok, and you have taken the inclusion maps and now you give the co-induced topology on  $Y$  that topology will be coherent maybe, but you started already with the topology there right. Some topology you started and took these subspaces, and then you took this co induced topology. That topology may not be equal to the original topology on  $Y$ . So, in that sense it may not be the coherent topology.

So, not all co induced topologies from the original subspaces may be coherent topology that is the meaning of this. A special case of importance which we have discussed earlier, when we have a countable family  $X_i$ 's of topological spaces one contained in the other and each  $X_i$  is closed subspace of  $X_{i+1}$ . In that case the co-induced topology gives you the same topologies which you started with.

So; that means, it is coherent topology. The co induced topology in this case we have seen earlier is coherent with respect to  $X_i$ . There are special cases when this can happen like open subsets that is another case or locally finite closed sub families etc, ok. So, there are cases wherein this can occur, also cases wherein this does not occur ok.

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The screenshot shows a video lecture interface. At the top, there is a title bar with the text "Coherent maps to the family  $\{X_i\}$ ". Below this is a navigation bar with the name "Anant B Shastri Retired Emeritus Fellow Department of Mathemat" and the course title "NPTEL-NOC An Introductory Course on Point-Set Topology, P". A small video window in the top right corner shows the speaker, Anant Shastri. The main content area is a blue table of contents with the following items:

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Below the table of contents is a text box with the following text:

(3) Another special case of importance is when  $\{X_i\}$  is the family of compact subspaces of a locally compact Hausdorff space. We shall discuss this example again, when we study locally compact Hausdorff spaces. Similarly, a lot more about quotient maps can be studied only after we get familiar with other topological notions.

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Another special case of importance is that  $X_i$  is the family of compact subspaces of a locally compact Hausdorff space. So, this I am mentioning that because it is so important. So, right now I just mentioned that we are going to study this one later on when we study compact spaces and locally compact Hausdorff spaces and so on ok. Right now, you do not know these terminologies let us say. So, do not worry about this.

Similarly, a lot more about quotient maps can be studied only after we get familiar with other topological notions ok. So, we have just begun the study of induced and co-induced topologies. So, we have explicitly five different cases of this ok; two for this two for the other five or four maybe. Subspace topology, product, quotient and this one is union right. So, and coherent topology and so on. Union under union coherent topologies under union itself anyway yeah?

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**Example 2.86**  
Here is an otherwise useless example, included only to test your understanding of co-induced topology.  
Let  $(X, \mathcal{T})$  be any topological space. Consider the family  $\{\eta_x\}_{x \in X}$  of inclusion maps  $\eta_x : \{x\} \rightarrow X$ . What is the co-induced topology on  $X$  by this family?

So, I will conclude it with a with an easy example here ok. But, this is only for testing the depth of our knowledge of co-induced topology alright. So, take a break that is all. This example is not of any use. Let  $(X, \mathcal{T})$  be any topological space ok. Consider the family  $\eta_x, x$  belonging to  $X$ , what are  $\eta_x$ 's? They are inclusion maps of the singletons inside  $X$ . Inclusion map from  $\{x\}$  to  $X$ .

Now these singleton  $x$  are topological spaces right. What is the co-induced topology from this family on the set  $X$ ? What is the co induced topology? That is what I am asking ok. By definition, something is open in  $X$  ok if and only if intersection with  $\{x\}$  is open in  $\{x\}$  for every  $X$ .

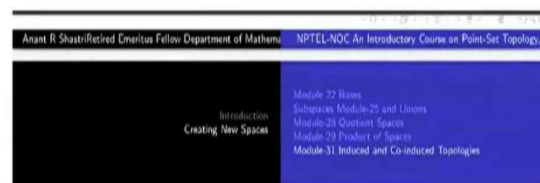
If this point is not in the set  $U$ , then the intersection is empty. So, it is open. If it is inside that, then the intersection is singleton  $x$ . Singleton  $x$  is an open subset of  $\{x\}$ . So, what I have concluded? I took an arbitrary subset of  $X$ , I have shown that it is open right? So, this will always give you the discrete topology on  $X$  right. This is a discrete topology on  $X$  ok.

For example, you know you can test some special case, you could have taken  $X$  to be  $\mathbb{R}$ , then all singletons will be actually closed subspaces. Yet, the co induced topology is discrete and not the original  $\mathbb{R}$  at all ok.

So, just closedness is not enough. See, in this theorem we had everything is closed. In this one and  $X_i$  is closed inside  $X_{i+1}$  increasing sequences etc we have right; or we had what is called, locally finite family of closed sets also ok, neither of the condition is satisfied here.

This family is not locally finite ok, in general. Inside  $\mathbb{R}$  for example, it is not locally finite. So, when it is locally finite it just means that the space is a discrete space you can prove that ok ah sorry, ah ah this is this is locally this is not locally finite, but it is point finite ok. So, this is also a nice example in a way, but let us leave it. I mean this is just for understanding what happens to the co-induced topology.

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**Remark 2.87**

In conclusion, starting with a topological space  $(X, \mathcal{T})$  and a cover  $X = \cup_i X_i$ , of some special importance, it is often the case that we consider the co-induced topology  $\hat{\mathcal{T}}$  on  $X$  which is usually finer than  $\mathcal{T}$ .



In conclusion, starting with a topological space  $(X, \mathcal{T})$  and a cover. Just like when you had a function  $X$  to  $Y$ , a surjective map in the case of quotient space. You take all the points otherwise some points are left out. There will not be any structure on that part. So,  $X$  is union of  $X_i$ 's of some special importance. These  $X_i$ 's you have chosen nicely you know to represent the whole.

They cover the whole thing, like a representative subsets. It is often the case that we consider the coinduced topology  $\hat{\mathcal{T}}$ , that is notation, which is usually finer than the given topology  $\tau$  ok. So, it is a finer topology than  $\tau$ . So, this is an important phenomenon. Many nice



mathematical ideas have been developed from this one. This can be used by politicians also perhaps, in the construction of states and you know countries and so on.

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Introduction  
Creating New Spaces

Module-28: Quotient Spaces  
Module-29: Product of Spaces  
Module-31: Induced and Co-induced Topologies

Anant Shrivastava

**Exercise 2.88**

Let  $X = \cup_i X_i$ , where each  $X_i$  is given a topology  $\mathcal{T}_i$ . Let  $\hat{\mathcal{T}}$  denote the topology co-induced from the collection  $\{\mathcal{T}_i\}$ . Put  $\tau_i = \hat{\mathcal{T}}|_{X_i}$ .

(a) How the two topologies  $\mathcal{T}_i$  and  $\tau_i$  on  $X_i$  compare with each other?

(b) Show that  $\hat{\mathcal{T}}$  is coherent with respect to  $\{(X_i, \tau_i)\}$ .

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NPTEL

Module 22: Basis  
Subspaces, Metrics, and Compact

So, here is an exercise now which you can immediately answer because I have explained the things behind it ok. Let  $X$  equal the union of  $X_i$  where each  $X_i$  is given the topology  $\mathcal{T}_i$ . Let  $\hat{\mathcal{T}}$  denote the topology co-induced from the collection. Put  $\tau_i$  equal to  $\hat{\mathcal{T}}$  restricted to  $X_i$ .

How the two topologies on  $X_i$  compare with each other? Next thing is show that this tau hat is coherent with respect to  $(X_i, \tau_i)$ , you started with arbitrary  $\mathcal{T}_i$  on each  $X_i$ , it may not be coherent with respect to, but  $\hat{\mathcal{T}}$  is coherent ok.

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Module 22: Bases  
Subspaces, Module 25 and Unions  
Module 28: Quotient Spaces  
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Introduction  
Creating New Spaces

Anant R. Shastri

### Exercise 2.89

Recall definition 2.43 and the remark that followed. In this context, we can define another concept. Let  $X$  be a topological space and  $\mathcal{A}$  be a family of subsets of  $X$ . We say  $\mathcal{A}$  is *point-finite* if each point  $x \in X$  belongs to at most finitely many members of  $\mathcal{A}$ . Now take  $X = \mathbb{R}$  with the usual topology and

$$\mathcal{A} := \{\{r\} : r \in \mathbb{R}\}.$$

Clearly,  $\mathcal{A}$  is a point finite family of closed subsets of  $X$ .

(a) Determine the co-induced topology on  $X$  with respect to the family  $\mathcal{A}$ .

(b) In the light of remark 2.44, what do you conclude?

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And this remark, this exercise all I have already explained it to you,  $X$  equal to  $\mathbb{R}$ , and I am taking  $\mathcal{A}$  equal to this family  $\mathbb{R}$ , it is a point finite family. But what happens you can understand ok. So, this exercise is already explained to you. You can write down the details. So, that is all. Next time, we will meet we will meet with the chapter 3 ok.

Starting in earnest to study topological properties we have reasonably enough number of examples now other than the standard you know non-linear spaces metric spaces and so on ok. So, we can start studying some topological properties from next time.

Thank you.