Introduction to Point Set Topology, (Part I)
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Lecture - 28
Quotient Spaces

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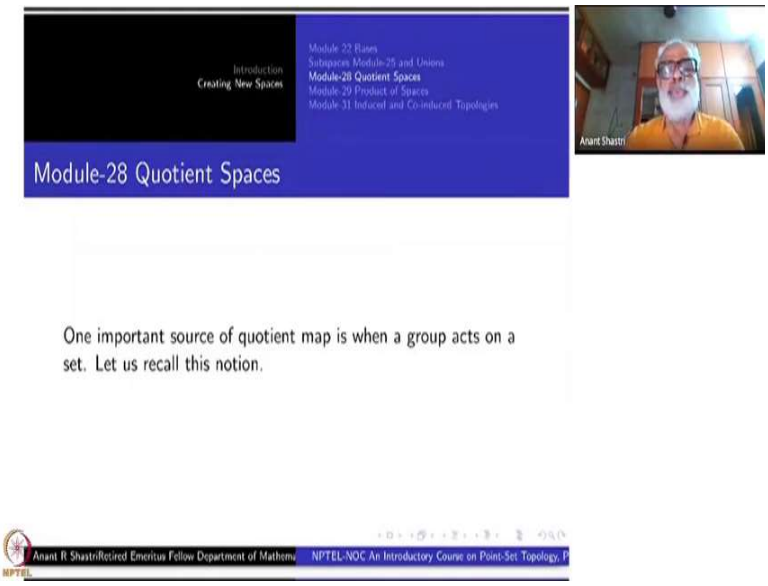

Welcome to module 28. We now introduce Quotient Spaces. Last time we try to understand what is the meaning of a quotient set in three different ways. Just giving a surjective function on a given set to another set is one way, defining an equivalence relation is another way and writing X as a disjoint union of non empty sets is the third way. So, all these things are equivalent. That is what we have seen.

One of the important sources of these quotients even at set theoretic level is what is called what is known as group action. Group action on a set you know gives you certain kind of decomposition of a set. And that happens to be a big source, an important source, an interesting source of quotient sets. So, let us recall this notion. Maybe you came across this, under a group actions while studying groups themselves.

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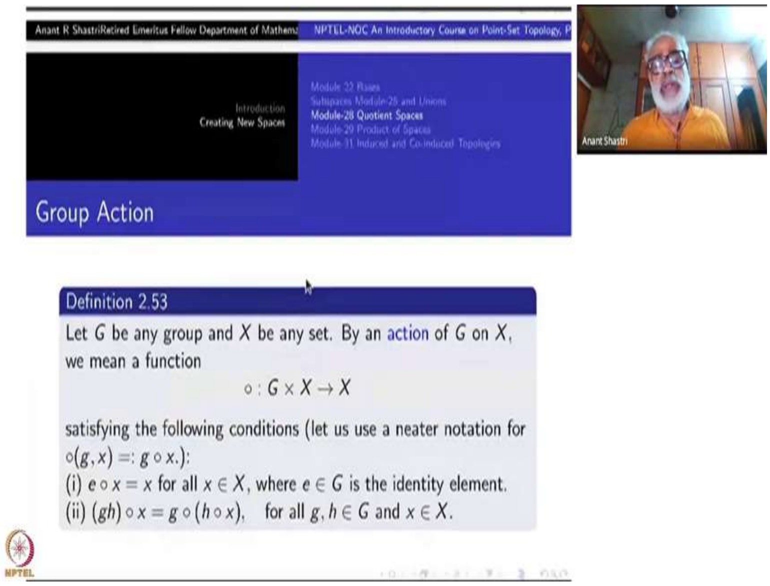

So, let us assume that you know what is the meaning of a group and so on. Let G be any group and X be any set, we have a map from $G \times X$ to X. It is not a binary operation in the sense that it is not on $X \times X$ to X, neither on $G \times G$ to G. It is $G \times X$ to X is mixture of this one and that one. So, G is acting on X is going to be defined via this map and this map is is denoted by not F or G or something, its standard notation is, just the $\circ$, Usually or dot. $\circ$ is used for writing compositions of functions, multiplication inside a group and so on and that is why this is coming here as well. ok?

So, what we will do is, we will have a manageable and easy to understand notation. the $\circ$ operating upon (g, x), we will write it as $g \circ x$ ok? When even this $\circ$ is too much to write, we will not write anything, we may write it merely as $g.x$ just like when we write multiplication of two real numbers or two complex numbers. So, that will be the next stage of simplification of notation right? Now we will keep this $\circ$ ok. So, $\circ(g, x)$ is written as $g \circ x$.

So, this operation ok whatever you want to say is an action if it has the following two fundamental properties: The first one is the identity: identity element e is an identity of G, identity operates identically, $e \circ x$ is x itself for all x. where e is the identity element of the group G.

The second one is the associativity gh operating upon x is the same as g later and h first, g operating upon $h \circ x$. So, $g \circ (h \circ x)$ is simply the $(gh) \circ x$; the bracket can be put in the first place or the other way around. So, that is the associativity. So, this must be true for all g and h inside G and all x inside X, that is all.

As soon as you have such a map $\circ$ defined, you can fix a $g \in G$, look at the function x going to $g \circ x$. That becomes a bijection of the set X, why? Because the inverse of this bijection; inverse of this function is nothing but x going to $g^{-1} \circ x$ ok? g is an element of the group. So, it has an inverse right? So, g^{-1} operating on $g \circ x$ is the same thing as $(g^{-1}g) \circ x$ which is identity operating on x. So, only these two operations are used.

So, you see that each g, you know the action of g, that is why it is called action, sends elements of X here and there, but in a bijective fashion, that means a permutation. So, each element of the group g can be thought of as a permutation of X. Then associativity again says that this association from X to the group of permutations, that itself is a homomorphism ok? Action of gh is the same as first h and then g. So, that is the meaning of this one.

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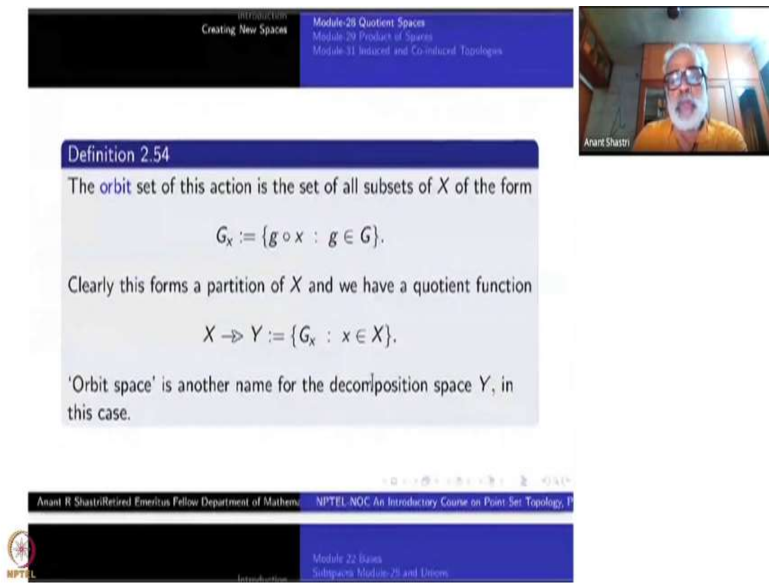

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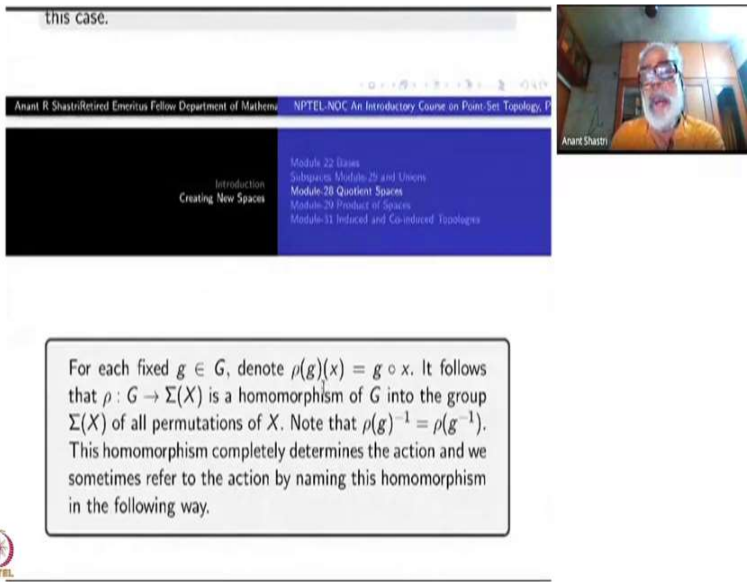

So, what you get is: first of all what you get this one: If I write $\rho(g)$ for the action of g, $\rho(g)(x) = g \circ x$; then this $\rho(g)$ is a function from X to X and it is invertible. So, it is an element of the permutation group $\Sigma(X)$. That association from G to the permutation group, that ρ itself becomes a homomorphism. Look at the identity elements. ρ of identity what? It is e operating upon x. So, it is x itself so $\rho(e)$ is identity function of X, identity permutation right? And $\rho(g^{-1})$ is the inverse of $\rho(g)$. That is why, these are all permutations ok?

Here what is the decomposition? I was talking precisely for that purpose. For each x, let us have this notation G_x, what is called the orbit of x. So, the word orbit is used as if these are satellites moving around you see. That is the whole idea. So, $g \circ x$ where g ranges over G, image of the point $x \in X$, $g_1 \circ x, g_2 \circ x, g_3 \circ x, \ldots$ look at all of then together. So, that is called the orbit of x and denoted by G_x ok. Clearly x belongs to G_x because $e \circ x = x$.

The point is if you take another y either G_y will be the whole of G_x itself or it will never intersect G_x So, that is it. That is obvious because of the group property of G ok?

So, that is why it forms a partition of X. It is easy to verify $G_x \cap G_y$ is nonempty means that element can be taken to span G_x as well as G_y. So, they are; they are equal, if the intersection is non empty. That means, it defines a partition, this is exactly the way you would have taken a subgroup in a group, its cosets, right cosets are all mutually disjoint. Similarly left cosets are also disjoint. So, orbits are like cosets here ok? So could have called orbits as G cosets of X ok?

But nobody uses that. There is a different word here they are orbits of x ok. So, clearly this forms a partition and we have a quotient function: put Y equal to the set of all orbits G_x, x belong to X. Then x goes to G_x is a surjective function. So, it is a quotient function. The partition is the same. Or you can also say that x_1 is related to x_2 if and only if there is a $g \in G$ which will push x_1 to x_2; $g \circ x_1 = x_2$. Then this becomes an equivalence relation, and these three concepts are the same.

So, you can use whichever one you like as pointed out last time for more general situation. Now here is a special case arising out of the action of G on X ok? The orbit space is another name for this decomposition space set of all orbits of the action. So, that set, I have denoted by Y temporarily. This X may be any set, it may be the set of real numbers $\mathbb{R}$, it may be $\mathbb{R}^2$, it may be $\mathbb{R} \setminus \{0\}$ and so on. Various examples are there ok? So, at least some examples we shall discuss.

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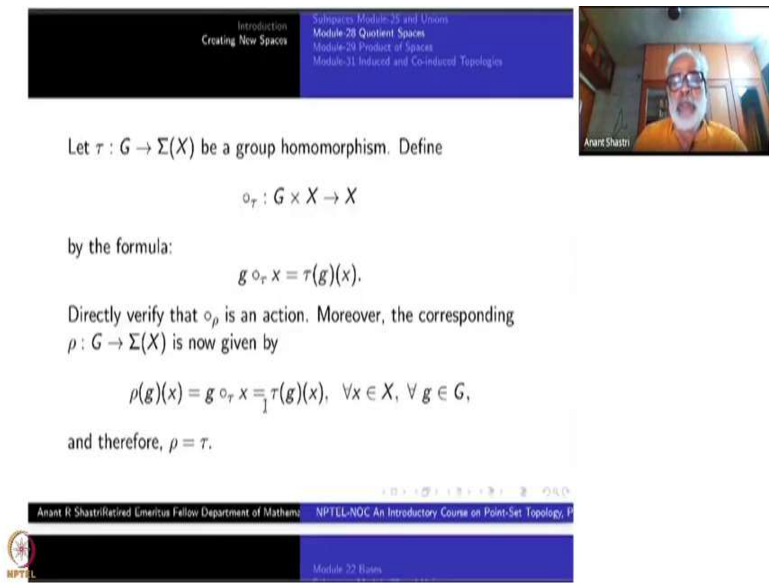

For each fixed g we have seen that $\rho(g)$, namely action of g on X is a permutation ok. Conversely suppose you start with a homomorphism τ from the group G to the group of permutations ok? Suppose this is a homomorphism, then I will define a $\circ$ operation corresponding to τ that is why I have written this as $\circ_\tau$ from $G \times X$ to X, by just similar to what we got here $\rho(g)$, so this is going to be $\tau(g)(x)$.

So, $\tau(g)(x)$ will give you the action namely $g \circ_\tau x$ which is the formula for the action. Now if you take the corresponding ρ here, it will be nothing but τ ok? Starting with the action I have defined a group homomorphism ρ right? Similarly starting with the group homomorphism now, I have define the action. And this will be a bijective correspondence one to one correspondence. Thus, when you want to mention an action you can just mention a group homomorphism from G to $\Sigma(X)$.

For instance, suppose this homomorphism is the trivial homomorphism; trivial homomorphism is genuine homomorphism it is allowed, what is the meaning of this? $g \circ x$ is equal to x for all x and for all g. So, this is called the trivial action, no action ok? So, it is the inertia action, so that is also allowed alright. Depending upon various properties of this homomorphism, there are various classes of actions, various types of actions ok? All that we will discuss whenever the situation arises.

We just do not want to get stuck with group actions. This is one source for quotients worth mentioning. We want to go ahead with our concept of quotient spaces ok right. So, there is a correspondence between group actions and group homomorphism from G to permutation group.

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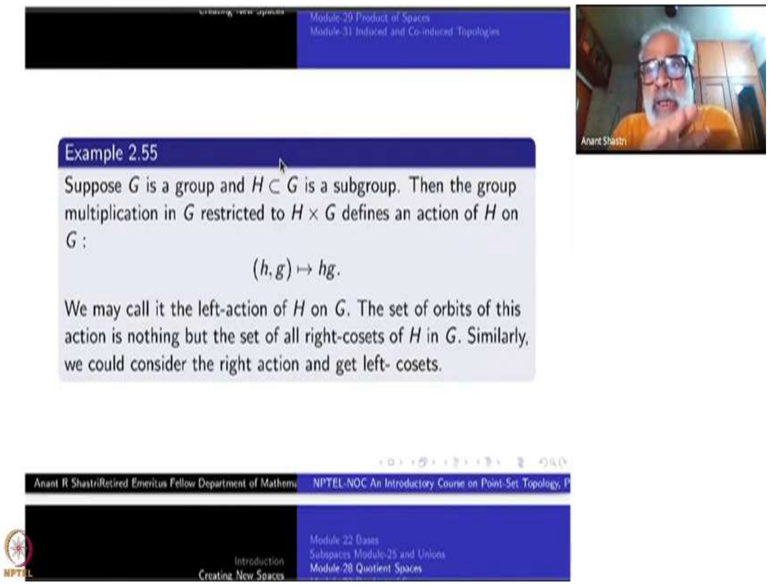

As special case, suppose you have subgroup H of a group of G. Then you have the group multiplication $G \times G$ to G restricted to $H \times G$ to G. You may treat now G as a set and H as a group but the multiplication is coming from G of course. So, (h, g) going to hg.

Now I am writing no $\circ$ extra here this is the action. This is called the left action of H on G, because h is getting multiplied on the left. I could have written it on the other side also (h, g) going to hg then that will be the right action ok? The left action right action are both possible. The corresponding orbits of the left action are nothing but the right cosets ok? So, you have to write Hg where g belongs to G, those are the orbits of this action.

So, they are right cosets. Similarly if you take right action you will get left cosets. So, be sure of that. So the things are left and right are getting interchanged here.

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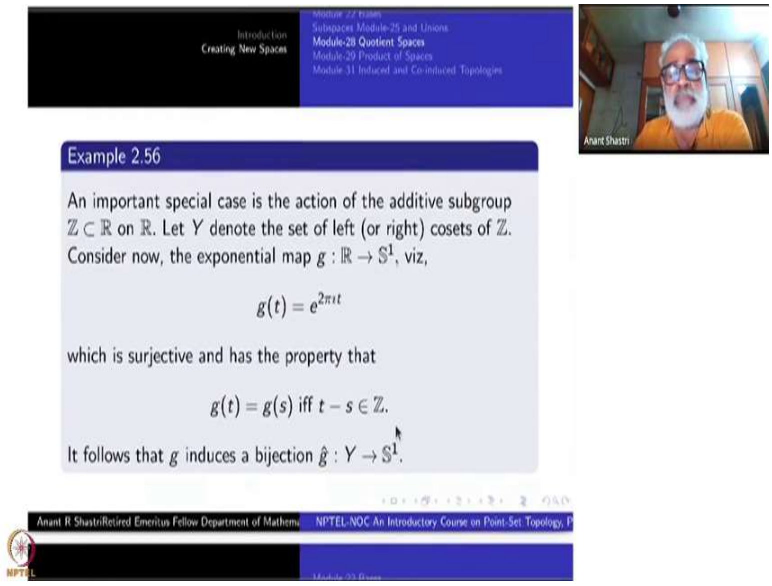

An important special case is the action of additive subgroup $\mathbb{Z}$ contained in the additive group $\mathbb{R}$ of real numbers ok? See first I considered arbitrary group acting on an arbitrary set. Then I took the set to be the group G and the group to be a subgroup H any sub group ok?

Now, I am taking a further special case namely $\mathbb{R}$ is the additive group and $\mathbb{Z}$ is the subgroup ok? So this is integers ok. Look at the left cosets here namely I have denoted again by Y, denote the set of left cosets or right cosets they will be same here because $\mathbb{R}$ is commutative, the additive group is an abelian group, so left and right do not make any difference here that is all. ok?

Look at this quotient function in a different way, entirely different way. Namely, consider the function g from $\mathbb{R}$ to the circle, $g(t)$ going to $e^{2\pi i t}$. $e^{2\pi i t}$ is an element of what? Complex number of modulus one which is $\mathbb{S}^1$ right. So, you get a map from $\mathbb{R}$ to $\mathbb{S}^1$. I have written as $e^{2\pi i t}$, we know that this is surjective right? We also know that $g(t)$ is equal to $g(s)$ if and only if the difference is an integer. I have put $2\pi i t$ here ok? iff integer i.e, iff $t - s \in \mathbb{Z}$ ok?

So, it follow now follows that the orbits of this action is in one-one correspondence with elements of $\mathbb{S}^1$. This $\hat{g}$ is a map from Y to $\mathbb{S}^1$ which is induced by map g from $\mathbb{R}$ to $\mathbb{S}^1$. The orbits of this $\mathbb{Z}$ action are nothing but the fibers of g, that is the meaning of this one you see?

they are fibers of this g: $g(t) = g(s)$ iff $t - s$ is an integer. That means s belongs to $\mathbb{Z} + t$ or t belongs to $\mathbb{Z} + s$.

So, they are nothing but the right cosets or left cosets of $\mathbb{Z}$ right. So, once you have an action, decomposition surjective map, or equivalence relation these are all equivalent concepts. Now I am thinking of this as a surjective function what is that function its precisely this. Integers are sent to 1 in $\mathbb{S}^1$ here. All the integers go to the complex number one here under this map. So, this is a wonderful map it has lot of topological, analytic properties that will be exploitedin our study right. This is a very fundamental example ok.

The right cosets of this action are in one one correspondence with the set of complex numbers of modulus one. This example we will meet again and again.

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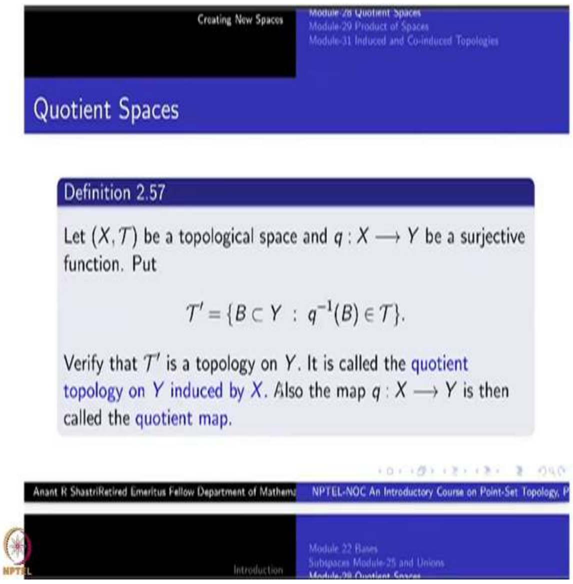

So, here there is a topology also. you see though we never bothered about topology so far. this $\mathbb{Z}$ is a sub group.

Student: Yeah yeah

Of the group $\mathbb{R}$. That is all I have used.

Student: Alright.

So, let us bring in the topology now that is, the quotient spaces.

Start with any topological space $(X, \mathcal{T})$, and a surjective function q, q set theoretic surjective function to Y. Put $\mathcal{T}'$ equal to all those subsets of Y, such that their inverse image under q is in X as an element of $\mathcal{T}$. That is $B \in \mathcal{T}'$ iff the inverse image of B under q is open in X ok? You see why we want such a thing.

We want this function q the important function, the surjective function which is going to define partition or whatever that function I want it to be continuous. There is a topology here. So, I am using that topology in that topology it must be continuous means I have to put a topology here on Y. So, that is the trick I am using here. Take $\mathcal{T}'$ equal to the collection of all B contained in Y such that $q^{-1}(B) \in \mathcal{T}$.

Verify that $\mathcal{T}'$ is a topology on Y. This kind of thing, we have done several times right? With several maps together we have also done like $\pi_i^{-1}(x_i)$ whatever. So, this verification is very straightforward. If you take the empty set, inverse image is empty. If you take the whole space Y inverse image with the whole space X, take $B_1 \cap B_2$ and then $q^{-1}(B_1 \cap B_2)$ is $q^{-1}(B_1) \cap q^{-1}(B_2)$. Taking inverse image of sets is a very well behaved operation ok?

So, you have a topology on Y now ok? This topology is called the quotient topology ok? Using the term `induced by $q$'  is actually wrong, but this is the term used. So, I am introducing terms `induced' and coinduced' correctly later on. Right now I have not told you that. This $\mathcal{T}'$ is actually the co-induced by the function q. That is the correct term.

So, now the function q itself is now continuous surjection. It is called the quotient map. As soon as I say that, it is not just an arbitrary continuous surjection, it is giving you the topology on Y very specifically. It prescribes what are open sets in Y; a subset B is open in Y if and only if $q^{-1}(B) \in \mathcal{T}$. Something is open if and only the q inverse of that is open, there is nothing else here. That is the only condition. That is the meaning of the word quotient map.

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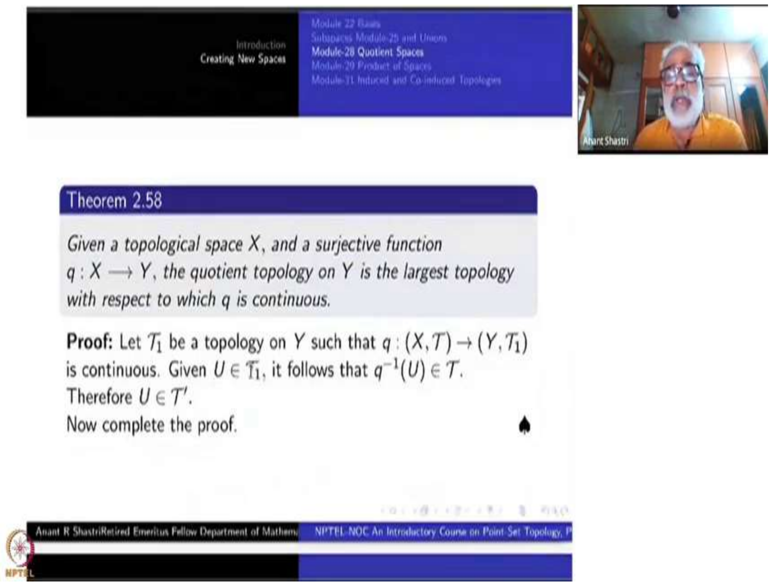

Given a topological space X and a surjective function q from $(X, \mathcal{T})$ to Y, the quotient topology on Y which you have defined just now is the largest topology with respect to which q is continuous. You see I could have taken the indiscrete topology on Y. Automatically q will be continuous maybe that is too small a topology, but it serves the purpose of making q continuous. So, there are many topologies. you take the biggest one ok? If you put more more open sets inside Y then q will not be a continuous function.

So that means by the biggest one, we take the biggest one which exists ok? So, how to we know that a biggest one exists. So, this is a claim this is not the definition, definition we have already got above. So, let $\mathcal{T}_1$ be a topology on Y such that q from $(X, \mathcal{T})$ to $(Y, \mathcal{T}_1)$ is continuous. Then I must show that $\mathcal{T}_1$ is contained inside this $\mathcal{T}'$, where $\mathcal{T}'$ is defined above. That is the meaning of saying that $\mathcal{T}'$ is the largest ok?

Given $U \in \mathcal{T}_1$, because q is continuous, $q^{-1}(U)$ belongs to $\mathcal{T}$, but the moment q inverse of something belongs to $\mathcal{T}$ that something will be inside $\mathcal{T}'$. So, if $q^{-1}(B)$ is equal to U and U is here is in $\mathcal{T}$, that means B is in $\mathcal{T}'$. Therefore $\mathcal{T}_1$ is contained in $\mathcal{T}'$ and hence $\mathcal{T}'$ is the largest topology ok?

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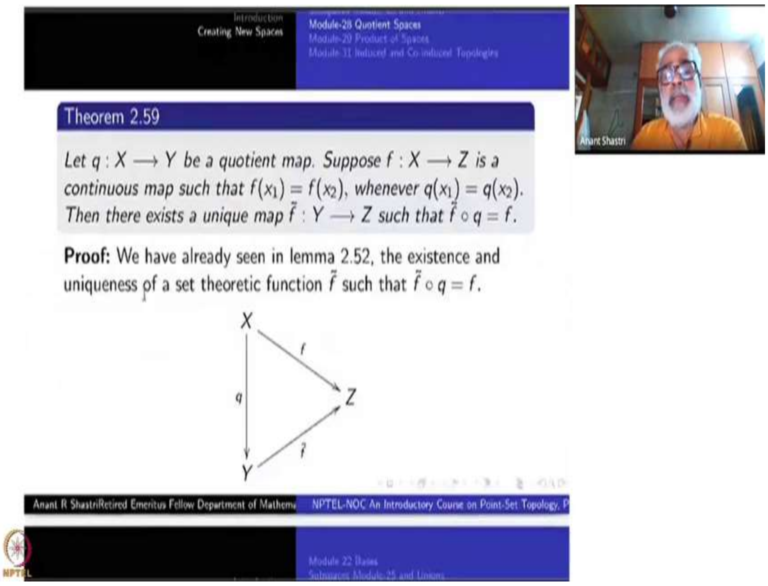

Next thing is: suppose you have this quotient map, now topology on X is fixed, topology on Y is also fixed. So, this is a quotient map ok? Suppose you have a function f from X to another space Z ok? It is a continuous map such that $f(x_1) = f(x_2)$ whenever $q(x_1) = q(x_2)$. Then there exists a unique map $\tilde{f}$ from Y to Z such that this $\tilde{f}$ composed with q here, first take q and then follow it by $\tilde{f}$, you get f.

The existence of this map as a function we have already seen, last time, ok? Whenever $q(x_1) = q(x_2)$, then $f(x_1) = f(x_2)$. This is the condition that gives you the function $\tilde{f}$. Why this $\tilde{f}$ is continuous? That is what you observe now, ok? Its existence and uniqueness we have already seen in yesterday's lecture.

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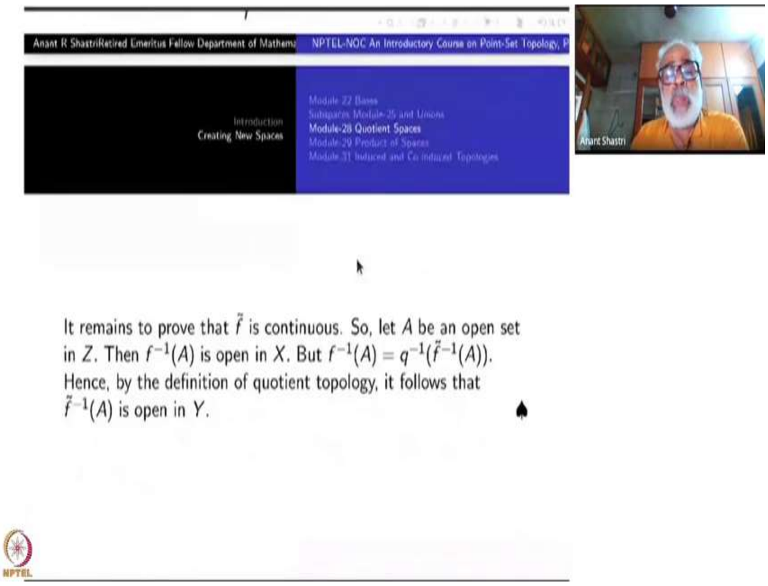

So, we have to prove that $\tilde{f}$ is continuous. So, let A be an open set in Z, this Z is an arbitrary space. I want to show that inverse image of A is open here in Y. When is a subset of Y open here? Its inverse image here must be open. So, what is $\tilde{f}$ inverse followed by q inverse, that is nothing but $(\tilde{f} \circ q)^{-1}$. But $\tilde{f} \circ q = f$. So, it is f^{-1}. But f is continuous. So, start with an open set in Z, its inverse image here under f is open because by the hypothesis f is continuous.

I repeat. Why $\tilde{f}$ is continuous? Take an open set here, come here under $\tilde{f}$ inverse. When you go all the way here that is nothing but f inverse of the open set. Now since f is continuous, we are done.

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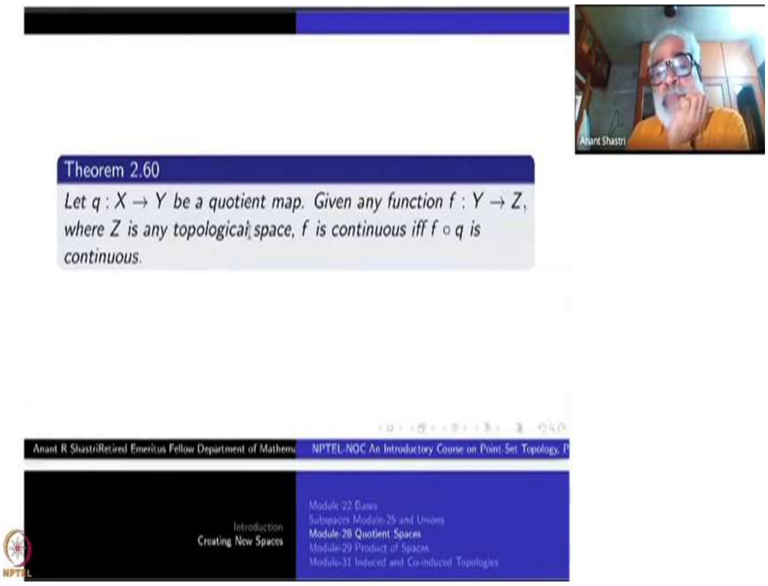

Take a quotient map q from X to Y. Given any function f from Y to Z where Z is a topological space this function is continuous if and only if its composition with q is continuous. Again in this picture you have some function here ok, Here when I have this function f ok, it introduces a function here which is continuous if this is continuous.

What I am doing? I am taking a function here. When this one is continuous what I am asking? If this is continuous, composed with q that will be continuous, right? Because composite of continuous function is continuous. But here is a case even if you do not assume this continuous but suppose the composite is continuous.

Then f itself is continuous. That is the statement here, given f from Y to Z, earlier f was from X to Z here. So, it is this f is playing the role of $\tilde{f}$ here ok. So, notations are different here alright. Since q is continuous and f is continuous implies the composite is continuous that is ok?

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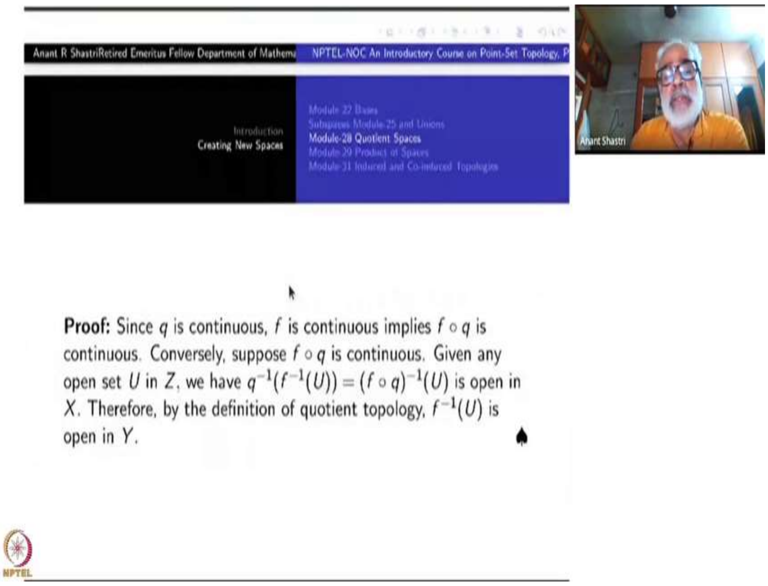

Conversely suppose $f \circ q$ is continuous. Now, I want to show that f is continuous. Given any open set U in Z, we have $q^{-1}(f^{-1}(U))$ is inverse of $f \circ q$ operating on U, the rule is the same you see now this is open is given. Therefore, $f^{-1}(U)$ is open here by definition of the quotient topology ok?

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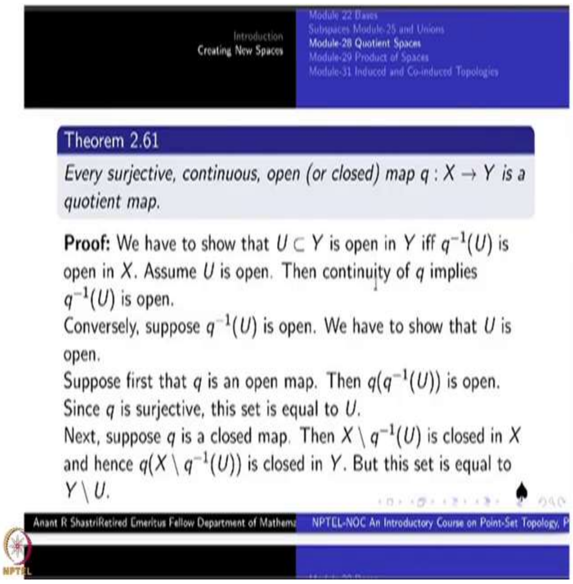

Next, the proof is similar but it is giving a different result here. Every surjective continuous open map is a quotient map. Instead of open you can put closed also, a closed map also satisfies this property.

When you say closed map I always mean a continuous and closed function, when I say open map I mean continuous and open ok. The extra thing is that they must be surjective. Unless you have surjectivity you cannot get a quotient right. So, surjectivity set theoritic and continuity are needed. Openness is extra that will ensure that it is a quotient map. Just a continuous surjection does not imply that the map is a quotient map.

So, let us go through the proof here. Start with any set here. Show that it is open if and only if q inverse is open. Both ways you have to show. start with any set U. If it is open then $q^{-1}(U)$ is open, if q inverse is open then U is open. That is what the statement is. So, first assume U is open. Then continuity of q will give you $q^{-1}(U)$ is open. So, here this is easy part. Now suppose $q^{-1}(U)$ is open. Why U must be open? If it is a quotient topology then it will be. But we do not know that. We are proving that it is a quotient topology that is why we have to prove this one.

Suppose $q^{-1}(U)$ is open. We have to show that U is open. Now first suppose you take the case when q is an open mapping. I have to consider the case for closed mapping also ok? but first take the case where q is open mapping. Open mapping means what? Image of an open set is open under q. Now $q^{-1}(U)$ is open? It is a lucky break that q operating upon $q^{-1}(U)$ is exactly U. Why? Because q is surjective. In general it is not true. It may be just a subset of U. So, surjectivity again plays an important role here ok?

So, this equal to U; the $q^{-1}(U)$ is open, q of that is open because q is open mapping. Therefore, what we have proved is that U is open inside Y if and only if its inverse image under q is open in X. This means that q is a quotient map.

Now the same thing can be done for closed map also. By just a little bit of De Morgan law you have to use. I will skip it you can read it.

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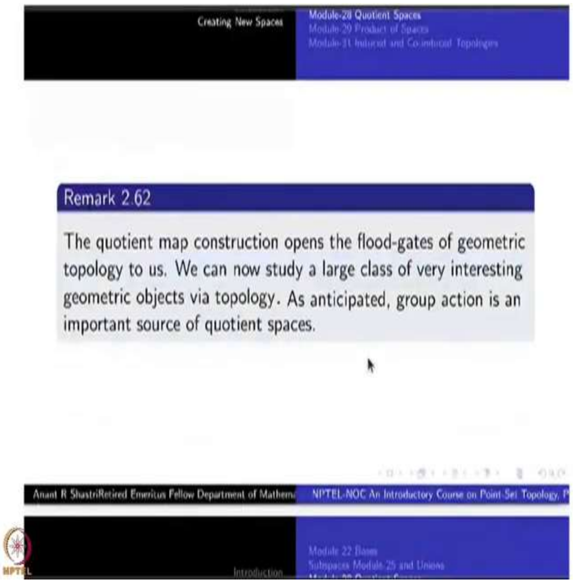

The quotient map construction opens the flood gates of geometric topology to us. We can now study a large class of very interesting geometric objects by our topology. As anticipated, group action is one of the important sources of quotient spaces, there are many other sources ok.

So, we have to study one by one some of these examples ok.

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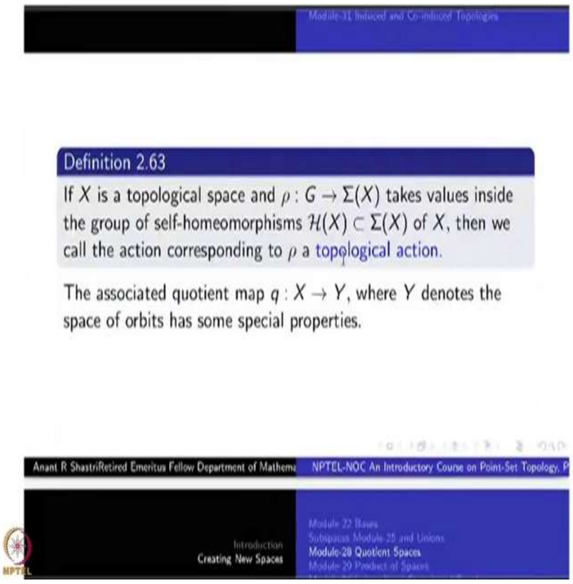

So, group action what is the topology coming where is its coming that is what I have to explain otherwise you have only quotient set right. So, this is where we make a definition. Take a topological space and an action of a group on it. namely, associated with a group of homomorphism like this. This group homomorphism I am assuming extra condition, I told you extra conditions on the group homomorphism will do different types of actions right.

So, I am putting in an extra condition here what is that, this homomorphism is actually taking its values inside a smaller subgroup namely, group of homeomorphisms, self homeomorphisms X to X. Any homeomorphism is a bijection, it is a permutation, but I do not want discontinuous ones, they must be continuous, inverse must be also continuous that is the definition of a homeomorphism.

So, take only the homeomorphisms, that forms a subgroup. That is very clear ok? because, composite of two homeomorphism is homeomorphism and inverse of a homeomorphism is the homeomorphism by definition ok.

When you have this we call it a topological action, you may wonder why all this is necessary you just take the quotient topological space is there ok. There is a quotient set there you can always give the quotient topology ok? So, that will have something to nothing to do with the group action, group action is just group theory, just algebra. But there is a topology, so the two different things you want to bring them together and this is one way. There may be many other ways also ok.

Depending upon the context that you want this is the way, take that now. What does this mean is that all these $\rho(g)$'s are homeomorphisms. Remember, all these $\rho(g)$'s are elements of the permutation groups, but now they are homeomorphisms. Take the associated quotient map q from X to Y where Y denote the orbit space of X. This will have some special properties which was not possible in the general situation of a group action ok.

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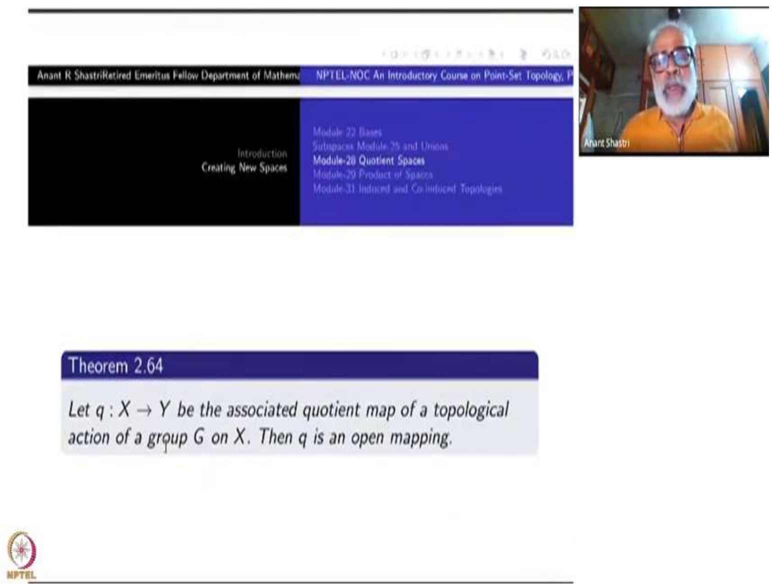

So, that is what we want to study now, let q from X to Y be the associated quotient map, of a topological action of a group G on the topological space X ok? Then this q itself is an open mapping. In the previous theorem we proved that continuous surjective open mapping is a quotient map ok? So, this quotient maps which arise here they are open mappings.

Remember for a quotient map there is no condition that it should be open if its open it will be quotient map is what we have verified ok? All quotients arising out of topological group action they are open mappings. So, they are stronger quotients in that sense they have better properties ok?

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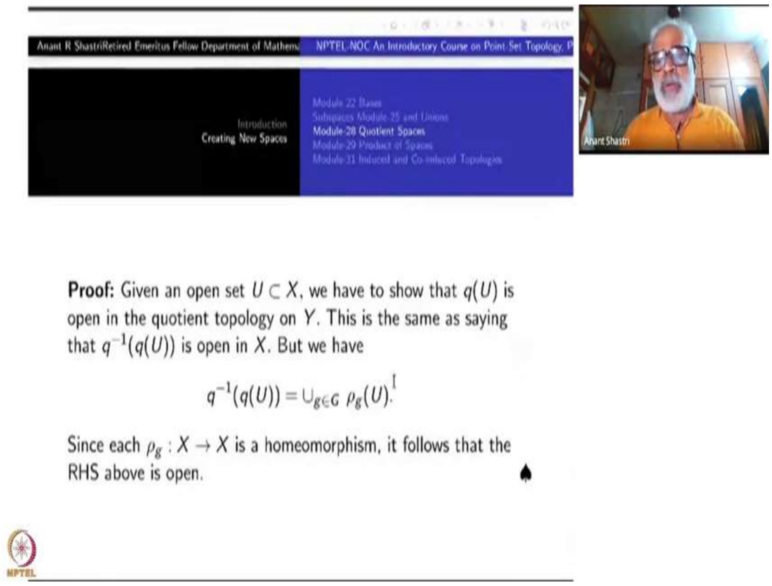

Let us verify why this is an open mapping. It is just one-line proof. See all of them are one-line proofs here. Take an open set U we have to show that $q(U)$ is open in Y. When this $q(U)$ is open? Inverse of that is open inside X. What is $q^{-1}(q(U))$ you have to understand that. That is nothing but all elements which are related to some element in $q(U)$, they will be there.

So, if you understand that then under rho g under this action whatever this is, just all the cosets gU ok taken together, that union of gU; g running over all of G. So, this is just easily verified. What is $q(U)$ for any subset U of X? This is a set of equivalence classes here. When you take a set of equivalence classes what is the inverse image? The union of all those classes.

That is the meaning of this ok? These are classes where members range over U so, it is union of all $\rho(g)(U)$. Now what is $\rho(g)(U)$, $\rho(g)$ is a homeomorphism from X to X just concentrate on that element g ok? Multiplication by that on the left or right there is an action that is a homeomorphism self-homeomorphism of X to X.

So, $\rho(g)(U)$ is also open for all g right? Therefore union is also open. So, that is all. So, we have proved that quotient map given by a topological action is an open mapping.

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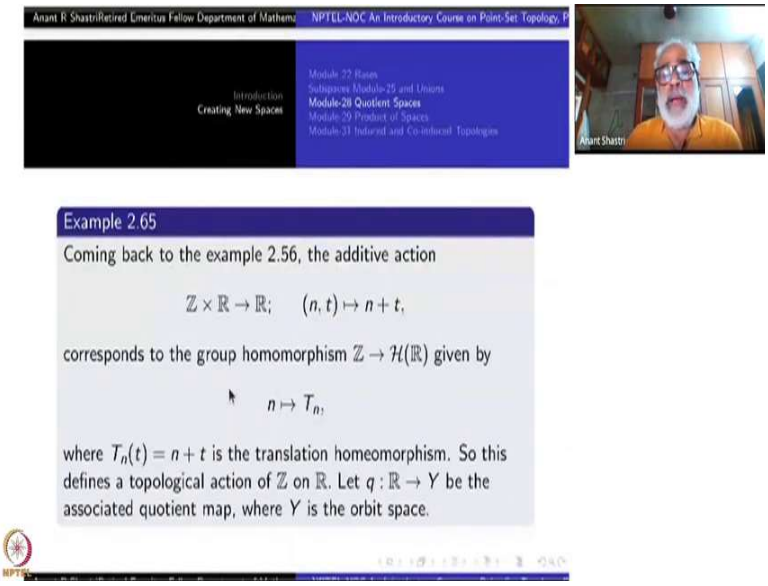

Again I come back to some examples here. This example is again the same as the we have considered before: $e^{2\pi it}$ ok? So, coming back to 2.56, the additive action $\mathbb{Z} \times \mathbb{R}$ to $\mathbb{R}$, namely integer (n, t) going to $n + t$. This is the additive action right?

Corresponds to the group homomorphism $\mathbb{Z}$ going to $\mathcal{H}(\mathbb{R})$; $\mathcal{H}(\mathbb{R})$ is what? All homeomorphisms of $\mathbb{R}$ to $\mathbb{R}$. Here, these are some special homeomorphisms, namely, the translation by n, so T_n, each n defines a translation. What it does here, t is pushed to $n + t$. That is all. These are called translations ok? So, $T_n(t) = n + t$, is a translation. What are the inverses? Inverse is T_n, which is T_{-n}. That is the inverse, the additive inverse ok? So, translations are homeomorphisms. So, inverse of T_n is T_{-n} here.

So, this defines a topological action of $\mathbb{Z}$ on $\mathbb{R}$. Let q from $\mathbb{R}$ to Y be the associated quotient map where Y is the orbit space ok. We have already seen that this Y is nothing but you know, it is in one one correspondence with $\mathbb{S}^1$, the unit complex numbers. We have already seen that right?

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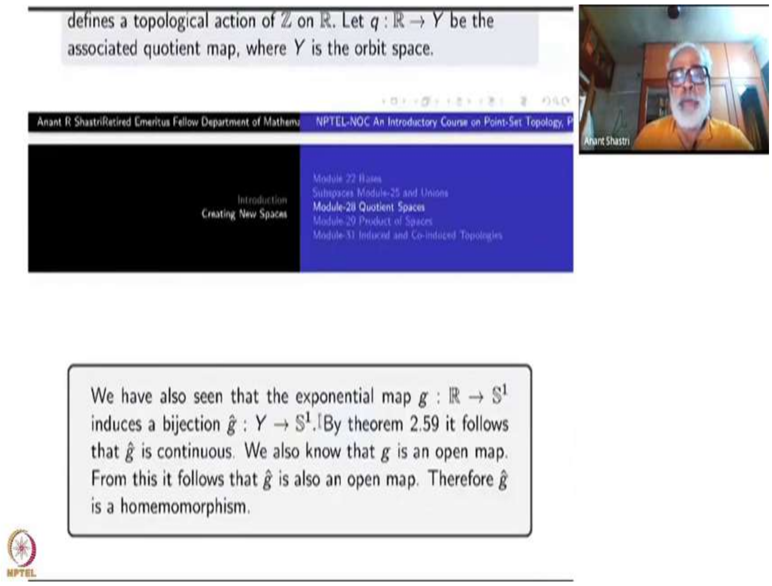

So, we have also seen that the exponential map g from $\mathbb{R}$ to $\mathbb{S}^1$ induces a bijection from Y to $\mathbb{S}^1$, the right cosets of this action.

So, by theorem 2.29, it follows that $\hat{g}$ is continuous, $\hat{g}$ which was just a bijection is now continuous ok? Now extra thing we have is, g is an open mapping, Why? Because this is given by this group action right? Therefore, g is an open mapping it follows that $\hat{g}$ is also an open mapping right?

See different notations may there for $\hat{g}$. If this is g; that is $\hat{g}$. This space happens to be $\mathbb{S}^1$ now ok. This is an open mapping why? Take an open set here in Y, inverse image is open here in $\mathbb{R}$. Take the image of that into $\mathbb{S}^1$ that is the same as image of this one ok?

So, since f is open this was open. So, g is an open mapping, so is $\hat{g}$. So, open continuous bijection. So, that is a homeomorphism. So, the orbit space here ok? It is actually homeomorphic to $\mathbb{S}^1$. We have proved that it is homeomorphic to this one. Thus under this homeomorphism, you can say that the quotient of $\mathbb{R}$ modulo the action of integers is precisely equal to $\mathbb{S}^1$, ok?

Here a bit of algebra is also coming. $\mathbb{R}$ is a group, $\mathbb{Z}$ is a subgroup which is actually normal because every subgroup, you know of an abelian group is normal. Quotient has a group structure also. What is the group structure? The multiplication of complex numbers. You know that exponential function takes addition operation to multiplication operation, $t+s$ goes to $e^{2\pi i(t+s)}$ which is the same thing as $e^{2\pi it}e^{2\pi is}$; it becomes multiplication right?

So, this $\mathbb{S}^1$ is the quotient group of $\mathbb{R}$. Just group theoretically. Now we have an extra condition. Topologically, it is the quotient space also ok. So, this example will be, again and again used, will be with us all the time.

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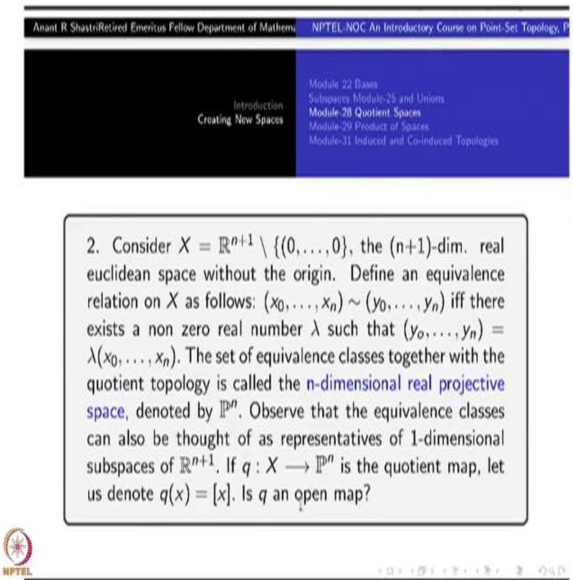

I will do one more example. Only touch upon this one a little bit. This again is by group action. First consider X to be the $n+1$ dimensional real Euclidean space minus the origin. you can put the origin for a while, but it does not help, so, minus the origin is better. Now, you define the action here by scalar multiplication.

Again throw away the 0 from the scalars also. So, $\mathbb{R}\setminus\{0\}$ ok scalar multiplication, that is what I am doing: $(x_0, x_1, \ldots, x_n)$ is an element of $\mathbb{R}^{n+1}$ and not equal to 0. Similarly $(y_0, y_1, \ldots, y_n)$; these two are equivalent (equivalence relation I am defining here), if there is a nonzero real number such that $(y_0, y_1, \ldots, y_n)$ is $\lambda(x_0, x_1, \ldots, x_n)$. So, this is a vector this is my scalar multiplication, this λ must be nonzero. Non zero real numbers form a group under multiplication. Of that group, I am taking the action of that group in on this space of non zero vectors.

Note that this time $\mathbb{R}^{n+1} \setminus \{0\}$ is not a group ok? This is a just a group action there is no subgroup and so on. This is not the special case as $\mathbb{Z}$ being a subgroup of $\mathbb{R}$ ok? The set of equivalence classes together with the quotient topology that becomes a topological space and is called n dimensional real projective space. That is the definition and the notation is $\mathbb{P}^n$.

There is a deliberate notation it is not a mistake, it is not a misprint you start with $n + 1$ and then you write the corresponding quotient $\mathbb{P}^n$ here.

Observe that the equivalence classes can also be thought of as representatives of one dimensional subspaces in $\mathbb{R}^{n+1}$, why? Because take a take an element take its equivalence class, all the elements lie on same line ok? You put back the origin just to make it a complete line a subspace. That is why I told you, you can keep the 0 for a while, but it does not help, instead of that whenever you want it you may put it back.

So, when you think of this as a line passing through origin, you put back the origin only for that purpose that is all ok. So, the equivalence classes represent lines. You see, you want to represent the whole line what do you want? You cannot represent it by 0, you take any other nonzero vector in the line. The whole line is determined. So, that is the geometry here. So, that is what we want to consolidate with this equivalence relation ok? Thus the equivalence classes can be thought of as representative of one dimensional subspaces of $\mathbb{R}^{n+1}$.

If q from X to $\mathbb{P}^n$ is the corresponding quotient map, let us denote $q(x)$ by the $[x]$,. Question is is q an open mapping? You should have a one-line proof here. What is that? I already told you that this equivalence relation etc is given by a group action. What is the group? $\mathbb{R}^{n+1} \setminus 0$. Is that action a topological action? That is what you have to see. What is meant by topological action? Multiplication by a non zero element should define a homeomorphism.

Of course it is continuous and its inverse is multiplication by λ^{-1} and that is also continuous. Therefore, there you are. Each multiplication defines a homeomorphism. Therefore, it is a topological action. Then you use the other theorem that we had, whenever you have a topological action the quotient map is an open map. So, that is the answer ok?

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There may be different ways of seeing it ok.? You try to think of some different way also. It is very good because you would like to have different views of this projective space which is difficult to imagine, and very difficult to draw pictures also ok? Once again, I tell you that we will come back to this example again to study more properties of this.

Thank you very much.