Introduction to Point Set Topology, (Part I)
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Module - 03
Lecture - 03
Metric Spaces

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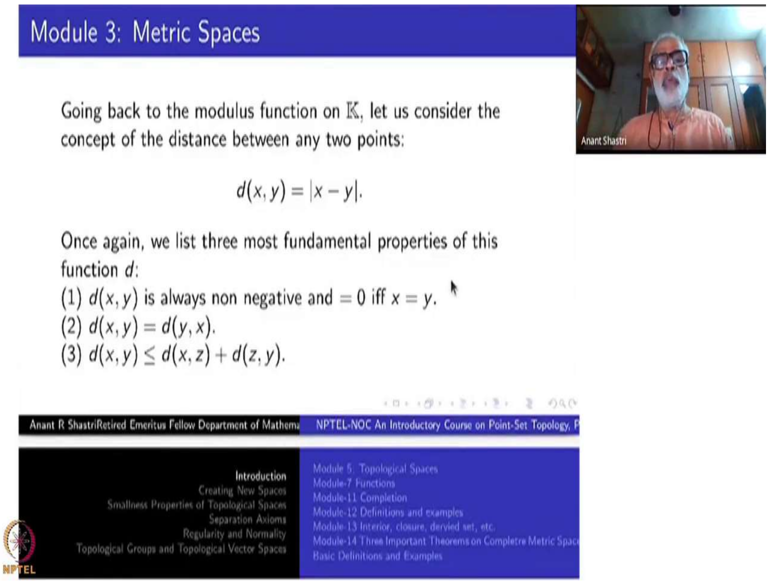

Welcome to module 3 of Point Set Topology course. Last time we had taken a cursory look on what are called as normed linear spaces. They where seen to be an immediate generalization of the modulus function on the field of real numbers or complex numbers, ok?

Today we are going to take one more step towards generalization. Once again we go back to this modulus function on $\mathbb{K}$, this $\mathbb{K}$ denotes, remember, the set of all real numbers or the set of all complex numbers, any one of them. So, take the distance function $d(x, y)$ where x and y are elements of $\mathbb{K}$ to be the modulus of $x - y$, ok?

Inside the real numbers or inside $\mathbb{R} \times \mathbb{R}$ maybe complex numbers this actually coincides with the standard notion of distance. So, it is called, you can call it a distance function. What are the fundamental properties of distance function? Just like, what we observed yesterday, about the modulus function itself, now those three properties will be picked up today also. What are they? The 1st thing is the distance is always a non negative real number and it is 0, if and only if x is equal to y.

The 2nd thing is a distance function is symmetric $d(x, y)$ is the same thing as $d(y, x)$. That is why it is called distance between x and y, distance from x to y is the same thing as distance from y to x so, that is the meaning of this. The 3rd one is, more clearly this time, is triangle inequality distance between x and y is less than or equal to distance between x and z plus the distance between z and y. So, if x, y, z are three points forming a triangle, this rule just says that the length of any side x to y is less than or equal the sum of the lengths of the other two sides namely, x to z and z to y. So, this is much direct as compared to the norm, norm of x plus y was less than or equal to norm of x plus norm of y. To interpret that rule as a triangle inequality you have to use the vector method. Represent x, represent y as vectors, viz. line segments from 0 to the point x, if you take the arrow from 0 to x, and then 0 to y and then take their sum that would be the third point.

So, now the end points become a triangle that is the meaning of the norm of x plus y is less than or equal to the norm of x plus norm of y ok. So, we take exactly those three, but formulated now in terms of d, ok? $d(x, y)$ equal to 0 or what does it mean modulus of $x - y$ equal to 0 ok, which is norm.

So, modulus of x minus y is equal to 0 implies x minus y is 0; that means, x is equal to y. So, this is identical with the condition 1, which we had called norm 1 (N1). The beauty of this one is now the condition is independent of the additivity on the vector space ok. So, that is what has happened.

Similarly, the 3rd condition is independent of the additivity, you do not have to take norm of x plus y there is no x plus y you know distance between x and y is less than or equal to x and plus this, plus is by the way the on the right-hand side this just real numbers in addition ok. So, the left-hand sides here or the starting things here the conditions they are independent of the vector space structure on V. And that allows us to make a sweeping generalization that is what we are going to do.

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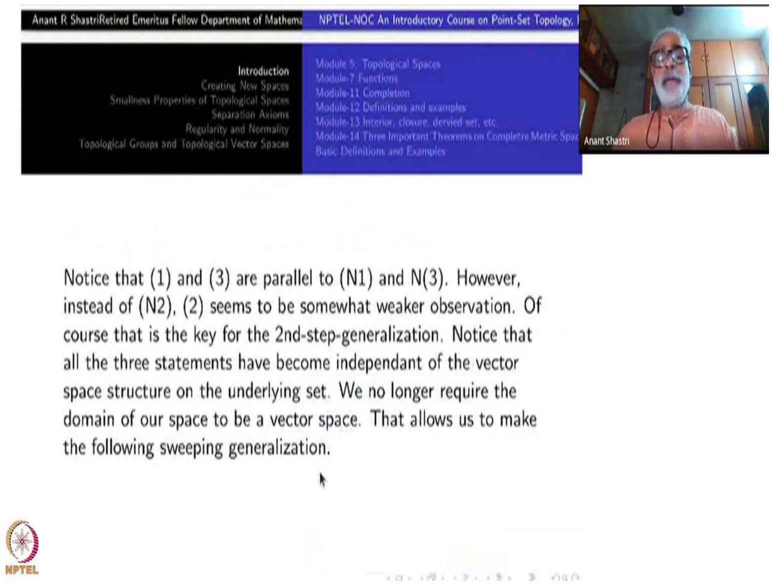

So, I am repeating it here once again, if you take the conditions 1 and 3, they here are similar to N1 and N3, this N1 and N3 were the norm for the norm condition same condition this was whatever you called positive definiteness this was triangle inequality. But, second one seems to be somewhat weaker right? whereas, the N2 there was something about alpha times x, if you take the norm of that modulus of alpha came out ok.

Norm of alpha x was equal to modulus of alpha into norm of x so that was quite a strong condition, but this is one does not seem to replace N2 in no way. here at all. it may be true, but why this is so fundamental? So, that is something which you have to think about ok. But how to derive this one that is not all that difficult ok we will check into that.

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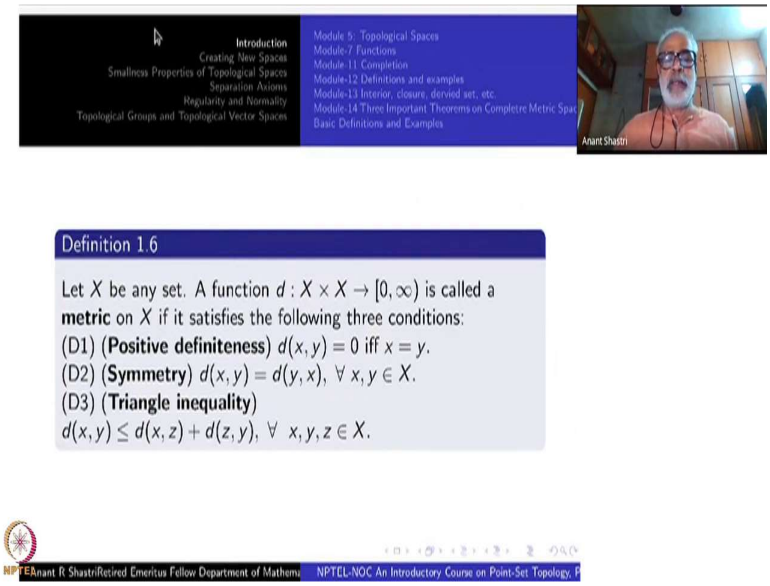

So, right now let us make the sweeping generalization namely start with any set X now. Remember for a normed linear space we have to start with a vector space V. So, there is no extra structure it is just a set X here. Now, take a function d from $X \times X$ to $[0, \infty)$ the norm was directly a function on the vector space; the distance is between two points that is why I have take $X \times X$ to $[0, \infty)$ a function like this is called a metric, that is a name we are going to give ok?

Or you could have called it a distance function. So, classically it has been called a metric. So, we are going to call it a metric, otherwise you could call it distance function also. So, what is the condition? There are three conditions, which you can call as axioms of metric space.

Positive definiteness which corresponds to $d(x, y)$ equal to 0 if and only if x equal to y. Symmetry $d(x, y)$ equal to $d(y, x)$; triangle inequalities $d(y, x)$ is less than or equal to $d(x, z) + d(y, z)$ for every x, y and z. So, take these three conditions here which are true for modulus function on $\mathbb{K}$, generalize it all sets, exactly these three conditions only, that is what we have done. Such a function will be called a distance function or a metric on X . ok?

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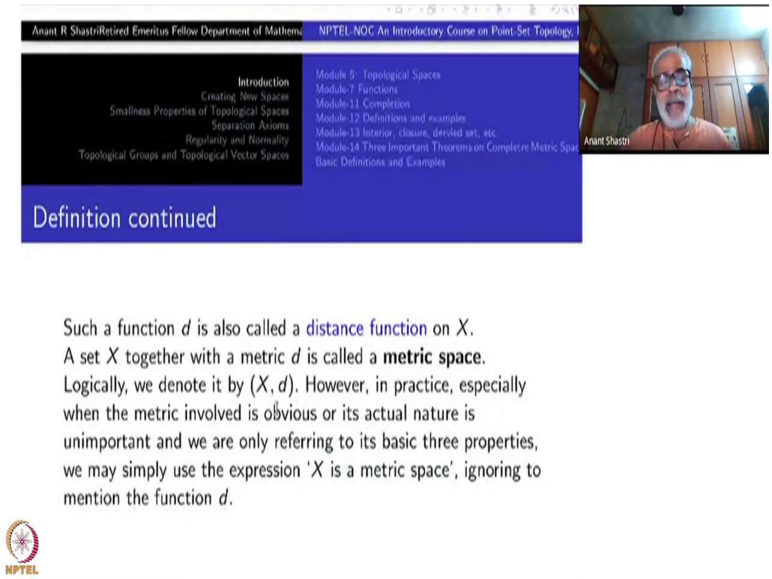

So, what we have is take any set X and a metric on it, the ordered pair (X, d) is going to be called a metric space ok. But, quite often we will just say X is a metric space without mentioning what is our d. Quite often the distance function, the metric is understood by the context ok?

Or it is mentioned just a few minutes back. So, you do not have to again and again mention it that is the only reason why you have to do that one, just to save some time; otherwise logically every time you have to say (X, d) is a metric or metric space then only it makes sense ok. Is that clear?

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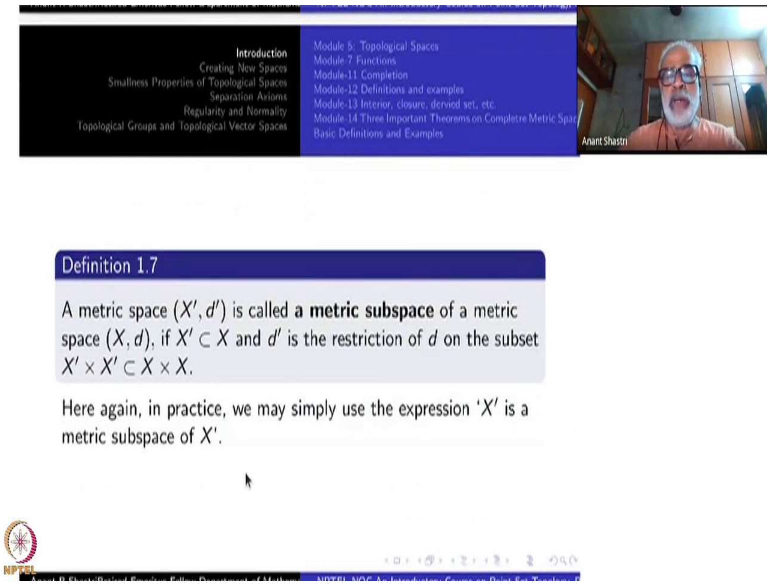

Now, one sweeping generalization definition is over that is all. So, slowly you have to build up the theory ok, just like what we have done were subspace of a normed linear space. What is it? It is a vector subspace and then the norm is restricted to the subspace. Similar way take any metric space ok (X, d) take a subset X' restrict the metric d to $X' \times X'$, say d', then you will get a metric subspace. So, then what you can say (X', d') is a metric subspace of (X, d) ok. So, this is a definition.

X' is subspace subset of X, d' is restriction of d on $X' \times X'$ which is a subset of $X \times X$; d is defined on $X \times X$, take the restriction. All those properties will be automatically true for (X', d') ok? So, we can just say X' is a sub metric space instead of my mentioned d' etc.

We can quite often write it as same d also quite often restricted function we use the same notation ok this is just to save time and you know instead of cumbersome, too many notation that is all. We can just say X' is a sub metric space of X ok. So, that is the just a second definition after making a metric space we have made a metric subspace. Then slowly we have to develop a number of terminologies and theories.

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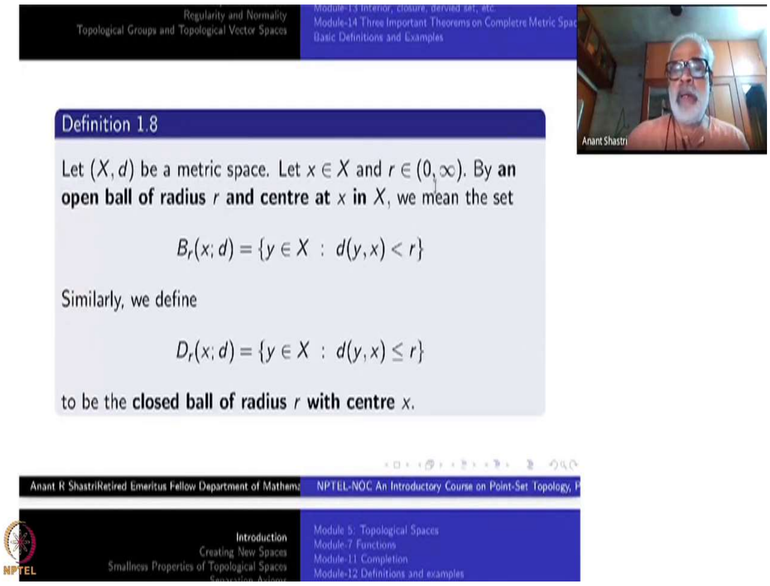

So, all based on similar experience with modulus function, and what it is doing for real numbers or complex numbers. Remember that ok? So, here are a few more terminology. So, start with a metric space take any point in the space, take any real number between 0 and ∞, any positive real number.

Now, by an open ball of radius r and center x in X; this is the definition now ok. What is the meaning of open ball? An open ball always has a centre and a radius, radius must be some positive number. It is not clear, what it is, you know open ball is an open ball ok?

What is it? It is the set of all points y inside X, which are at a distance less than r from x, $d(y, x)$ is less than r. That is an open ball. Just put in equality also here, $d(y, x)$ less than or equal to r allow equality also here then you get a closed ball. So, we will use the notation B_r for open ball and $D_r(x, d)$ for closed ball this x denote the center, r denote the radius and d denotes the metric.

In the same space X, same set X. If you change the metric d to some other metric these balls will be different, obviously, ok? So, that is why you have to put that d also there, but if you understand what this d is, just like I can call X is a metric space here also I do not need to mention d, then I will have a shorter notation.

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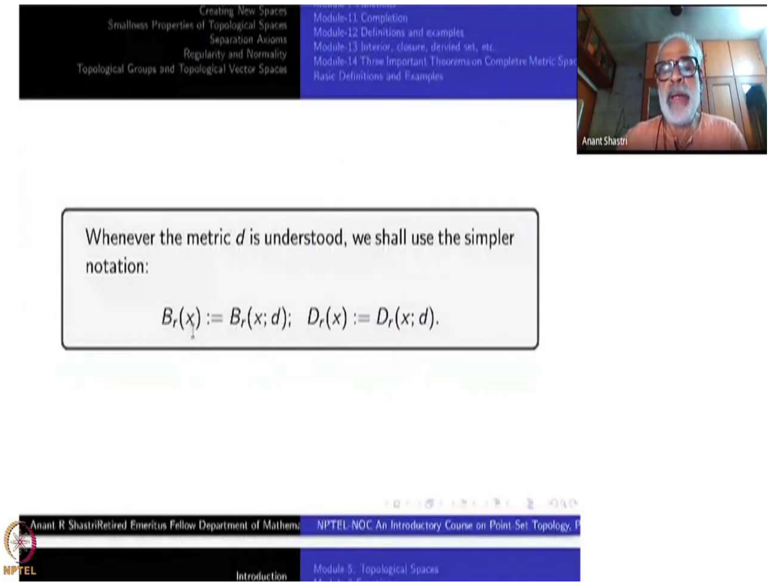

Namely $B_r(x)$ is $B_r(x;d)$ that d is not mentioned right that d is understood. Similarly, $D_r(x)$ is $D_r(x;d)$ ok. This is provided you know what metric with respect to which we are speaking. Alright? So, now, you know a few more definitions.

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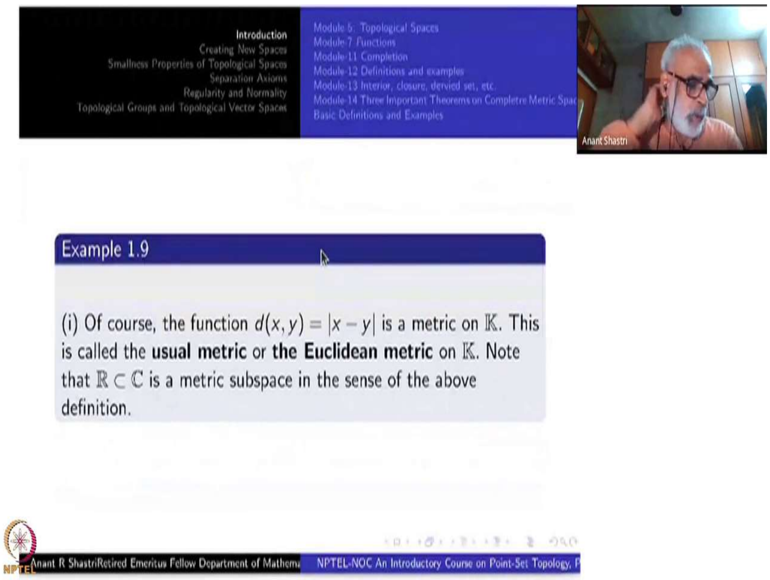

So, let me again go back to emphasize that our standard example with which we begin is just the field $\mathbb{K}$ itself together with $d(x,y)$ equal to $|x-y|$ this was the starting example after all. So, this first example, this is called the usual metric on $\mathbb{R}$ or $\mathbb{C}$ whatever coming out of the modulus function ok, it is also called Euclidean metric because it is so ancient, goes back all the way to Euclid ok, more than 2000 years old, ok.

So, in that sense if you take whatever we have defined namely take a metric on $\mathbb{C}$ restrict it to $\mathbb{R}$, then you will get a metric subspace, but this function is the same ok? When you take x and y as real number, when you take the modulus of $x - y$ it is the same thing as when you take them as complex numbers ok. So, this is the first example of a metric space and a sub metric space $\mathbb{R}$ is a sub metric space of $\mathbb{C}$, ok with the usual metric.

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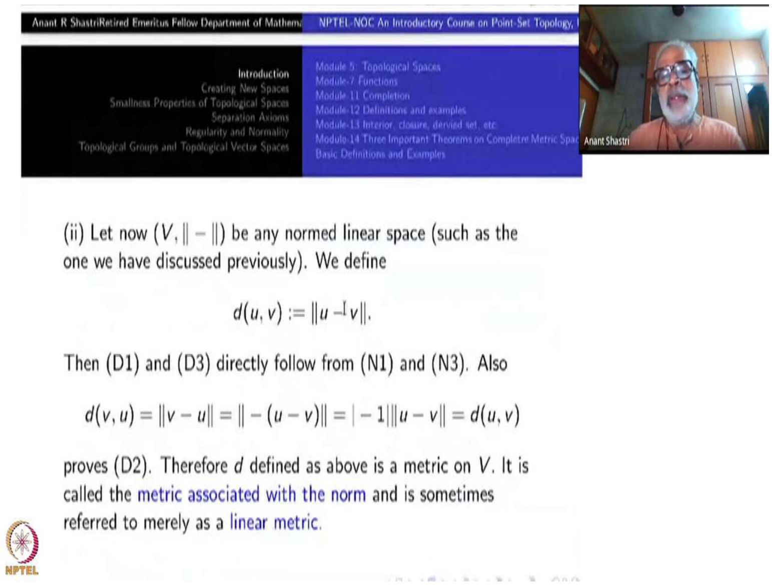

So, the second example here is: Take any normed-linear space that we have studied last time a vector space together with a norm. Then you define $d(u, v)$ just like we defined modulus of $x - y$ you define norm of $u - v$; $d(u, v)$ is norm of $u - v$. Then D1 is first one follows directly from N1 right, what is D1? $d(u, v)$ is 0, if and only if u equal to v, $d(u, v) = 0$ means norm of $\|u - v\| = 0$ then $u - v = 0$; that means, u is equal to v so, that is N1.

Similarly, N3 which is triangle inequality stated in terms of vectors is equivalent to triangle inequality stated in terms of the distance function D3. So, they are fine, but what we wonder is this number 2. How do you get it. $d(u, v)$ equal to $\|u - v\|$, that is u here $d(v, u)$ is by definition I want to show that it is equal to $d(u, v)$ right, $d(v, u)$ is norm of $\|v - u\|$; it has the symmetry property. that is a point.

It is minus 1 times norm of u minus v and v minus u can be written like this, but this now -1 comes out with a modulus that is now just 1 so, that is equal to $d(u,v)$. So, the symmetry property of the norm function, which is much weaker than ok modulus of alpha times something coming out alpha coming out and so on, that is just forgotten you do not need that one.

But, we have retained something else namely the symmetry ok. So, that is the beauty. So, this is definitely going to be something more general ok? If I just write $d(\alpha u, \alpha v)$ here, there is no way this α will come out; $|\alpha|$ will not come out in general there is that is not part of the axiom whereas, if your metric is defined using a norm then that is true ok. So, that is what I want to repeat here. First of all, whenever we have a metric coming out of a norm then we call it a metric associate with the norm.

In short, we can call it a linear metric. The linearity is coming from norm of αx is equal to $|\alpha|$ times norm of x. And of course, triangle inequality; triangle inequality is always there. So, number 2 makes it a linear metric here ok, but in general this is not a part of the definition of the metric we have to be careful about that, ok.

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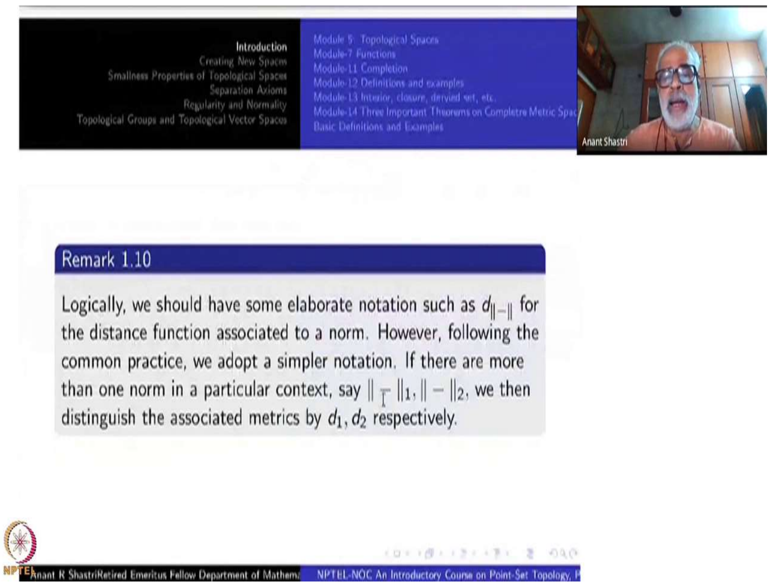

And whenever such a property holds, we may indicate it by writing $d_{\|-\|}$ whatever norm you have taken. Suppose there are two different norms on the same vector space v they will give rise to two different metrics. So, which one? So, you better tell that. So, in that case suppose if norms are written as $\|-\|_1$ and $\|-\|_2$, two different norms then I can say correspondingly d_1 and d_2 are the metrics. So, this is the way we will treat these metrics related to the corresponding norms, ok.

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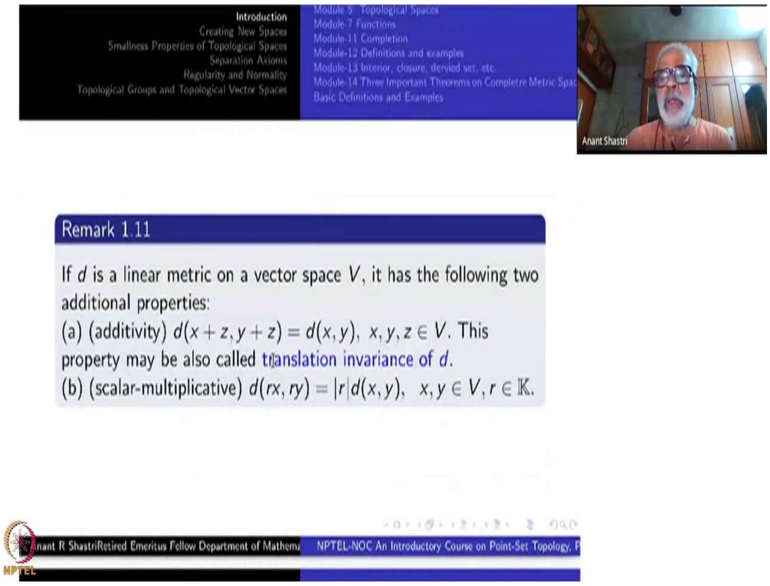

If d is a linear metric, it has two additional properties. So, I repeat it. I have already told you one is this $d(rx, ry)$ is $|r|$ times $d(x, y)$, where x, y belong to V, r belong to $\mathbb{K}$ ok. The second property which is also hidden there is that distance between $x + z$, and $y + z$ (note that $x + z$ and $y + z$ makes sense, because I am working inside a vector space V now) is same thing as what? Norm of x minus y; x plus z minus y plus z, z cancels away is equal norm of $x - y$, which is equal to $d(x, y)$. So, this property is additivity you can directly state ok, on a vector space if you start with the metric suppose we can put this as a condition. So, one can have this as condition it is satisfied with the linear metric.

But, in addition I can put this as a condition for an arbitrary metric on a vector space, then it will be called translation invariant metric. A linear metric is automatically translation invariant ok this condition itself is called translation invariance of d. Similarly, it is multiplication invariant; you can say this is scalar multiplicative namely r does not come out directly r comes out as $|r|$ ok.

So, something is not a linear metric if even this condition or this condition is not satisfied, that is easy way of checking whether a given metric comes from a norm or not ok. If it is then this will be satisfied, but that does not guarantee that it is coming from then there is a likelihood that it may be coming from ok. On the other hand, if these any of these condition is not satisfied then you are happy ok? this is not a norm.

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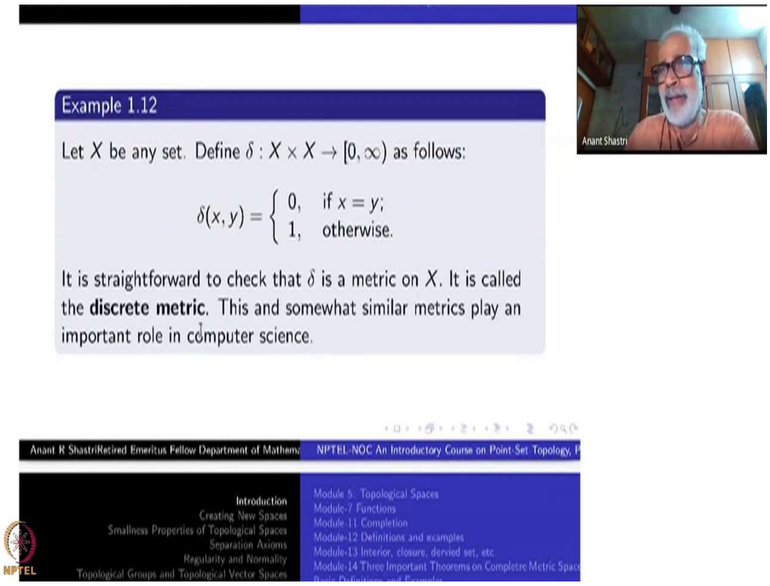

So, let us have something now which is not coming from a norm at all or maybe coming from norm, but we have not bothered, we can directly define them ok. So, there are lots and lots of such metrics, which arise without reference to any norm. Let us see a few of them ok. Start with any set X. Define this delta. $\delta(x, y)$ equal to 0 if $x = y$ and equal to 1 otherwise; as soon as x and y are different, it is 1.

This delta is actually the opposite of the dirac delta function in analysis ok? So, that is the, you know contrast here. It is straightforward to check that this is a metrics. What you have to check? $d(x, y)$ 0 implies x is equal to y that is built-in in the definition here ok, $d(x, y)$ is equal to $d(y, x)$ that is also built-in in the definition.

Triangle inequality takes two more seconds. You just think about it, why this satisfies triangle inequality? It does not take more than two seconds that is all you have to think about ok. So, this is a metric and it has a name. It is called discrete metric not Dirac metric ok yeah. So, this is discrete metric.

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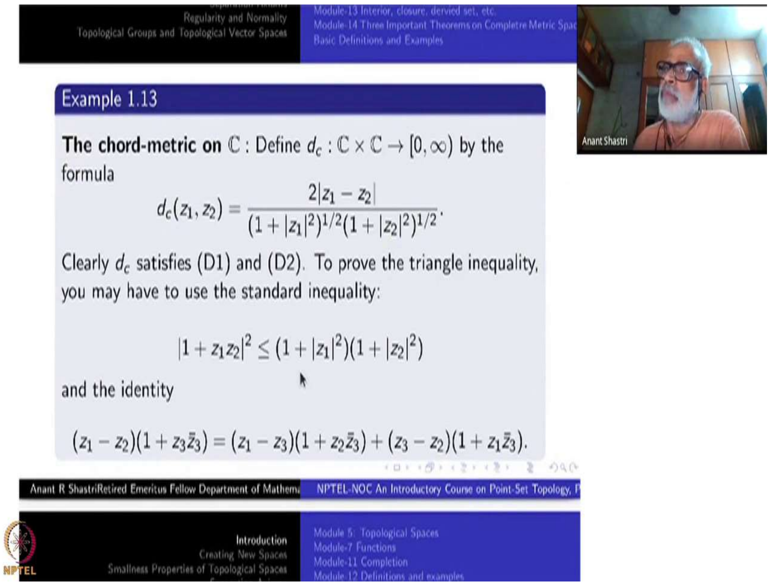

Next, I will give you something which is quite non trivial out of the blue. I will not explain today, how it came, but I will just give this much, today you have only to understand the computational aspect of it here. I will do part of it only and leave the rest as an exercise. So, what is this? The chord metric. It is defined on the set of complex numbers it is not coming from any norm ok, at least not on $\mathbb{C}$ as a vector space.

So, this d_c, c for chord d_c from $\mathbb{C} \times \mathbb{C}$ into $[0, \infty)$ is defined as $d_c(z_1, z_2)$ is twice the $|z_1 - z_2|$.... The numerator is quite close to a norm now its twice the norm of module of $z_1 - z_2$.

But, there is a denominator here which will kill all these properties ok. What is it? $(1 + |z_1|^2)$ into $(1 + |z_2|^2)$ whole thing raised to half, the square root of the product of these two $(1 + |z_1|^2)$ and $(1 + |z_2|^2)$, ok. So, that is the denominator this is the definition see I have divided, but this is never 0.

So, I can divide. So, this makes sense. So this is now a real number ok? a non negative real number. If it is 0, the numerator must be 0; that means, z_1 is equal to z_2; $|z_1 - z_2|$ is 0. So, z_1 is actually equal to z_2, ok? So, this satisfies condition D1. Next, look at $d(z_2, z_1)$ is perfectly symmetric here because $|z_1 - z_2|$ is $|z_2 - z_1|$ and these things are you can change this stuff to here that is all. So, this is symmetric.

So, what is difficult here? The difficulty is; not very difficult either, but it is not quite easy either is in proving the triangle inequality. By the way, in all these 'new' examples triangle inequality verification is the most difficult part ok? Other things if they are there true, come easily. If they are not true, you cannot help it. they are not there. Over, ok. So, so let me help you to see how the triangle inequality comes here. I will not do the entire computation.

The first thing is there is some inequality of complex numbers take z_1 and z_2, any two complex number ok. Then $|1 + z_1 z_2|^2$ is less than or equal to $(1 + |z_1|^2)$ into $(1 + |z_2|^2)$. Where is this coming from? Look here. It is a square of the denominator there. If you take the square root so, modulus of $1 + z_1 z_2$ is less than or equal to this one. So, this is the first inequality that you have to see. So, this denominator here is bigger than $|1 + z_1 z_2|$. The square is not there now.

This itself is not difficult. You have to just expand the left hand side. $1 + z_1 z_2$ modulus square is nothing but $1 + z_1 z_2$ into $1 + \overline{z_1 z_2}$, which is same thing as $1 + \bar{z}_1 \bar{z}_2$. Now, expanded, and collect the two central terms namely twice real part of $z_1 z_2$. It can be replace $|z_1|^2 + |z_2|^2$. Then you will get this ok.

So, that is the explanation for this one. There is another one here. Though looking somewhat threatening, but this is much simpler this is an identity. It is a product of two terms on the LHS here and it is a product of two terms plus product of two terms on RHS here. So, there are 4 terms here and 8 terms there. But you can immediately see that four of the terms cancel out in pairs-- minus z_3 here plus v here and z 1 z 2 z 3 bar minus z 2 z 1 2 z 3 bar.

So, something you will cancel out what you are left with, what you will get is same term here so, this is an equality, ok. Once you do this snd combine the two, triangle inequality will come very easily ok? So that part I will leave it to you. So, I leave the further calculations to you as an exercise alright?

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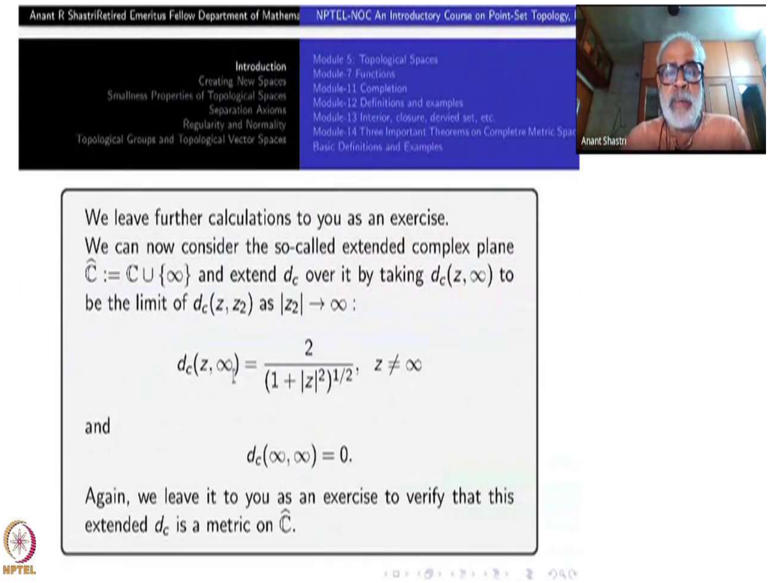

Now, we go one step ahead here look at what is called the extended complex plane. In this you take the complex numbers along with the infinity one single infinity ok? Don't do much of algebra there, but you can do a lot of topology and analysis that is why it is there, this extended complex plane. So, one extra point set theoretically that is all, but now I want to extend the metric, the chord metric to this set $\hat{\mathbb{C}}$. What you have to do is: whenever both of the points are in $\mathbb{C}$, this is the formula.

So, if one of them is infinity I have to define what is the meaning of this namely, I have to define take any z in the first slot and in the second you have ∞, what is $d_c(z, \infty)$ is what I have to define. So, what I do take z_1, z_2 here. And let z_2 go to infinity. That take the limit of this expression as z_2 going to infinity, ok?

What we get is this formula of course, instead of z_1 I am writing z here that is all. So 2 divide by $(1 + |z|^2)$ raised to half. Work it out. It is not very difficult. How do you take the limit as z_2 infinity? This is infinity by infinity method right? Do not go to differentiation etc. Just pull out z_2 that is all.

Pull out mod z_2 here in the numerator, in the denominator, $|z_2|^2$ will come then raised to half. So, $|z_2|$ that will cancel out. Then take the limit it will be easy it will be just this much ok. This will define d, whenever one of them is infinity.

Finally, when both the points are infinity what should I do? I am forced to define that equal to 0 there is no other choice ok, take this as definition. Now, you have to verify that for this extended thing also the triangle inequality is true ok.

So, z infinity you take some other point here ok. z_1, z_2 some other z' and then you have to show that this inequality this is not very difficult. So, that also I will leave it to you as an exercise. So, extended complex plane has a metric here. So, because of this denominator here we were able to do that.

If you just take the linear metric $z_1 - z_2$ you could not take this limit as z tends to infinity. Now, you see why the you know the geometry or analysis whatever you want to say is reflected here so, ok. More about this one I will tell you when the time comes, alright.

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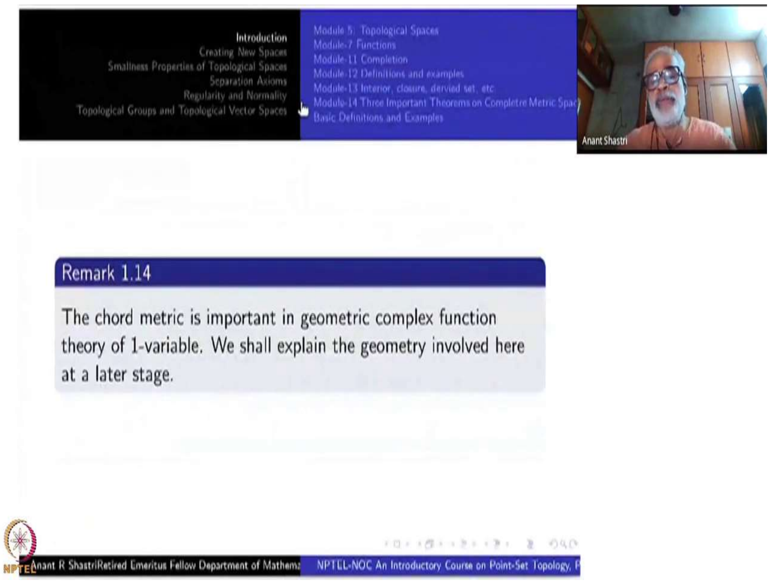

The chord metric is very important in the geometric function theory of 1-variable complex numbers ok. So, the geometry involved here will be explained to you a little later alright. So, let us take a break here today we have done some good work. So, next time we will see more.

Thank you.