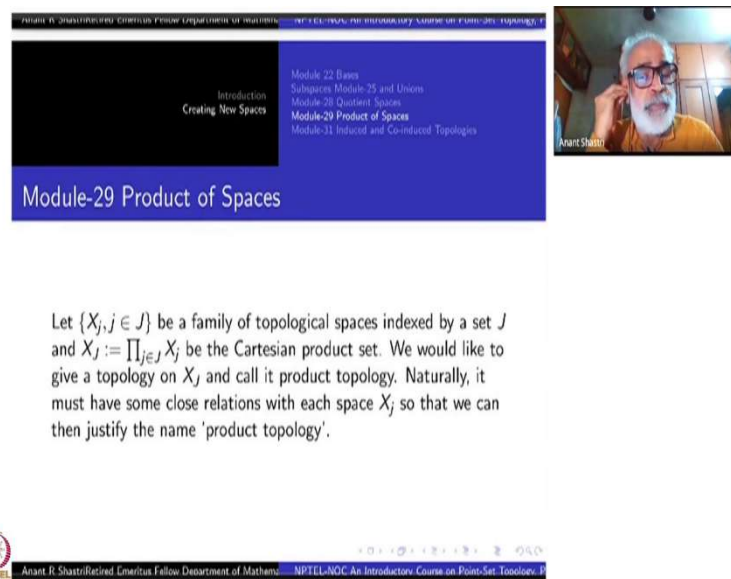


Introduction to Point Set Topology, (Part I)
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Lecture - 29
Product of Spaces

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Module-29 Product of Spaces

Introduction
Creating New Spaces

Module-22 Bases
Subspaces, Metric 25 and Uniform
Module-28 Quotient Spaces
Module-29 Product of Spaces
Module-31 Induced and Coinduced Topologies

Let $\{X_j, j \in J\}$ be a family of topological spaces indexed by a set J and $X_J := \prod_{j \in J} X_j$ be the Cartesian product set. We would like to give a topology on X_J and call it product topology. Naturally, it must have some close relations with each space X_j so that we can then justify the name 'product topology'.

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Welcome to module 29 of Point Set Topology part 1 course. So, today we will pick up another topic Product of Spaces. Again within the our general topic of producing new topologies new topological spaces out of the old and so on. So, recall that we have already defined what is called the box topology on atleast finite products right? In fact, we have also done it for infinite products.

Now, we want to bring in the most natural way of doing this namely, the function theoretic approach to the products not through sub base and basis of course. We will also see sub base and basis here finally, just like we did while understanding, set theoretic understanding of quotient spaces.

It is very much important to understand the set theoretic aspect of of a product. Namely infinite products ok? An order implicit order on the indexing set is present in the back ground

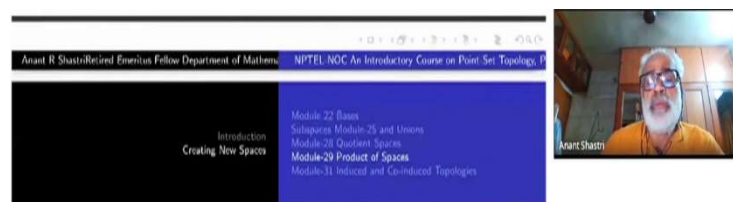
when you take finite products you write it as $X_1 \times X_2 \times X_3 \times \dots$ the indexing set has been given a very nice order 1, 2, 3, 4 and so on right.

We should come out of that you know, it is like a baby starts learning bicycle, she is given the support wheels to begin with. So, at that time support wheels are necessary. The order on the indexing set is like support wheels to understand what is going on. To begin with, but it is a hindrance later, in the general case ok?

So, we should get rid of the support wheels. So, let X_j where $j \in J$, J is the indexing set, be a family of topological spaces ok. Then look at this product space the Cartesian product space. What is the product set first of all. What is it that is what I want to understand. What exactly is the set theory? Ok?

And then on each X_j if you have a topology then what is the corresponding topology on the product ok? One of them we have defined, but is there only one, or there are some other way of doing it. So, that is the kind of thing you want to understand ok? And the one which comes out from our pursuit here will be called the product topology. Earlier one which you had got was called box topology. We shall also compare these two things.

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First of all we must understand correctly what is the Cartesian product of the sets X_j 's. In case J is finite, we put an order on this indexing set and think of the product as the set of all ordered n -tuples $x = (x_1, \dots, x_n)$ with the condition that $x_j \in X_j$. This is what we understand by Cartesian product set

$$X_1 \times X_2 \times \dots \times X_n.$$

We also know that the choice of order on the indexing set is often immaterial in the sense that the permutation of coordinates defines a bijection.



First of all we must understand correctly what is the Cartesian product set ok, where X_j s are some indexes families of sets, ok? I repeat. If J is a finite set, the indexing set, we usually put an order on this indexing set and think of the product as the set of ordered n tuples ok? (x_1, x_2, \dots, x_n) , where x_1 comes from X_1 , x_2 comes from X_2 and so on x_n comes from X_n . That is what we understand by Cartesian coordinate set as such. And then this notation can be used $X_1 \times X_2 \times \dots \times X_n$, ok.

We also know the choice of order on the indexing set is immaterial $X_1 \times X_2$ can be written as $X_2 \times X_1$ also, for most of the time unless you are really doing some geometry something like binary theory and so on. So, there it may be of some value some importance not to change the order ok. Certainly in topological considerations you do not see any role of this one right? ok.

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So, let us be done away it. We shall altogether get rid of this order so that this concept can be easily generalized. Recall that ordered n tuple x , I am writing single x for (x_1, x_2, \dots, x_n) , what is it? It is a sequence of length n taking values in X_1, X_2, \dots, X_n right?

So, if we get rid of the order we can still think of this as a function on the set J of n elements which was the indexing set there ok? And the sequence taking values inside the union of all

X_j 's of course, with additional condition that the j^{th} image of x now I am thinking of x as a function, which I write it as x_j , the j^{th} coordinate, that is an element of X_j ok? We see that in this form, it immediately generalizes to describe the product when J is an arbitrary set also ok?

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Definition 2.67

The Cartesian product X_J of the family of sets $\{X_j : j \in J\}$ is defined to be the subset of the set of all functions $x : J \rightarrow \cup_j X_j$, which satisfy the condition such that $x(j) \in X_j$. We allow ourselves the notation $X_J := \prod_{j \in J} X_j$ for this set and for each member $x \in X_J$, the notation $x_j := x(j)$. The element x is also denoted by (x_j) following the practice of ordered n -tuples. The assignment $x \rightarrow x_j$ defines a function $p_j : X_J \rightarrow X_j$ which we call the j^{th} coordinate projection. Note that p_j can also be thought of as the evaluation map on the element $j \in J$.

The Cartesian product we can write it as X_J , instead of $X_1 \times X_2$ etc, the whole product set X_J , I am writing ok? Or this other notation $\prod_{j \in J} X_j$, this is defined to be the subset of all functions from indexing set J into the disjoint union of $X_j, \cup_j X_j$ with one extra condition that the image of j under x is inside X_j , for each j in J , this must be true ok?

So, we will allow this you know product notation which is independent of any order on j ok this the left hand side notation is a short notation temporarily for us. Also element of this product being functions, we will represent by $x(j)$ like this one like a sequence as a sequence. So, it is no order here, but we can use this notation ok generalized notation.

What it means is j^{th} coordinate function j^{th} value of x , or $x(j)$, that is the meaning ok? So, they are functions now on the indexing set. An element x is also denoted by this sequence following the practice of ordered n tuples or sequences. The assignment x going to $x(j)$

defines the functions p_j (or π_j , I shall use both) from X_J to X_j , which we call j^{th} coordinate projection ok?

Note that π_j can also be thought of as the evaluation map. Take a function here look its value on the element j that is evaluation map ok? So, these are different ways of looking at the same concept here.

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The slide is titled "Remark 2.68" and contains a table of contents and three bullet points. The table of contents lists: Introduction, Creating New Spaces, Module 22: Bases, Subspaces, Module 23: and Unions, Module 24: Quotient Spaces, Module 25: Product of Spaces, and Module 26: Induced and Co-induced Topologies. The bullet points are:

- (i) Observe that the product set is empty even if one of the components X_j is empty. Thus we shall always assume tacitly that each X_j is non empty while discussing the product.
- (ii) It is easily verified that a set theoretic function $f : Y \rightarrow X_J$ is uniquely determined by a family $\{f_j\}_{j \in J}$ of functions $f_j : Y \rightarrow X_j$ such that $\pi_j \circ f = f_j$. These are called the coordinate functions of f .
- (iii) Of particular importance is the case when $X_j = X$, the same set for all j . Then we can use the notation X^J to denote X_J . A further simplification in notation is in practice, viz., when $\#(J) = n$ is finite, we denote X^J by X^n also.

Observe that the product is empty even if one of the X_j is be empty, because there is no function into an empty set. Thus we shall always assume tacitly that each X_j is non empty while discussing the product. Also we can further assume that the indexing set is non empty. Otherwise there is nothing to work on alright?

Not only that we will assume quite often that indexing set has atleast two elements. If there is only one element it is just X_j . There is no need to worry about all this concept. So, we are not discussing anything whatever about products in these trivial cases. Usually for the indexing set must have at least two elements, and each X_j must be nonempty ok.

It is easily verified that a set theoretic function f from Y to X_J is uniquely determined by the families $f_j; j$ belong to J of functions f_j from Y to X_j .

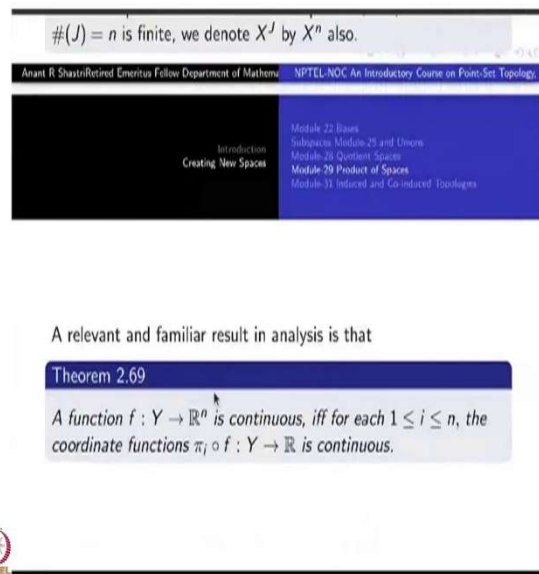
This is by definition, f_j is equal to $\pi_j \circ f$. If you know all these functions then you know f and conversely ok?

Of particular importance is the case when all the X_j 's are same set X , ok? Nevertheless while defining the product we will treat them as if they are disjoint copies of X ok? This is just for logical reason there is nothing more than that. The condition that the j^{th} coordinate of x must be inside X_j , that must be clear. We can then use another notation X^J , X you can write it as X^2 right? As we are doing with Euclidean spaces right? $\mathbb{R}^2, \mathbb{R}^3$ etc., we do not keep on writing $\mathbb{R} \times \mathbb{R} \dots$, we use a shorter notation.

So, the same notation is available in the genral case, only when all X_j 's represent the same set. Then you can write X^J ok? What is this? This is just set of all functions from J to X that is all, the extra condition disappears. A further simplification in notation in practice is when cardinality of J is n . Instead of writing X^J you can write X^n , just like we do for euclidean spaces \mathbb{R}^n , right?

So, that will also do. Then what is exponent n ? It is a natural number. Strictly speaking a natural number n is an equivalence class of a set with finitely many elements, depends on how many elements are there in that set. So, that is why this notation is also valid this is all quite logical no problems there.

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#(J) = n is finite, we denote X^J by X^n also.

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Introduction
Creating New Spaces

Module 22: Bases
Subspaces, Module-25 and Unions
Module-26: Quotient Spaces
Module-29: Product of Spaces
Module-31: Induced and Co-induced Topologies

A relevant and familiar result in analysis is that

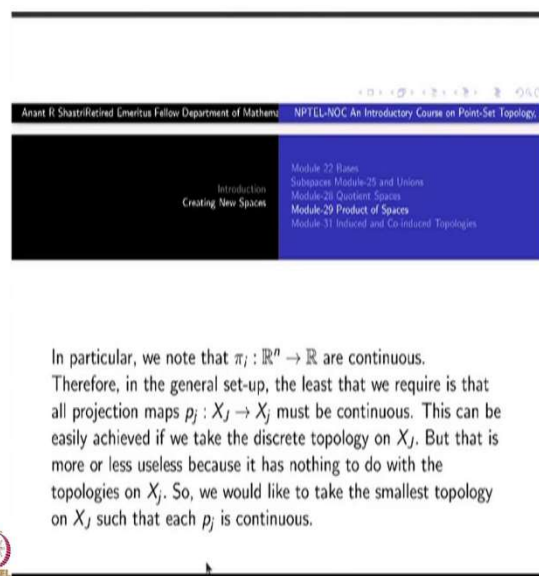
Theorem 2.69

A function $f : Y \rightarrow \mathbb{R}^n$ is continuous, iff for each $1 \leq i \leq n$, the coordinate functions $\pi_i \circ f : Y \rightarrow \mathbb{R}$ is continuous.

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The relevant and familiar result in analysis is that a function f from Y to \mathbb{R}^n is continuous if and only if each coordinate function $\pi_i \circ f$ which we have just denoted by f_i ok? Each of them is continuous. I am not going to prove this one. This we all know. We are taking that for granted. This \mathbb{R}^n is given the usual topology which is induced by a metric, there are several metrics giving the same usual topology right?

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Introduction
Creating New Spaces

Module 22: Bases
Subspaces, Module-25 and Unions
Module-26: Quotient Spaces
Module-29: Product of Spaces
Module-31: Induced and Co-induced Topologies

In particular, we note that $\pi_i : \mathbb{R}^n \rightarrow \mathbb{R}$ are continuous.

Therefore, in the general set-up, the least that we require is that all projection maps $p_j : X_J \rightarrow X_j$ must be continuous. This can be easily achieved if we take the discrete topology on X_j . But that is more or less useless because it has nothing to do with the topologies on X_j . So, we would like to take the smallest topology on X_j such that each p_j is continuous.

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In particular we note that π_i the coordinate functions themselves are continuous. Therefore, in a general setup when we move from \mathbb{R} to X , and then from X to X_1, X_2, \dots, X_n different X_i 's also, what we want to retain is the coordinate functions must be continuous ok?

In other words we have started with a family X_j 's of topological spaces. We are hunting for some nice topology on this product set, that topology should satisfy that each coordinate function must be continuous. That is the minimum requirement we want to have. Like when you take the quotient set and a quotient function we wanted the quotient function to be continuous and when you take subspace, the inclusion map must be continuous etc. Without that you do not want to go further. Similarly here the coordinate functions must be all continuous. This condition can be easily achieved if we take discrete topology on the domain. If you take discrete topology on the domain everything is continuous.

So, why bother then. Because, that is more or less useless. Because it has nothing to do with the topological spaces that we started with on each X_j 's ok. So, why are you taking this topology at all you will be asked. So, the solution should have something to do with X_j . So, discrete topology will satisfy the condition, ignoring all the topologies on individual X_j 's. It is like you know using an atom bomb to kill a fly right? So, that is not what we want.

So, we would like to take the smallest topology on X_J such that each π_j is continuous. So, that is our motivation now. Experience already shows that such a thing is possible right? So, let us verify let us go ahead and verify this theorem.

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Theorem 2.70

There is a unique topology on X_J satisfying the following properties:

(i) each projection map $p_j : X_J \rightarrow X_j$ is continuous, $j \in J$.
(ii) Given any topological space Y , any function $f : Y \rightarrow X_J$ is continuous iff $p_j \circ f : Y \rightarrow X_j$ is continuous for all $j \in J$.

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Introduction
Creating New Spaces

Module 22: Bases
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Module 28: Quotient Spaces
Module 29: Product of Spaces
Module 31: Induced and Co-induced Topologies

There is a unique topology on X_J satisfying the following properties: each projection map is continuous and given any topological space Y and any function f from Y to X_J , f is continuous if and only if the coordinate functions $p_j \circ f$ are all continuous for each j ok?

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Proof: Let \mathcal{T}_j denote the given topology on X_j . Let

$$\mathcal{S} = \{p_j^{-1}(U_j) : U_j \in \mathcal{T}_j, j \in J\}.$$

Take the topology \mathcal{T} on X_J generated by \mathcal{S} as a subbase. We claim that \mathcal{T} has the required properties.

Property (i) is obvious.

If $f : Y \rightarrow X_J$ is continuous, then clearly $p_j \circ f$ is continuous. Now suppose $p_j \circ f$ is continuous for all $j \in J$. To see that f is continuous, we can take the subbase \mathcal{S} and check that $f^{-1}(V)$ is open in Y for every $V \in \mathcal{S}$. But $V = p_j^{-1}(U_j)$ for some $U_j \in \mathcal{T}_j$. Therefore, $f^{-1}(V) = (p_j \circ f)^{-1}(U_j)$ and by continuity of $p_j \circ f$, it follows that $f^{-1}(V)$ is open in Y .

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Introduction
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So, this is the claim. So, let us see how to prove that. Start with all this X_j 's each with a topology. Let us call them \mathcal{T}_j . Till now we did not need a notation for them. Now let \mathcal{T}_j be the given topology on X_j . Put \mathcal{S} equal to what? Start with U_j in \mathcal{T}_j , take $p_j^{-1}(U_j)$, why I am

taking that? Because I want p_j to be continuous, inverse image of all these open sets must be open.

So, I am putting them here, for all j . I have to put them for all j and all U_j inside \mathcal{T}_j , take $p_j^{-1}(U_j)$. So, that is the collection. Call that collection \mathcal{S} . Whatever topology we want to have on this X_J the product space, must contain this family \mathcal{S} . Therefore, take $\mathcal{T}_{\mathcal{S}}$ to be the topology. Take the topology on X_J generated by \mathcal{S} . Remember the notation $\mathcal{T}_{\mathcal{S}}$. For this one I am not using this one here somehow does not matter this \mathcal{T} is $\mathcal{T}_{\mathcal{S}}$ ok.

We claim that this \mathcal{T} has the required property. First one is that all projection maps are continuous. Well that is how we have managed it here ok? that is continuous fine ok. Now let us look at the second one. Suppose f from Y to X_J is continuous. Then since each p_j is continuous, which we verified just now, the composite will be continuous alright. So, we are only to check the converse. Suppose all these $p_j \circ f$ are continuous. Then will f be continuous? So, converse part is what we have to worry ok?

Suppose Y to X_J is continuous then this is continuous follows because p_j 's are continuous composite is continuous. Now we assume $p_j \circ f$ is continuous for all j . To see that f is continuous, we can take members V of the subbase \mathcal{S} and check that inverse image of V under f are open in Y .

See here on X_J , there is some topology τ that topology has this \mathcal{S} as a subbase right? inverse image of subbasic open sets are open is enough to check that something is continuous. You do not have to check continuity on the whole of \mathcal{T} , ok? That is what we have seen earlier we have used that one several times. So, to see f is continuous, we must check that $f^{-1}(V)$ is open in Y for every V in \mathcal{S} .

But what is V ? It is $p_j^{-1}(U_j)$ for some $U_j \in \mathcal{T}_j$ ok? But then $f^{-1}(V)$ will be $f^{-1}(p_j^{-1}(U_j))$ which is nothing but $(p_j \circ f)^{-1}$ operating on U_j . And $p_j \circ f$ is continuous by assumption. Therefore, $f^{-1}(V)$ is open in Y . So, very straightforward right? One-step proof. So, it remains to prove the uniqueness of \mathcal{T} .

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Suppose \mathcal{T}' is another topology on X_J satisfying the same two conditions ok. By one, it follows that \mathcal{T} is contained inside \mathcal{T}' , because this \mathcal{S} is contained inside \mathcal{T}' because \mathcal{T}' satisfy that p_j 's are continuous. Once the subbase is contained inside another topology, the whole topology is contained inside that one. So, \mathcal{T} is contained inside \mathcal{T}' , ok?

This means that identity function from (X_J, \mathcal{T}') to (X_J, \mathcal{T}) is continuous. So, this the meaning of \mathcal{T} is contained in \mathcal{T}' . You can take an element here in \mathcal{T} . What is inverse image on under the identity map, it is the same set, right? It must be in \mathcal{T}' . So, \mathcal{T} contained in \mathcal{T}' is the same thing as identity map the other way is continuous.

Now, consider identity from (X_J, \mathcal{T}) to (X_J, \mathcal{T}') , the other way around ok. If this also continuous then two topologies will be the same. So, when is something continuous here? since p_j composite with identity is continuous see I have made the hypothesis that \mathcal{T}' to satisfies those two conditions. Apply condition 2 here. Condition 1, I have used. Now condition 2, I am going to use.

See condition one says that it is large enough. condition 2 puts restriction it brings down. So, that is what is happening here ok. So, $p_j \circ f$ they are all continuous. So, to verify this identity

map is continuous, I have to verify p_j composite identity is continuous, but p_j composite identity is p_j itself ok.

So, p_j for each j in J , property 1 tells you, is continuous. Therefore, \mathcal{T}' is contained inside \mathcal{T} ok? Identity map here is continuous means that. Observe how we got the uniqueness of the topology. So, instead of saying that this is the smallest topology etc.,

What I have said is I have put another condition here. The condition gives you that \mathcal{T} is large enough to make all p_j 's continuous. The second condition puts some restriction on \mathcal{T} . That is continuous functions from Y into X_J . So, first was from X_J to other coordinates spaces. This second condition is for function from any Y into X_J , a function is continuous if and only if their coordinate projection coordinate functions are continuous. So, these two properties define the topology uniquely this is the theorem.

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The screenshot shows a video lecture interface. At the top, there is a header with the name 'Anant B Shastri' and 'NPTL NOC An Introductory Course on Point Set Topology'. Below this is a table of contents with the following items:

Introduction	Module 22: Boxes
Creating New Spaces	Subspaces, Module 25: and Discrete
	Module 26: Quotient Spaces
	Module 29: Product of Spaces
	Module 31: Induced and Co-induced Topologies

Below the table of contents is a blue box titled 'Remark 2.71' containing the following text:

The proof of the above theorem gives two descriptions of \mathcal{T} viz.,

- (i) it is the smallest topology of X_J such that all p_j 's are continuous, and
- (ii) it is the topology $\mathcal{T}_{\mathcal{S}}$ with the subbase

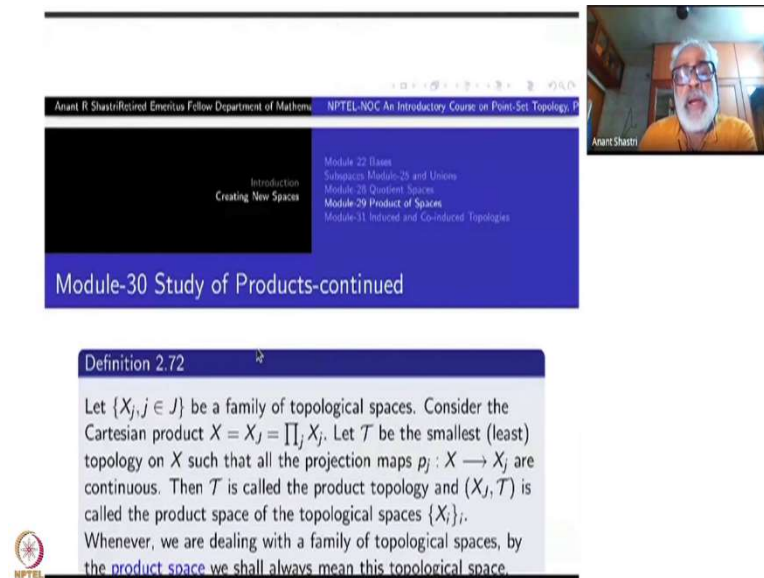
$$\mathcal{S} = \{p_j^{-1}(U_j) : U_j \in \mathcal{T}_j, j \in J\}.$$

At the bottom left of the slide is the NPTL logo.

The proof of the above theorem gives two descriptions of tau two different pictures of \mathcal{T} , half the picture on this side other other side that will be complete picture. what are they? One: \mathcal{T} is the smallest topology of X_J such that all p_j 's are continuous. Second: it is the topology $\mathcal{T}_{\mathcal{S}}$ with the subbase \mathcal{S} equal to the collection of all $p_j^{-1}(U_j)$, where U_j belonging to \mathcal{T}_j ok. So,

this description is more handy for working you know for working out things this will help.
The first one will help for conceptual understanding alright?

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The image shows a video lecture interface. At the top, there is a navigation bar with the text "Anant R Shastri Retired Emeritus Fellow Department of Mathemat... NPTEL-NOC An Introductory Course on Point-Set Topology, P...". Below this is a menu with the following items: "Introduction", "Creating New Spaces", "Module 22 Bases", "Subspaces, Mappings, and Unions", "Module 28 Quotient Spaces", "Module 29 Product of Spaces", and "Module 31 Induced and Co-induced Topologies". The main title of the slide is "Module-30 Study of Products-continued". Below the title is a section titled "Definition 2.72" which contains the following text: "Let $\{X_j, j \in J\}$ be a family of topological spaces. Consider the Cartesian product $X = \prod_j X_j$. Let \mathcal{T} be the smallest (least) topology on X such that all the projection maps $p_j : X \rightarrow X_j$ are continuous. Then \mathcal{T} is called the product topology and (X, \mathcal{T}) is called the product space of the topological spaces $\{X_j\}_j$. Whenever, we are dealing with a family of topological spaces, by the product space we shall always mean this topological space." The NPTEL logo is visible in the bottom left corner of the slide.

So, next time we shall study product spaces in more detail.

Thank you.