# Introduction to Point Set Topology, (Part I)

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## Lecture - 30
## Study of Products - Continued

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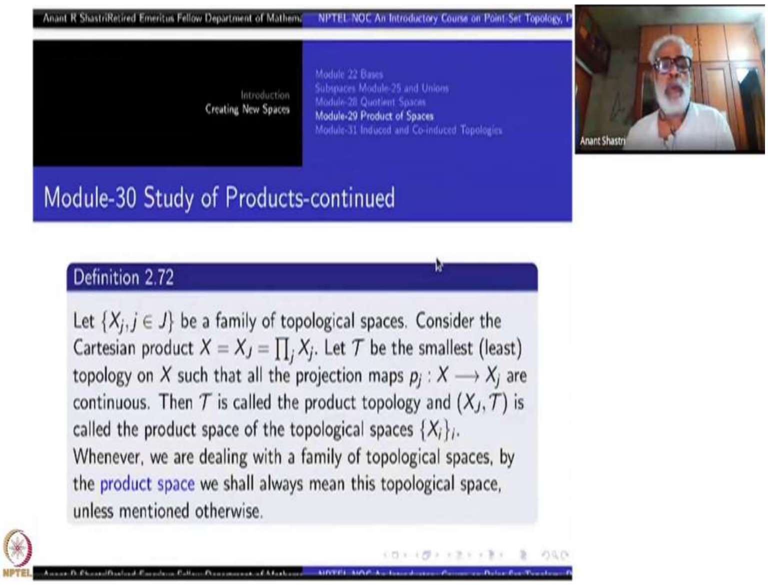


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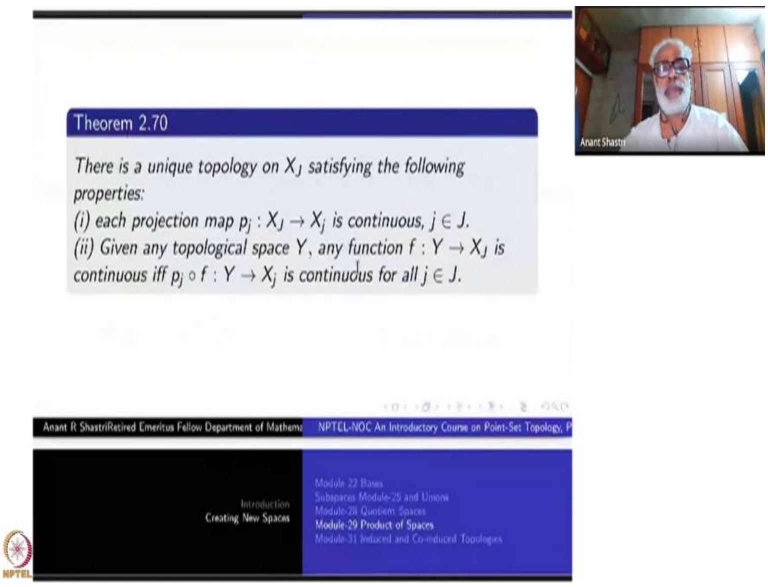

Welcome to module 30 of Point Set Topology Part I. Last time, we introduced the study of product sets basically and started just the meaning of putting some topology on the product set. I will recall this theorem which we did last time. And, then we will go ahead. So, this was the theorem: On $X_J$ which is the product of the family $X_j$'s ok. There is a unique topology satisfying these two conditions.

The projection maps $p_j$'s are all continuous and given any topological space $Y$ a function $f$ is continuous from $Y$ to $X_j$ if and only if $p_j \circ f$ which are so called coordinate functions of $f$. They are all continuous for every $j$. So, this was proved last time.

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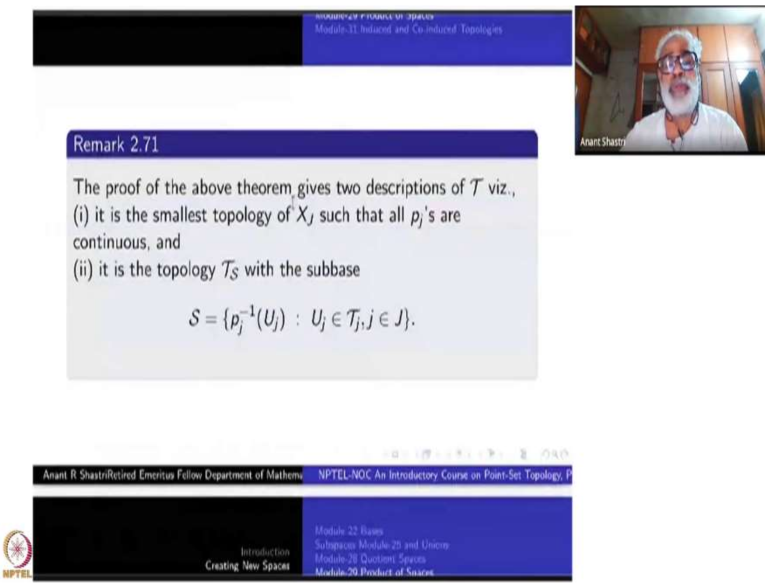


And, we observed that this proof gives you two descriptions of this topology namely you can just say that the topology that we have got here is the smallest topology with respect which all the $p_j$'s are continuous or you can just describe it by the subbase. Take any open subset in $\mathcal{T}_j$ take $p_j^{-1}$ of that. Put all such sets in one single collection $\mathcal{S}$. So, that will be a subbase for this topology it is $\mathcal{T}_\mathcal{S}$ means what? generated by $\mathcal{S}$. So, this is the other description.

So, now let us continue the study of the product spaces to some extent whatever we can do in half an hour that is all. Of course, during the entire course, we will keep coming back to the study of products spaces again and again.

So, start with a family of topological spaces and take the Cartesian coordinate space which we have defined as set of all functions from the indexing set j into the union of X j's with certain property ok.

So, start with a family of topological spaces and take the Cartesian coordinate space which we have defined as set of all functions from the indexing set $J$ into the union of $X_j$'s with certain property ok. Let $\mathcal{T}$ be the smallest typology such that all the projection maps are continuous. So, this is the theorem that we had. So, I am just recasting this as the definition now. Then, $\mathcal{T}$ is called the product topology and $(X_J, \mathcal{T})$ is called the product space of what of the collection $\mathcal{T}_j$, you know $X_j$'s.

Whenever we are dealing with a family of topological spaces by the word product space we shall always mean this topological space unless mentioned otherwise. Why I am telling this one because though $X_i$'s are given, there may be many different ways of putting topology on $X_J$. When you say product space you should take this that is the convention now. Just like when we are taking $\mathbb{R}$ and then we say usual topology if the topology induced by the distance function there right. So, that is the convention.

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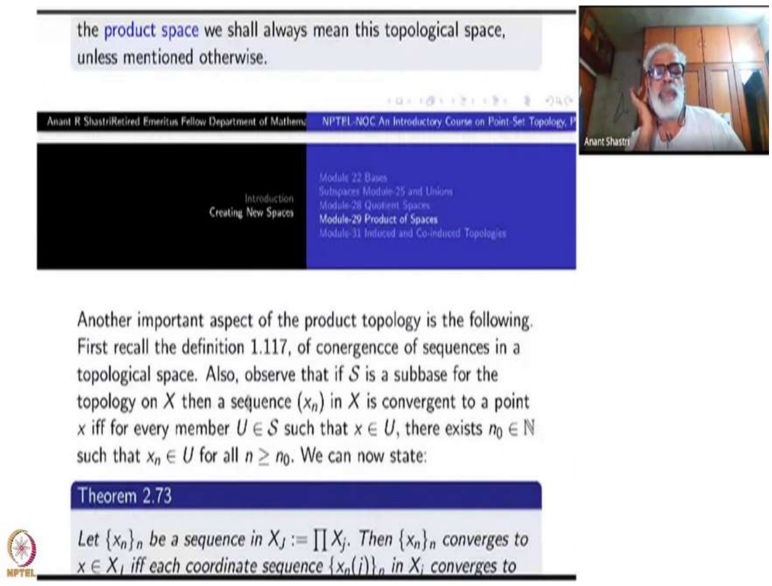

Another important aspect of the product topology is the following. First recall this definition of convergence of sequences in a topological space.

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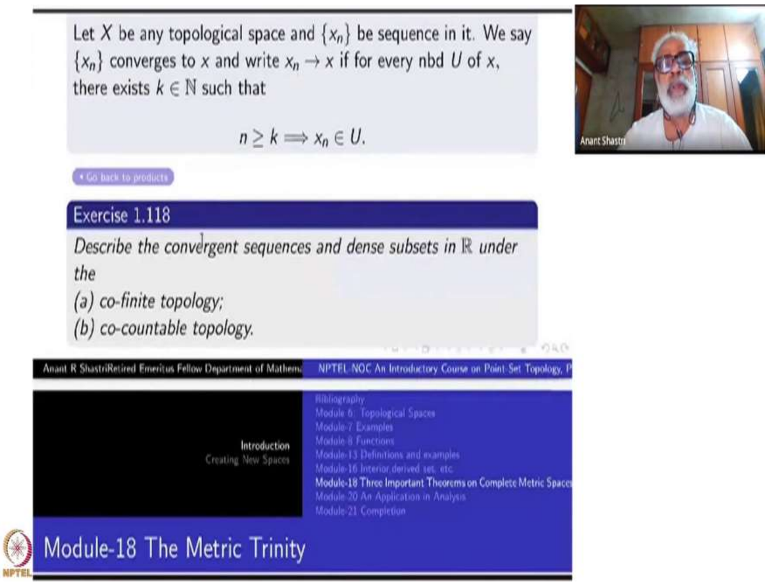


What is that? Here given any topological space $X$ a sequence $\{x_n\}$ converges to $x$, if only if for every neighborhood $U$ of $x$ there exist an integer $k$ such that $n \geq k$ implies $x_n$ is inside $U$ ok? I am just recalling this definition then I am going to use it here now.

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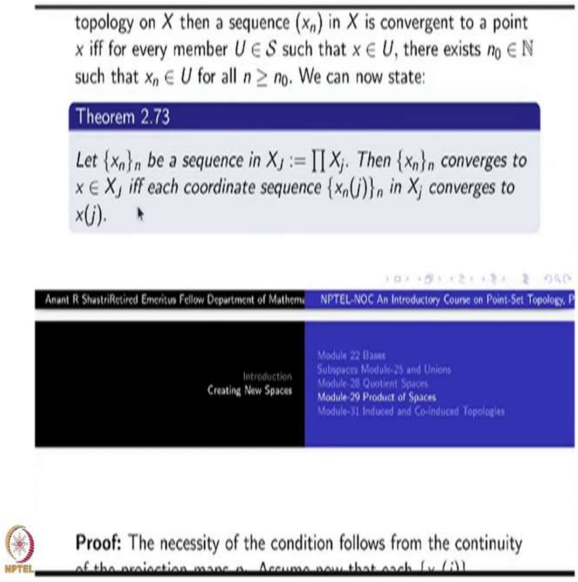


So, a sequence $\{x_n\}$ inside the product space $X_J$ ok. this will converge to $x$ belonging to $X_J$ if and only if each coordinate sequence is convergent. Evaluate each element at the $j^{th}$ coordinate then you get a sequence $x_n(j)$ in $X_j$. So, that must converge to $x_j$. What is $x_j$? $x_j$ is the $j^{th}$ coordinate of $x$ ok.

So, this is another of more or less characterizing the topology, but that is not what we are going to do. We are going to do only one way. Characterization is little more stringent. They do not work in complete generality. So, if you take the product topology then it has this property is what we are going to see.

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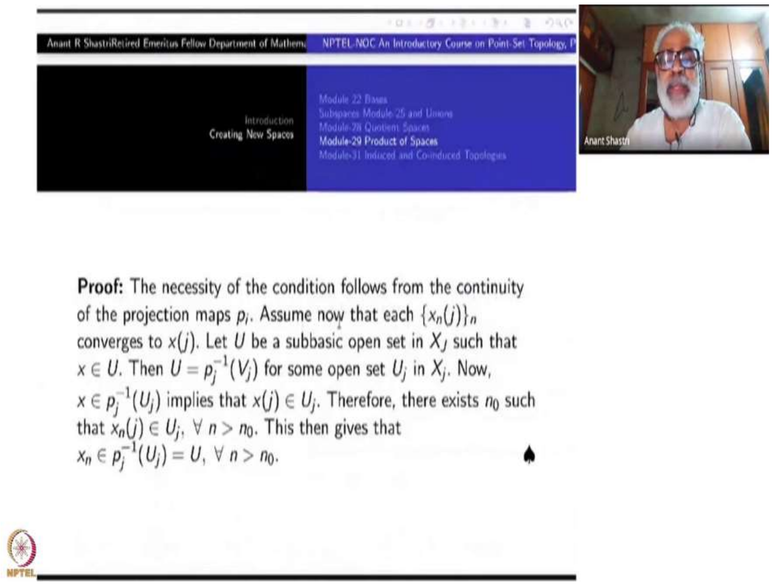


The necessity of the condition follows, from the continuity of projection maps. once this sequence is convergent $p_j$ of that sequence must be convergent because $p_j$'s are continuous $p_j(x_n)$ of this sequence is actually the sequence $x_n(j)$. And where they should converge? To $p_j(x)$. So, the continuity of $p_j$'s will ensure this if this is a convergent sequence then each $x_n(j)$'s is a convergent in $X_j$.

Converse is what we have to take care of. That is also a one-line proof. First of all you must observe that for every $U$ containing $x$ we must verify something, but it is enough to do it for subbasic open sets. Once you have done it for subbasic open sets, it gets verified for finite intersections of subbasic open sets, by taking the maximum of this $n_1, n_2, \ldots, n_k$ integer got for each of the case we have taken. So, that will give you the verification for all basic open sets. Once it is verified for basic open sets around each point in an open set, there is always a basic open around that point and contained in the given open set. So, the property gets verified for all open sets.

So, this is the elaborate way of putting the whole thing, but we have seen all these things, the role of bases and sub bases. So, here I am going to use it for the first time. So, this is the way economically we can do a lot of work that is the whole idea of base and subbase after all. So, let $U$ be a sub basic open set in $X_j$ such that $x$ belongs to $U$, then I must find an $n_0$ such that for $n \geq n_0$ all the $x_n$'s are inside this open set. So, that is what I have to do.

Since $U$ a subbasic open set it will look like $p_j^{-1}(U_j)$ for some open subset $U_j$ in $X_j$ for some $j$, right? Now, $x$ belongs to $p_j^{-1}(U_j)$ is the is the hypothesis right? Because $x$ is inside $U$. $U$ is this one ok. So, this implies that $p_j(x)$ is inside $U_j$ in $X_j$. But now convergence of the sequence $x_n(j)$ will tell you that there is an $n_0$ such that all the $x_n$'s are in $p_j^{-1}(U_j)$ for $n \geq n_0$

Because, sorry all the $x_n(j)$'s are inside $U_j$ for $n \geq n_0$, but that is the same thing saying, $x_n$ belongs to $p_j^{-1}(U_j)$ and that is $U$. So, $U$ is arbitrary open subset a subbasic open set and we have verified the property for all of them. So, this is done ok.

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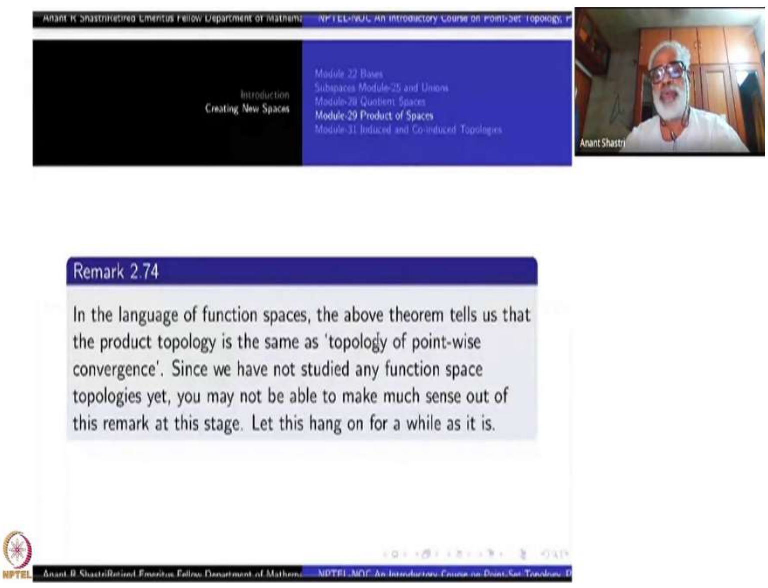


So, let us go ahead with the number of remarks here. In the language of function spaces, the above theorem tells us that product topology is the same as topology of pointwise convergence. If you do not know this terminology from function spaces you are excused from understanding this ok. When you come to that and your analysis teacher says this is pointwise convergence then you will say oh this, I know that is the other way around; it is actually what is happening.

Since we have not studied any function space topology, you may not be able to make much sense out of this remark at this stage ok. So, therefore, let this remark hang for awhile ok? Do not throw it away it will be useful when you study functional analysis or any function space topology and so on.

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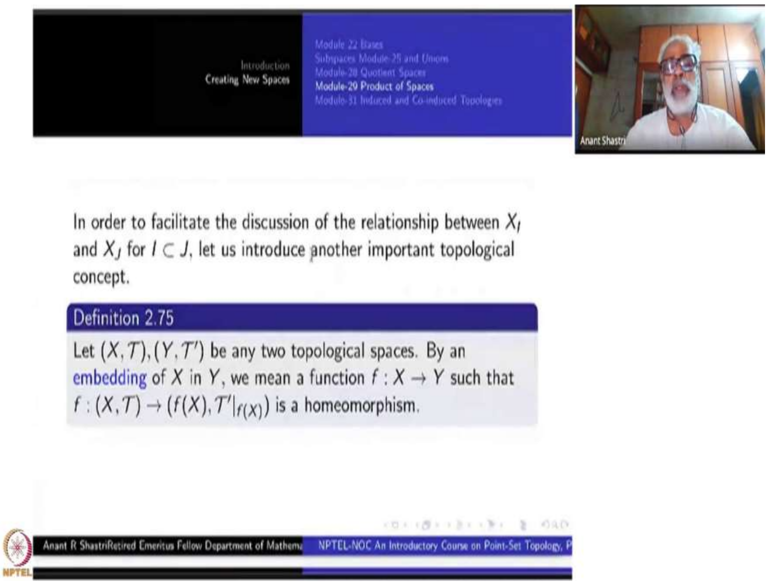


In order to facilitate the discussion of relationships between $X_I$ and $X_J$ where $I$ is a subset of $J$, let us introduce an important typological concept here. This is just a stopgap definition, but the concept itself will be useful elsewhere also. So, I am taking this opportunity to introduce that one here. Start with any two topological spaces ok, By an embedding of $X$ in $Y$, we mean a function $f$ from $(X, \mathcal{T})$ to $(Y, \mathcal{T}')$, such that $f$ from $(X, \mathcal{T})$ to $f(X)$ is a homeomorphism.

First part I am taking this subspace $f(X)$, the subspace topology is $\mathcal{T}'$ restricted to $f(X)$. So, this is the notation for subspace topology, remember that. So, now, instead of $(Y, \mathcal{T}')$, I am taking just $(f(X), \mathcal{T}')$ restricted to $f(X)$. So, this must be a homeomorphism ok.

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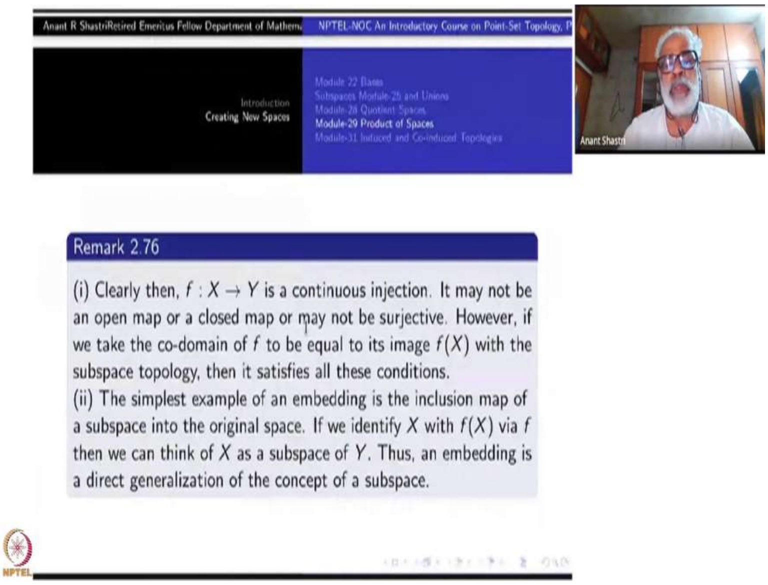


So, let me elaborate on this one namely, if $f$ is a continuous injection ok first of all, why because the same $f$ is there it is a homeomorphism. So, it must be an injection and it must be continuous from $X$ to $f(X)$. From $f(X)$ to $Y$, there is the inclusion map composite with that, that is also $f$. So, there are two different things there this is $f$ restriction right. So, I am using the same notation $f$ for both of them. So, the original $f$ and this one is $f$ to $f(X)$ here and then $f(X)$ to $Y$ inclusion map. So, if this is continuous that will be also continuous. So, $f$ must be continuous first of all and injective map right? So, this must be necessary.

It may not be an open map, may not be a closed map, may not be surjective. All these things are true when you come down here to $f(X)$. Because it is a homeomorphism. If you are here only continuity and injectivity is still there right. So, this is what you have to assume. So, it may not be an open map or a closed map or may not be a surjective; however, if you take the co-domain of $f$ to be equal to the image of $f$, i.e., $f(X)$ with the subspace topology, then it satisfies all these conditions. So, that is the definition of an embedding.

The simplest example of an embedding is the inclusion map itself from a subspace into the original space ok. If we identify $X$ with $f(X)$ via the map $f$ namely $x$ goes to $f(x)$, then we can think of $X$ as a subspace of $Y$ right? Because $f(X)$ is a subspace of $Y$ now $X$ is $f(X)$ is replaced by $X$ and so on.

Thus an embedding is a direct generalization of the concept of a subspace ok. You can almost confuse it with a subspace, but do not confuse it because we want to have some separate identity for $X$, sometimes ok. So, whenever it is convenient you can identify this $X$ with $f(X)$, that is the whole idea here, because they are after all homeomorphic, homeomorphic to each other ok.

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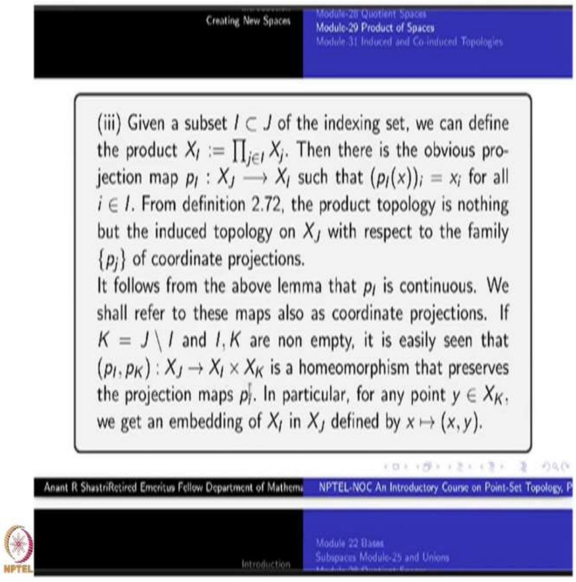


Let me continue with some more remarks: Given a subset $I$ of $J$ (oh this was my aim ok?) of the indexing set. So, I am taking a subset of the indexing set we can also define $X_I$. What is $X_I$? product of all $X_i$'s where $i$ is inside $I$ ok. This is some independent product set and so on. So, what is the relation between the $X_I$ and $X_J$ then? There is the obvious projection maps from the larger's this one $X_J$ to $X_I$.

What does it do? Just ignore all the other coordinates. Suppose $J$ consists of only two elements. Take $X_1$ and $X_2$ ok? From $X_1 \times X_2$ to $X_1$ or $X_1 \times X_2$ to $X_2$, you have the projection maps right. Similarly what do you do in the general case? You just drop out one of the coordinates or several of the coordinates, retain the rest of them as they are. So, this is what is $p$ is doing here $p_I$ of $X_J$ to $X_I$ retains all the $I$ coordinates inside $I$ ignores all of them inside $J \setminus I$. $p_I(x)$ equal to $x_I$, this is all I have.

Because there are no other coordinates here. All those coordinates which are here namely, all those $j$'s inside I they are there ok. Once again from the definition of the product topology, this topology this product is nothing but the induced topology on $X_J$, with respect to the family $p_j$ of coordinate projections.

It follows from the above lemma that all the $p_I$'s are continuous ok? No matter which subset $I$ of $J$ you take ok? This $I$ is fixed now this $p_I$ is continuous, why? Because the $j^{th}$ coordinate of this one $p_I$ nothing but the old $p_j$. So, all the $j^{th}$ coordinates for $j \in I$ they are continuous. So, $p_I$ is continuous we shall refer to these maps $p_I$ also as coordinate projections.

It is like $X_1, X_2, X_3$ going to $X_1, X_2$ ok. So, we have been using all these things in the case of $\mathbb{R}^n$ and $\mathbb{C}^n$ and so on. So, only those practices we are putting in a general setup that is all ok. So, we shall refer to these maps also as coordinate projections.

Let now $K$ is $J \setminus I$; that means, the left out indices here. And let us assume that both $I$ and $K$ are non empty. This is a standing assumption of course, it is easily seen that the map $(p_I, p_K)$ from $X_J$ to $X_I \times X_K$, is a homeomorphism. This is a 1-1 map is clear. Onto is also clear. So, bijection is obvious here set theoretically. Why this is continuous? Look at any coordinate projections here ok, they are all the old one of the $p_I$'s or $p_K$ belonging to $K$.

So, they are continuous if you go from here to this way $X_J$, the same coordinate functions will give you that that is also continuous ok. So, this is a homeomorphism that preserves the projection maps the $I^{th}$ projection here the $I^{th}$ projection here this same thing it is more or less like the identity map ok.

So, this is the way we identify $\mathbb{R}^2 \times \mathbb{R}^2$ with $\mathbb{R}^4$, right. The beauty of this notation is now it is independent of the order on the indexing sets. Therefore, I can write $X_I \times X_J$ or $X_J \times X_I$. It will be the same $X_J$ up to homeomorphism is what you want to say ok. In particular, for any fixed $y \in X_K$, we get an embedding of $X_I$ in $X_J$ defined by $x$ going to $(x, y)$, see $x$ is varying over $X_I$, $y$ is fixed which is inside $X_K$.

So, the whole thing will be in $X_J$. So, this is just like the example wherein you take $x$ going to $(x, 0)$ or $x$ going to $(x, 1)$ or $x$ going to $(x, 1500)$ or the other way round you can take $y$ going to $(0, y)$ right. So, all those are embeddings. So, that is why the word embedding was defined just before ok these are all embedding. So, the partial products can always be thought of as subspaces in various ways, you may say horizontally and vertically and so on. So, these coordinates can be thought of as horizontal and these are as vertical, when you partition the indexing set into two subsets, $I$ and $K = J \setminus I$.

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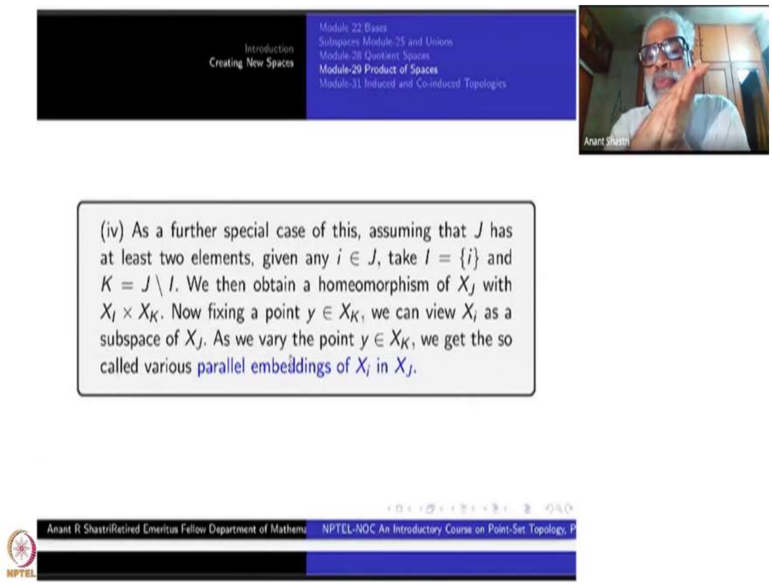


As a further special case of this, assume that $X_J$ has at least two elements then given any $i$ belonging to $J$, take $I = \{i\}$ ok. Then you have to fix an element $y$ in the rest of them $X_K$, then what it will give? It will give you various embedding parallel embeddings, of $X_i$. Inside $X_J$, instead of $X_I$, it will be a smaller $i \in J$. So, this is a special case which I have already explained with examples, alright.

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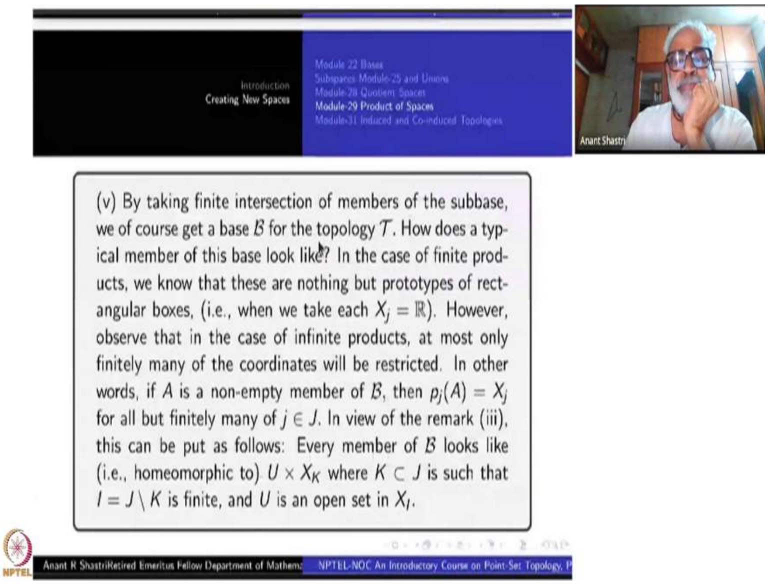


There is more interesting thing, I have to tell you. By taking finite intersection of members of a subbase, we of course get a base $\mathcal{B}$, for the topology. How does a typical member of this base look like? In the case of finite product we know that these are nothing but prototypes of rectangular boxes, interval cross interval cross interval and so on right, right.

Especially, when each $X_j$ is $\mathbb{R}$; however, observe that in the case of infinite products at most only finitely many of the coordinates will be restricted. What is $p_j$ inverse of... let us say what is $p_1^{-1}(U_1)$, where $U_1$ is an open subset of $X_1$? Just $U_1$ first coordinate cross the entire of $X_{J\setminus\{1\}}$, in our notation here ok.

Take $K = J \setminus \{1\}$ right. In one particular case here $K \setminus I$, you have taken $I$ as to be singleton to fix one point there then you have got embeddings. Now, I am looking at inverse image of whole set $U$. So, the second coordinate onwards will be completely free. So, that it will look like $U_1$ cross the entire of the space built upon $J$ except 1.

Suppose similarly you take $p_2^{-1}(U_2)$, $X_1$ will be free again only $U_2$ will be restricted $U_2$ coordinate will be restricted $p_2^{-1}(U_2)$ will have only $U_2$ restricted all other coordinates are free. Therefore, the intersection will have only $U_1 \times U_2$ and then rest of the coordinates are all free no conditions there.

So, this is not a box this could be especially so when $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$, we are taking all of them are $\mathbb{R}$, the other coordinates are freely varying infinitely unbounded right? So, this is what we have to careful here. Except the finitely many case is fine just like the box, but in the case of infinite products at most only finitely many of the coordinates will be restricted.

In other words, if $A$ is a non empty member of $\mathcal{B}$, then $p_j(A)$ just assume that $A$ is non empty that is all; $p_j(A)$ ok, equal to $X_j$ for all but finitely many $j$ belong to $J$ what I am assuming? $A$ is a basic open set ok? I am making the comment on basic open sets here. In view of the remark (iii) this can be put as follows. This I have already told you how every member of $\mathcal{B}$ looks like; that means what?

Homeomorphic to, $U \times X_K$ where $K$ contained inside $J$ is such that $I$ which is $J \setminus K$ is finite. So, this is co-finite and $U$ is an open subset of $X_I . I$ is finite $U$ itself will look like $U_1 \times U_2 \times \cdots \times U_K$, only $U_1 \times U_2 \times \cdots \times U_K$, but every member of $\mathcal{B}$ looks like that $U_1 \times U_2 \times \cdots \times U_K \times XK$.

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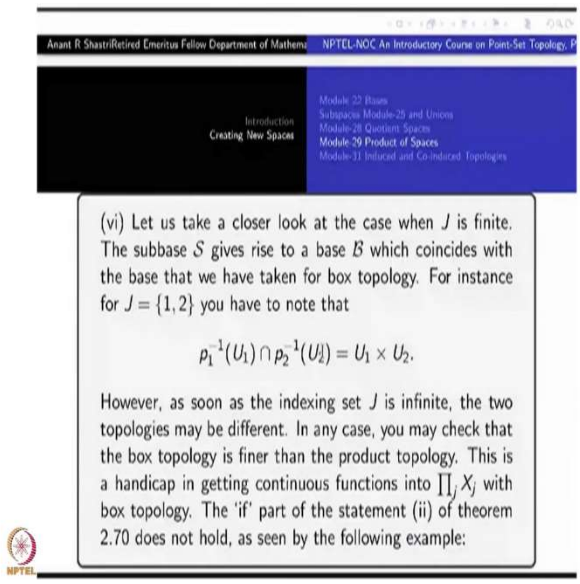


So, coming back to the finite case. Suppose $J$ is finite. The subbase $\mathcal{S}$ gives rise to a base $\mathcal{B}$ which coincides with the base that we have taken for box topology namely take all $U_1 \times U_2 \times \cdots \times U_K$'s ok. For instance, when $J$ is just two element set you have to note that

$p_1^{-1}(U_1) \cap p_2^{-1}(U_2)$ is just $U_1 \times U_2$ just intersection which this will says the first coordinate is restricted inside $U_1$, second coordinate restricted to $U_2$ therefore, it is $U_1 \times U_2$.

However, as soon as the indexing set j is infinite the two topologies may be different. In any case you may check that the box topology is finer than the product topology. This is a handicap in getting continuous functions into the product of $X_j$'s infinite products with box topology. The 'if' part of the statement 2 of theorem 2.70 does not hold, as seen by the following example.

See in the product topology we have the condition number 2 there. A function is continuous if and only if all the coordinate functions $p_j \circ f$ those are continuous ok. So, I will show you now an example very easy example which will violate this one in the case of box topology ok.

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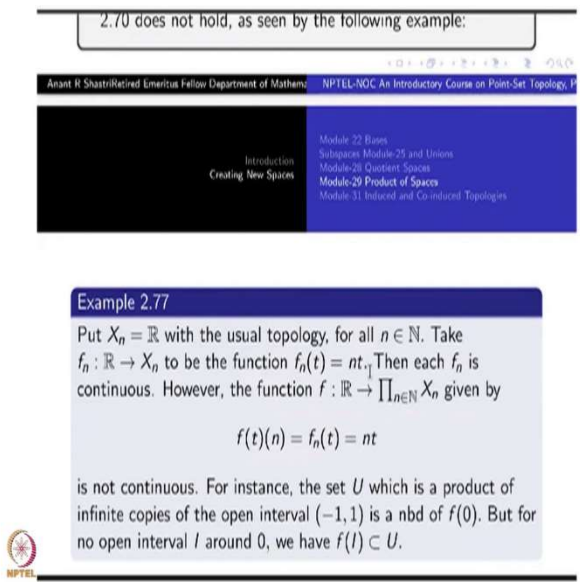


So, put $X_n$ equal to $\mathbb{R}$ with the usual topology for all $n$ and $X$ equal to product of $X_n$'s. So, I am taking a countable family here ok indexed by $n$.

Take $f_n$ from $\mathbb{R}$ to $X_n$ be the function $f_n(t) = nt$ ok. I think I wanted $1/nt$ here. Let us see which one we work $nt$ would not to do $t/n$ is what I wanted. Then, each $f_n$ is continuous ok; however, the function $f$ from $\mathbb{R}$ to $\prod X_n$ the product space given by $f(t)(n) = f_n(t)$ is not continuous. That is the claim. For instance the set $U$ which is a product of infinite copies of the interval $(-1, 1)$ ok.

Look at the interval $(-1, 1)$ is a neighborhood of $f(0)$ in the box topology. What I am taking? Infinite product of $(-1, 1)$ is neighborhood of $0$ in every interval $(-1, 1)$. So, the product will be a neighborhood of $(0, 0, \dots)$ in the box topology All the coordinates $0$, and that is $f(0)$ because all the coordinate $f(0) = n0 = 0$.

But for no open interval $I$ around $0$, we have $f(I)$ is contained inside $U$, can you see that? So, for that what I needed here is $t$ going to $nt$ not $t/n$.

So, $f(I)$ is a bounded interval we take some interval ok. So, I should say that given an open interval here ok; $f$ of something must be inside $U$, that is the continuity. So, that is violated this is not true ok.

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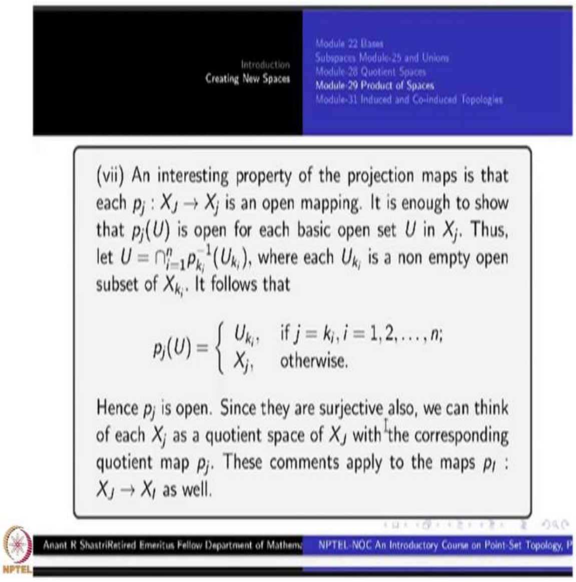

So, an interesting property of the projection maps is that each $p_j$ from $X_J$ to $X_j$ is an open mapping. It is enough to show that $p_j(U)$ is open for each basic open set $U$ inside $X_J$. So, here you have to be careful showing only for sub basic is not enough here. Because you are taking intersections and image of intersections may not be intersection of the images ok. You have to basic open sets, unions will be ok. So, for basic open set if you show that you are done ok, you will be careful. $p_j(U)$ where $U$ is a basic open set I should show that it is open inside $X_j$ for every $j$ ok.

So, let $U$ equal to intersection of $p_{k_i}^{-1}(U_{k_i})$, $i$ ranging from $1$ to $n$. This is how a typical a basic open set looks like. Finite intersections of subbasic open sets. So, this will be basic open set each $U_{k_i}$'s non empty open subset of $X_{k_i}$. Now, what is $p_j(U)$? $p_j(U)$ depends upon what $j$ is.

If $j$ is one of the $p_{k_i}$'s here, then $p_j(U)$ is precisely $U_{k_i}$. Remember in the standard notation this will be nothing but $U_{k_1} \times U_{k_2} \times \cdots \times U_{k_n}$ whatever $n$ terms cross the rest of the all other indices, the entire spaces $X_j$'s therefore, its $j^{th}$ projection would be just $U_{k_i}$ where $j$ is equal to $k_i$, $i$ equal to $1, 2, 3, \ldots, n$.

If $j$ is not equal to any of these first $n$ coordinates ok? If $j$ is one of the other indices, then $p_j(U)$ will be just the whole of $X_j$ ok. In any case, they are all open subsets on the right hand side. Therefore, $p_j$ is an open map alright. All coordinate projections are surjective maps because I am assuming that all the $X_j$'s are non empty that is important here.

Otherwise the product will be empty even though some of the $X_j$'s may not be empty even if one of them is empty the entire product is empty. So, you are assuming that then each $p_j$ is a surjective map also we have seen that a surjective open continuous map is a quotient map. In other words, each coordinate space here $X_j$ with which you begin they are all quotients of the product space ok.

So, this any surjective continuous function then you can give the product we can give the base the quotient topology if you do that first take these $X_j$'s take the product and give quotient space you will get back $X_j$ the original topological space, that is the meaning of this. How did you do this? Just by showing that each projection map is open map ok. The same thing applies instead of one single $j$ you can do it for a bunch of $j$'s. $X_J$ to $X_I$ which we have discussed earlier because that is also surjective open map.

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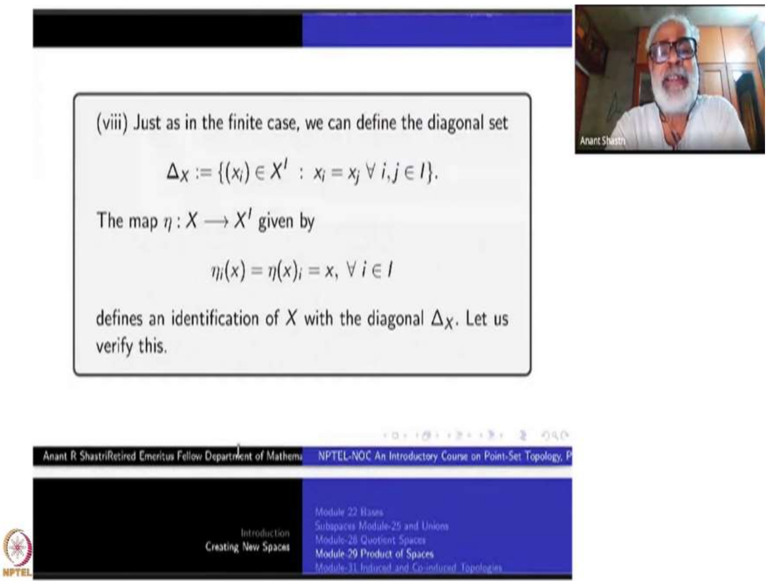


So, I want to discuss a few more things here. Just as in the finite case you have $X \times X$, let us consider the case for example, all the $X_j$'s are $X$ now, one single space ok. Then, we have introduced this notation remember that $X^I$ or $X^J$ whatever now I am taking the indexing set as $I$, it is $X^I$ is what? Is product over $i$, but all the $X_i$'s are equal to $X$, the same topological space, same set same topological space ok.

Just like in the finite case, we can define the diagonal set to be all those $X$'s such that all of them are equal to each other equal to one single $x$ belonging to $X$, ok? That is the definition of diagonal. The map $\eta$ from $X$ to this $X^I$ given by $\eta_i(x) = x$; see when you want to define a map into $X^I$, what you have to do? You have to just mention all the coordinate functions here. $\eta_i(x)$ which you can write it $\eta(x_i)$ ok that must be $x$ for all $i$ that is the definition of this map $\eta$.

So, it will give you an identification of $X$ with the diagonal by which I mean a homeomorphism or here in the case it is an embedding of $X$ inside $X^I$ ok. With respect to this embedding, you can identify $X$ with the diagonal in $X^I$, ok.

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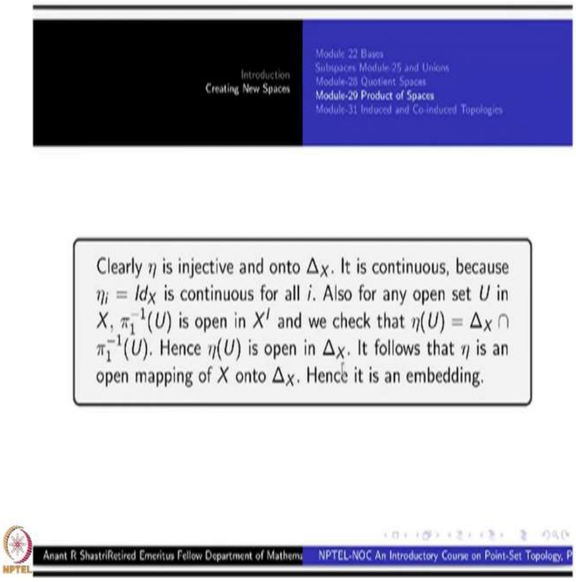


So, I am just giving you more elaborate explanation here; $\eta$ is injective and it is on to $\Delta_X$. It is continuous because $\eta_i$ is identity for all $i$, the coordinate projections $\eta \circ p_i$ is $\eta_i$, this is what I have defined.

So, they are all continuous. So, $\eta$ is continuous, also for any open set $U$ in $X$ ok? $\pi_i^{-1}(U)$ is open in $X^I$ ok? That is by definition is a subbasic open set. How does $\eta(U)$ look like? You start with an open set $U$ in $X$, consider the projection map $p_i$. Now, you take the inverse image of $U$ is an open set in $X^I$. That open set intersection with $\Delta$ is precisely $\eta(U), \eta(U)$ is $\Delta_X \cap \pi_i^{-1}(U)$.

Hence, $\eta(U)$ is an open set inside $\Delta_X$ which means $\eta$ is an open mapping into $\Delta_X$. So, $X$ to $\Delta_X$, $\eta$ is a homeomorphism ok?

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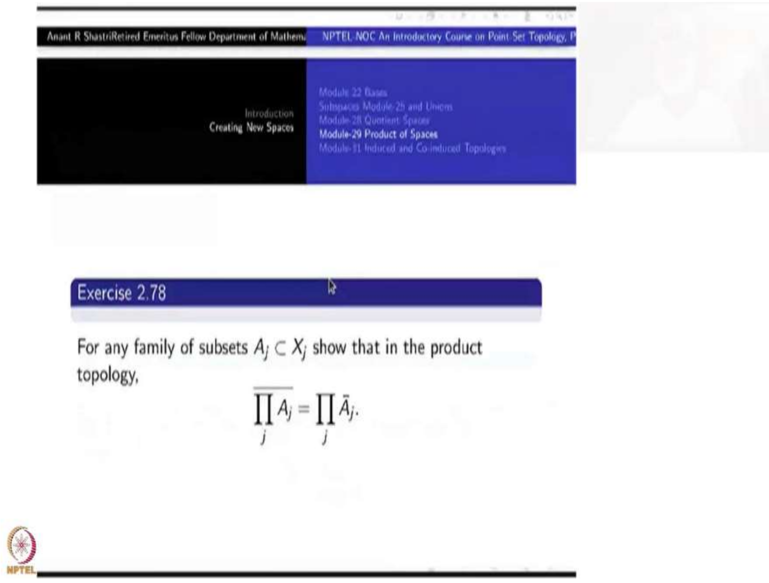


So, let us stop here. Let me just give you one exercise here, namely take subsets $A_j$ inside $X_j$ especially this exercise is for infinite products ok. Look at the product set $A_j$ ok? This is like a box the $j^{th}$ coordinate is $A_j$, $A_1 \times A_2 \times \ldots$.

But this infinite product I am taking, its closure is the same thing as closure of $A_j$'s then take the product. You know whenever infinite processes are there, you have to be careful with the closures right. So, you better do not believe this one, but you know try to disprove it or try to prove it only after proving it you believe it alright? So, this is the exercise, any doubts? Ok. Let us stop here.