Introduction to Point Set Topology, (Part I)
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Lecture - 29
Product of Spaces

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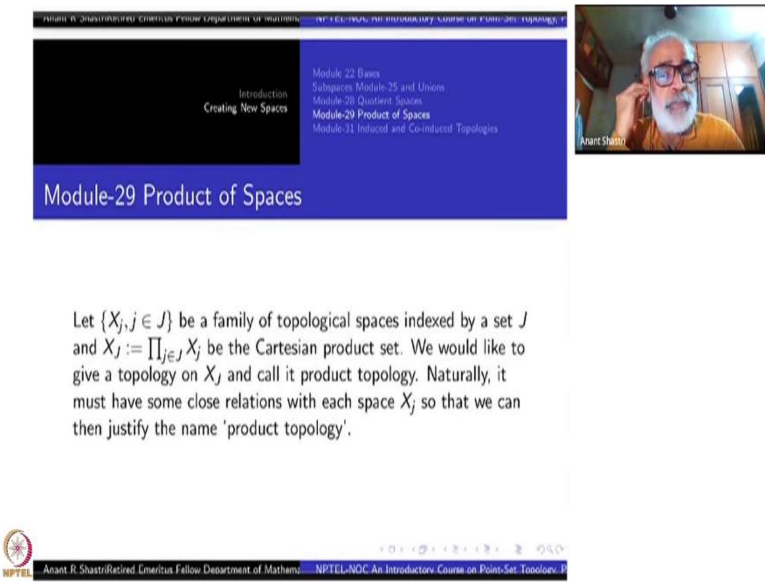

Welcome to module 29 of Point Set Topology part 1 course. So, today we will pick up another topic Product of Spaces. Again within the our general topic of producing new topologies new topological spaces out of the old and so on. So, recall that we have already defined what is called the box topology on atleast finite products right? In fact, we have also done it for infinite products.

Now, we want to bring in the most natural way of doing this namely, the function the theoretic approach to the products not through sub base and basis of course. We will also see sub base and basis here finally, just like we did while understanding, set theoretic understanding of quotient spaces.

It is very much important to understand the set theoretic aspect of of a product. Namely infinite products ok? An order implicit order on the indexing set is present in the back ground when you take finite products you write it as $X_1 \times X_2 \times X_3 \times \ldots$ the indexing set has been given a very nice order $1, 2, 3, 4$ and so on right.

We should come out of that you know, it is like a baby starts learning bicycle, she is given the support wheels to begin with. So, at that time support wheels are necessary. The order on the indexing set is like support wheels to understand what is going on. To begin with, but it is a hindrance later, in the general case ok?

So, we should get rid of the support wheels. So, let X_j where $j \in J$, J is the indexing set, be a family of topological spaces ok. Then look at this product space the Cartesian product space. What is the product set first of all. What is it that is what I want to understand. What exactly is the set theory? Ok?

And then on each X_j if you have a topology then what is the corresponding topology on the product ok? One of them we have defined, but is there only one, or there are some other way of doing it. So, that is the kind of thing you want to understand ok? And the one which comes out from our pursuit here will be called the product topology. Earlier one which you had got was called box topology. We shall also compare these two things.

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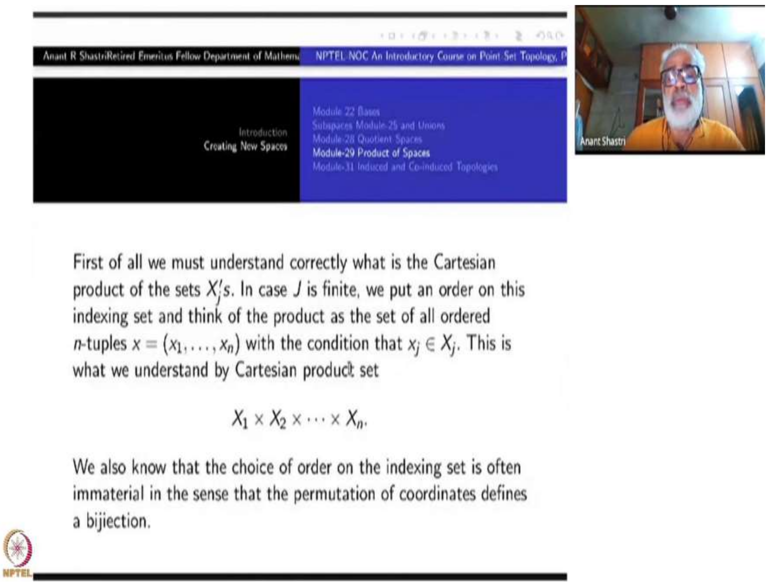

First of all we must understand correctly what is the Cartesian product set ok, where X_js are some indexes families of sets, ok? I repeat. If J is a finite set, the indexing set, we usually put an order on this indexing set and think of the product as the set of ordered n tuples ok? $(x_1, x_2, \ldots, x_n)$, where x_1 comes from X_1, x_2 comes from X_2 and so on x_n comes from X_n. That is what we understand by Cartesian coordinate set as such. And then this notation can be used $X_1 \times X_2 \times \cdots \times X_n$, ok.

We also know the choice of order on the indexing set is immaterial $X_1 \times X_2$ can be written as $X_2 \times X_1$ also, for most of the time unless you are really doing some geometry something like binary theory and so on. So, there it may be of some value some importance not to change the order ok. Certainly in topological considerations you do not see any role of this one right? ok.

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So, let us be done away it. We shall altogether get rid of this order so that this concept can be easily generalized. Recall that ordered n tuple x, I am writing single x for $(x_1, x_2, \ldots, x_n)$, what is it? It is a sequence of length n taking values in $X_1, X_2, \ldots, X_n$ right?

So, if we get rid of the order we can still think of this as a function on the set J of n elements which was the indexing set there ok? And the sequence taking values inside the union of all

X_j's of course, with additional condition that the j^{th} image of x now I am thinking of x as a function, which I write it a x_j, the j^{th} coordinate, that is an element of X_j ok? We see that in this form, it immediately generalizes to describe the product when J is an arbitrary set also ok?

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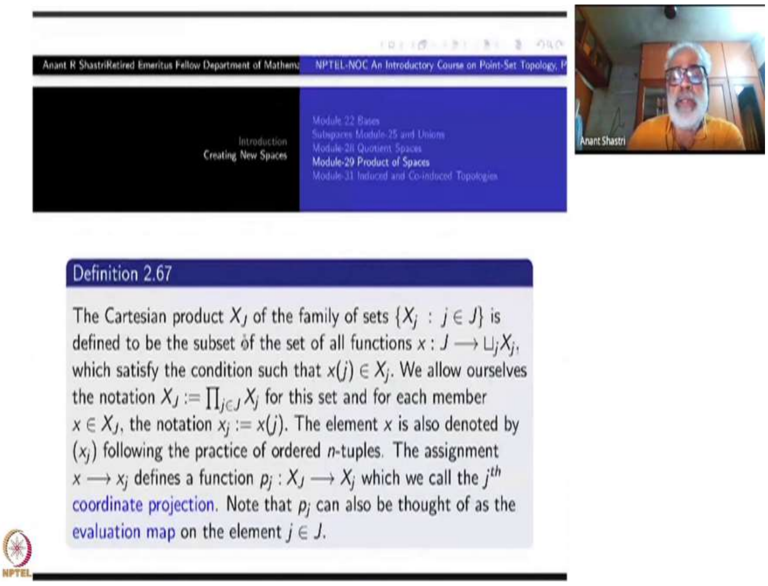

The Cartesian product we can write it as X_J, instead of $X_1 \times X_2$ etc, the whole product set X_J, I am writing ok? Or this other notation $\prod_{j \in J} X_j$, this is defined to be the subset of all functions from indexing set J into the disjoint union of $X_j, \cup_j X_j$ with one extra condition that the image of j under x is inside X_j, for each j in J, this must be true ok?

So, we will allow this you know product notation which is independent of any order on j ok this the left hand side notation is a short notation temporarily for us. Also element of this product being functions, we will represent by $x(j)$ like this one like a sequence as a sequence. So, it is no order here, but we can use this notation ok generalized notation.

What it means is j^{th} coordinate function j^{th} value of x, or $x(j)$, that is the meaning ok? So, they are functions now on the indexing set. An element x is also denoted by this sequence following the practice of ordered n tuples or sequences. The assignment x going to $x(j)$

defines the functions p_j (or π_j, I shall use both) from X_J to X_j, which we call j^{th} coordinate projection ok?

Note that π_j can also be thought of as the evaluation map. Take a function here look its value on the element j that is evaluation map ok? So, these are different ways of looking at the same concept here.

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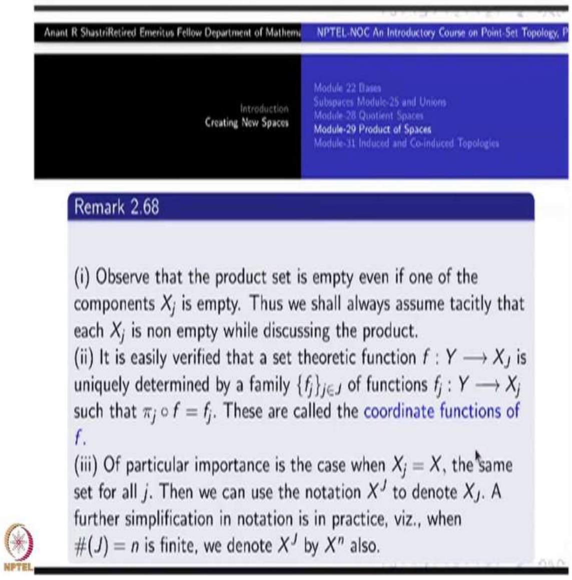

Observe that the product is empty even if one of the X_j is be empty, because there is no function into an empty set. Thus we shall always assume tacitly that each X_j is non empty while discussing the product. Also we can further assume that the indexing set is non empty. Otherwise there is nothing to work on alright?

Not only that we will assume quite often that indexing set has atleast two elements. If there is only one element it is just X_j. There is no need to worry about all this concept. So, we are not discussing anything whatever about products in these trivial cases. Usually for the indexing set must have at least two elements, and each X_j must be nonempty ok.

It is easily verified that a set theoretic function f from Y to X_J is uniquely determined by the families $f_j; j$ belong to J of functions f_j from Y to X_j.

This is by definition, f_j is equal to $\pi_j \circ f$. If you know all these functions then you know f and conversely ok?

Of particular importance is the case when all the X_j's are same set X, ok? Nevertheless while defining the product we will treat them as if they are disjoint copies of X ok? This is just for logical reason there is nothing more than that. The condition that the j^{th} coordinate of x must be inside X_j, that must be clear. We can then use another notation X^J, X you can write it as X^2 right? As we are doing with Euclidean spaces right? $\mathbb{R}^2, \mathbb{R}^3$ etc., we do not keep on writing $\mathbb{R} \times \mathbb{R} \ldots$, we use a shorter notation.

So, the same notation is available in the genral case, only when all X_j's represent the same set. Then you can write X^J ok? What is this? This is just set of all functions from J to X that is all, the extra condition disappears. A further simplification in notation in practice is when cardinality of J is n. Instead of writing X^J you can write X^n, just like we do for euclidean spaces $\mathbb{R}^n$, right?

So, that will also do. Then what is exponent n? It is a natural number. Strictly speaking a natural number n is an equivalence class of a set with finitely many elements, depends on how many elements are there in that set. So, that is why this notation is also valid this is all quite logical no problems there.

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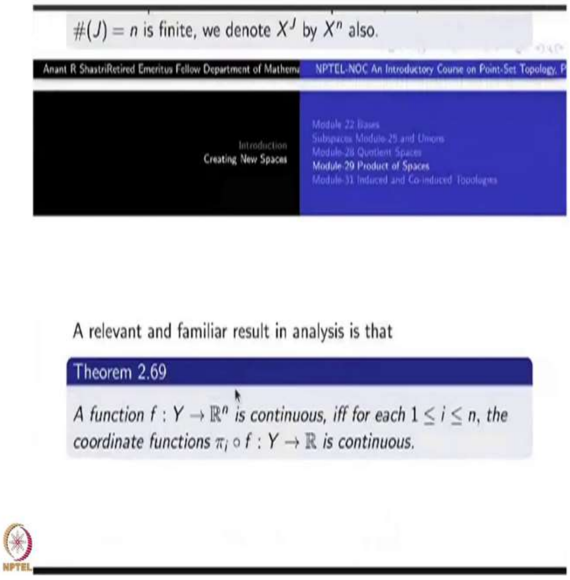

The relevant and familiar result in analysis is that a function f from Y to $\mathbb{R}^n$ is continuous if and only if each coordinate function $\pi_i \circ f$ which we have just denoted by f_i ok? Each of them is continuous. I am not going to prove this one. This we all know. We are taking that for granted. This $\mathbb{R}^n$ is given the usual topology which is induced by a metric, there are several metrics giving the same usual topology right?

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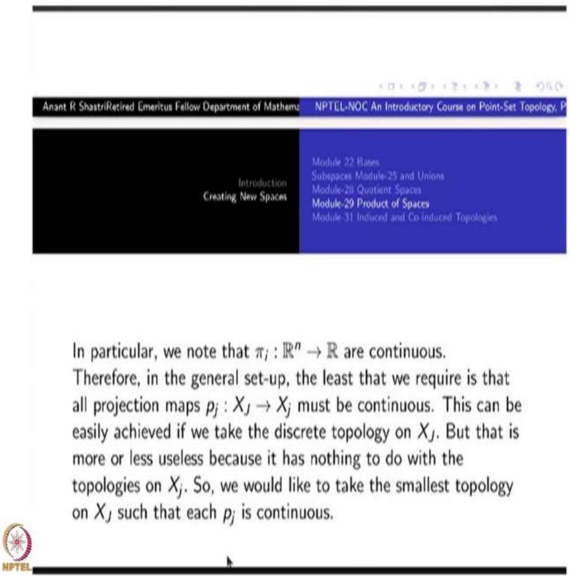

In particular we note that π_i the coordinate functions themselves are continuous. Therefore, in a general setup when we move from $\mathbb{R}$ to X, and then from X to $X_1, X_2, \ldots, X_n$ different X_i 's also, what we want to retain is the coordinate functions must be continuous ok?

In other words we have started with a family X_j's of topological spaces. We are hunting for some nice topology on this product set, that topology should satisfy that each coordinate function must be continuous. That is the minimum requirement we want to have. Like when you take the quotient set and a quotient function we wanted the quotient function to be continuous and when you take subspace, the inclusion map must be continuous etc. Without that you do not want to go further. Similarly here the coordinate functions must be all continuous. This condition can be easily achieved if we take discrete topology on the domain. If you take discrete topology on the domain everything is continuous.

So, why bother then. Because, that is more or less useless. Because it has nothing to do with the topological spaces that we started with on each X_j's ok. So, why are you taking this topology at all you will be asked. So, the solution should have something to do with X_J. So, discrete topology will satisfy the condtion, ignoring all the topologies on individual X_j's. It is like you know using an atom bomb to kill a fly right? So, that is not what we want.

So, we would like to take the smallest topology on X_J such that each π_j is continuous. So, that is our motivation now. Experience already shows that such a thing is possible right? So, let us verify let us go ahead and verify this theorem.

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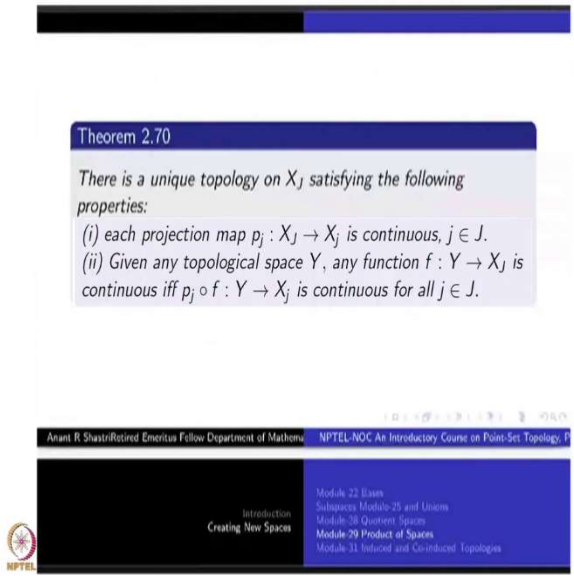

There is a unique topology on X_j satisfying the following properties: each projection map is continuous and given any topological space Y and any function f from Y to X_J, f is continuous if and only if the coordinate functions $p_j \circ f$ are all continuous for each j ok?

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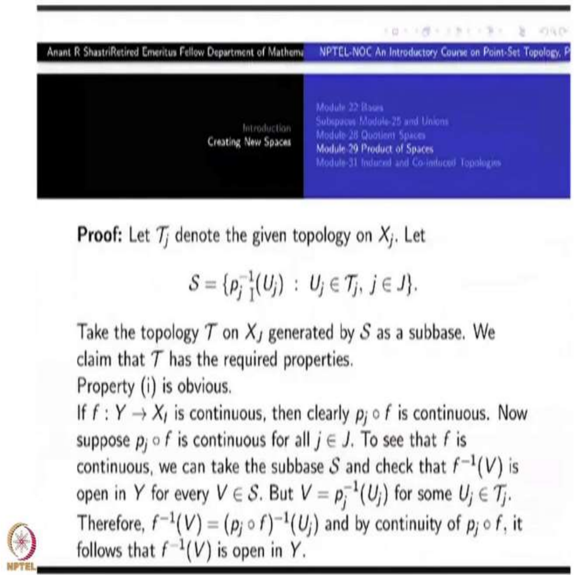

So, this is the claim. So, let us see how to prove that. Start with all this X_j's each with a topology. Let us call them $\mathcal{T}_j$. Till now we did not need a notation for them. Now let $\mathcal{T}_j$ be the given topology on X_j. Put S equal to what? Start with U_j in $\mathcal{T}_j$, take $p_j^{-1}(U_j)$, why I am taking that? Because I want p_j to be continuous, inverse image of all this open sets must be open.

So, I am putting them here, for all j. I have to put them for all j and all U_j inside $\mathcal{T}_j$, take $p_j^{-1}(U_j)$. So, that is the collection. Call that collection $\mathcal{S}$. Whatever topology we want to have on this X_J the product space, must contain this family $\mathcal{S}$. Therefore, take $\mathcal{T}_\mathcal{S}$ to be the topology. take the topology on X_J generated by $\mathcal{S}$. Remember the notation $\mathcal{T}_\mathcal{S}$. For this one I am not using this one here somehow does not matter this $\mathcal{T}$ is $\mathcal{T}_\mathcal{S}$ ok.

We claim that this $\mathcal{T}$ has the required property. First one is that all projection maps are continuous. Well that is how we have managed it here ok? that is continuous fine ok. Now let us look at the second one. Suppose f from Y to X_J is continuous. Then since each p_j is continuous, which we verified just now, the composite will be continuous alright. So, we are only to check the converse. Suppose all these $p_j \circ f$ are continuous. Then will f be continuous? So, converse part is what we have to worry ok?

Suppose Y to X_J is continuous then this is continuous follows because p_j's are continuous composite is continuous. Now we assume $p_j \circ f$ is continuous for all j. To see that f is continuous, we can take members V of the subbase $\mathcal{S}$ and check that inverse image of V under f are open in Y.

See here on X_J, there is some topology tau that topology has this $\mathcal{S}$ as a subbase right? inverse image of subbasic open sets are open is enough to check that something is continuous. You do not have to check continuity on the whole of $\mathcal{T}$, ok? That is what we have seen earlier we have used that one several times. So, to see f is continuous, we must check that $f^{-1}(V)$ is open in Y for every V in $\mathcal{S}$.

But what is V? It is $p_j^{-1}(U_j)$ for some $U_j \in \mathcal{T}_j$ ok? But then $f^{-1}(V)$ will be $f^{-1}(p_j^{-1}(U_j))$ which is nothing but $(p_j \circ f)^{-1}$ operating on U_j. And $p_j \circ f$ is continuous by assumption. Therefore, $f^{-1}(V)$ is open in Y. So, very straightforward right? One-step proof. So, it remains to prove the uniqueness of $\mathcal{T}$.

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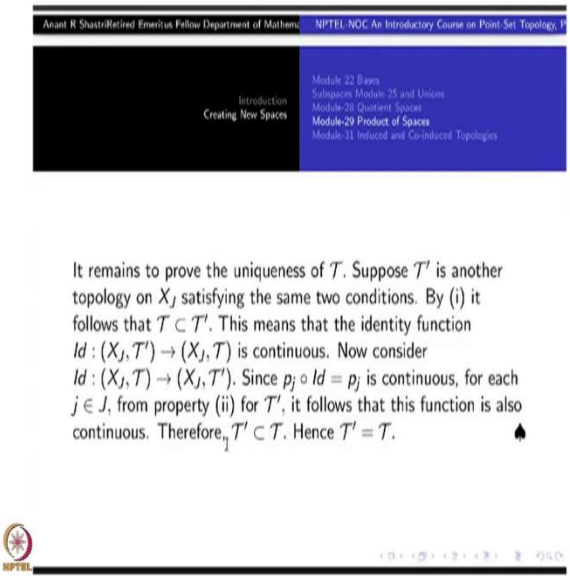

Suppose $\mathcal{T}'$ is another topology on X_J satisfying the same two conditions ok. By one, it follows that $\mathcal{T}$ is contained inside $\mathcal{T}'$, because this $\mathcal{S}$ is contained inside $\mathcal{T}'$ because $\mathcal{T}'$ satisfy that p_j's are continuous. Once the subbase is contained inside another topology, the whole topology is contained inside that one. So, $\mathcal{T}$ is contained inside $\mathcal{T}'$, ok?

This means that identity function from $(X_J, \mathcal{T}')$ to $(X_J, \mathcal{T})$ is continuous. So, this the meaning of $\mathcal{T}$ is contained in $\mathcal{T}'$. You can take an element here in $\mathcal{T}$. What is inverse image on under the identity map, it is the same set, right? It must be in $\mathcal{T}'$. So, $\mathcal{T}$ contained in $\mathcal{T}'$ is the same thing as identity map the other way is continuous.

Now, consider identity from $(X_J, \mathcal{T})$ to $(X_J, \mathcal{T}')$, the other way around ok. If this also continuous then two topologies will be the same. So, when is something continuous here? since p_j composite with identity is continuous see I have made the hypothesis that $\mathcal{T}'$ to satisfies those two conditions. Apply condition 2 here. Condition 1, I have used. Now condition 2, I am going to use.

See condition one says that it is large enough. condition 2 puts restriction it brings down. So, that is what is happening here ok. So, $p_j \circ f$ they are all continuous. So, to verify this identity map is continuous, I have to verify p_j composite identity is continuous, but p_j composite identity is p_j itself ok.

So, p_j for each j in J, property 1 tells you, is continuous. Therefore, $\mathcal{T}'$ is contained inside $\mathcal{T}$ ok? Identity map here is continuous means that. Observe how we got the uniqueness of the topology. So, instead of saying that this is the smallest topology etc.,

What I have said is I have put another condition here. The condition gives you that $\mathcal{T}$ is large enough to make all p_j's continuous. The second condition puts some restriction on $\mathcal{T}$. That is continuous functions from Y into X_J. So, first was from X_J to other coordinates spaces. This second condition is for function from any Y into X_J, a function is continuous if and only if their coordinate projection coordinate functions are continuous. So, these two properties define the topology uniquely this is the theorem.

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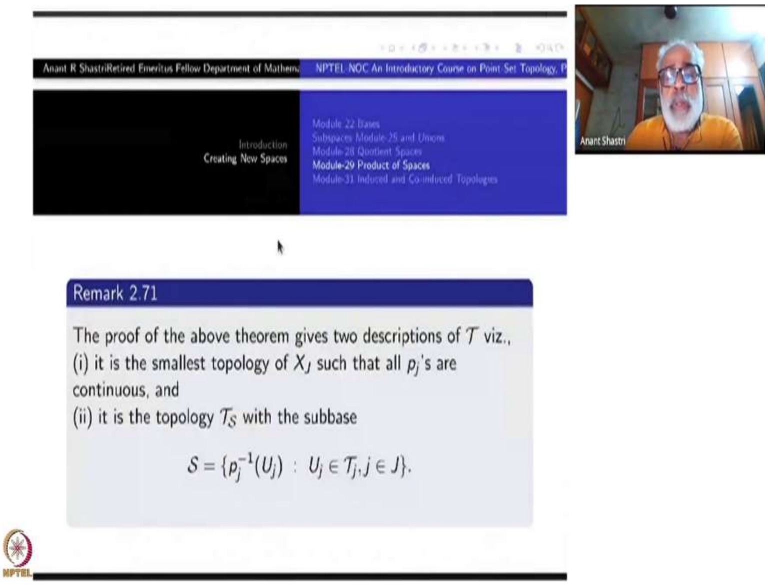

The proof of the above theorem gives two descriptions of tau two different pictures of $\mathcal{T}$, half the picture on this side other other side that will be complete picture. what are they? One: $\mathcal{T}$ is the smallest topology of X_J such that all p_j's are continuous. Second: it is the topology $\mathcal{T}_{\mathcal{S}}$ with the subbase $\mathcal{S}$ equal to the collection of all $p_j^{-1}(U_j)$, where U_j belonging to $\mathcal{T}_j$ ok. So, this description is more handy for working you know for working out things this will help. The first one will help for conceptual understanding alright?

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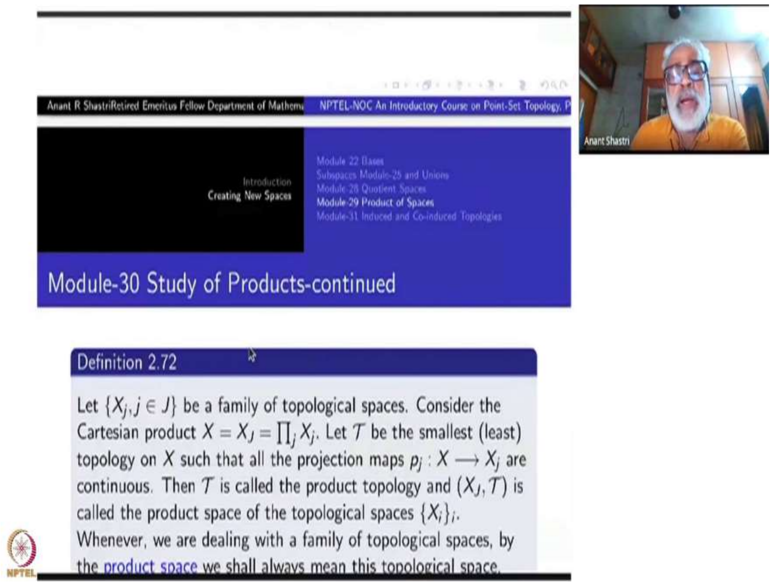

So, next time we shall study product spaces in more detail.

Thank you.