Introduction to Point Set Topology, (Part I)
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Lecture - 27
Extending nbds

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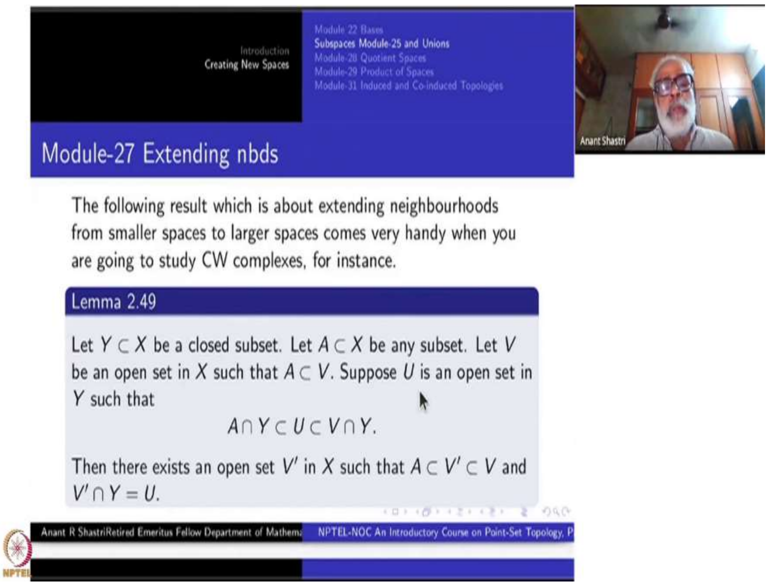

Welcome to module 27 of Point Set Topology, Part 1 course. Last time we have studied the union of open sets, union of topological spaces, coherent topology and such things. So, those things are much more used when you go to manifolds or more generally simplicial complexes and CW complexes and so on. One of the problems there again is that you have some neighbourhoods of the point in the smaller space. You want to extend it to a larger space and then again larger space and again larger larger and so on.

So, the first step of that we will study here which is purely you know elementary point set topology. If you are stuck with this one there then that will be more difficult ok, here is the lemma; take Y contained inside X a closed subset ok. X is topological space, Y is a closed subset with its topology coming from X now.

So, as a subspace also you can think of this one. Now, take A contained inside X any set, let V be an open set in X, such that A is contained inside V ok. So, V is an open subset of X. Now, suppose U is an open subset of Y ok, such that $A \cap Y$ is contained inside U the whole thing is contained inside V. Therefore, they are contained inside $V \cap Y$ also, ok.

The question is whether you can find an open set V' in X such that A is contained inside V' and contained in V, I mean I am not going out of V, but now this V' is such that when you intersect with Y it is exactly equal to U. The Y part of A, this $A \cap Y$ ok, was contained inside U.

So, U is a neighbourhood of $A \cap Y$ inside Y. Now, I want to extend it, extension means what? when you take intersection, it will come back to U right? V' is an extension of U to open subset of X. So, set it becomes a neighbourhood of A in X ok.

If A is a single point this will be easy to do ok, but if A is not a single point and not contained inside Y and so on, then this is somewhat complicated. That is why I have put it here, actually I feel like you know having explained this one, feel like giving it to you as an exercise. But this kind of point set topology is somewhat unusual, many people have difficulty in this one. So, I am going to explain this one completely.

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Let us understand the problem correctly, we start with a topological space X and this Y is a closed subspace ok? This subspace and as a subset it is closed also inside X. Now you have an arbitrary subset of A here, A is an arbitrary subset of X contained in open subset V of X ok? When you take the intersection of A with Y that is contained in an open subset U of Y and whole thing does not go out of V, which means when you take $V \cap Y$, it contains U.

What I want is if this were equality then I am happy right? But you do not know that it is an equality. So, what I want is maybe something smaller here V' such that $V' \cap Y$ will be equal to U. So, can you bring it down to a subset a smaller one, such that equality holds is the question ok? The assertion is that it is true. There is this question mark here, and the answer is affirmative. So, this is the lemma.

So, you can find a V' here open contained inside V, containing A when you take the intersection with Y it will be exactly equal to U. That is why this is an extension of U ok? You do not call V an extension because when you intersect with Y, you do not get the set U, but you get something larger maybe, that is all. So, what is V', how to get it?

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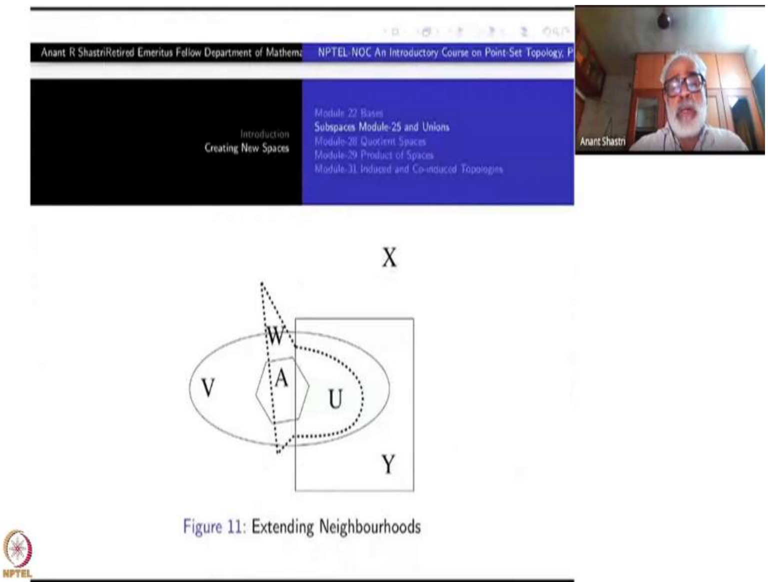

Figure 11: Extending Neighbourhoods

So, I have put it directly here, but here is a picture showing what all kinds of things can go wrong. So, this is your X, this square is your Y, in the picture ok. The picture cannot be too bad after all, this is your V which is an open subset of X. It is not contained in Y, of course, then everything will be trivial. This is your A this hexagon which is also not contained inside X. U contained inside Y ok? Part of it is inside Y and part of it part is outside Y ok.

Now what is U? One second. So, see U is an open subset of Y and contains $A \cap Y$. So, this is $A \cap Y$ ok? And this dot dot dot up till here is an open subset of Y and it contains $A \cap Y$ ok? Can you extend it to an open neighbourhood which will contain the whole of A like this and when you intersect it with Y, it should be only this part.

Since U is open in in Y, U will be equal to something like W if full thing dot dot dot intersect with Y, that is the definition of open subsets of Y ok. But this W has gone out of V and it does not contain A either. So, this W the choice of W here by the definition of that U is a open inside Y is not doing the job, it neither contains A nor contained inside V, I want both of them.

Since U is open in Y, U will be equal to something like W, the full thing dot dot dot intersect with Y, that is the definition of open subsets of Y ok. But this W has gone out of V and it does not contain A either. So, this W the choice of W here by the definition, that U is an open inside Y, is not doing the job, it neither contains A nor contained inside V, I want both of them.

So, what should I do? this is the point. So, I have to use the fact that Y is a closed subset of X ok? I have to use that and I do not want to go out of V ok? V is an open subset Y is closed. So, $V \setminus Y$ will be an open subset. So take this open subset. This V is open, this Y is closed inside X, so, complement of Y is open inside of X. So, $V \cap Y^c$ will be open inside X itself. So, this portion will be inside V, but still, it is not covering A right?

So, how to cover A? So, think properly and that will give you an answer. How to cover A? So, this W is there, I can intersect it with $V \setminus Y$ ok, that will give you an open subset contained in V, but I should find a nbd containing A also right? So, the picture will tell you what we have to do. But here I have a written down the answer anyway.

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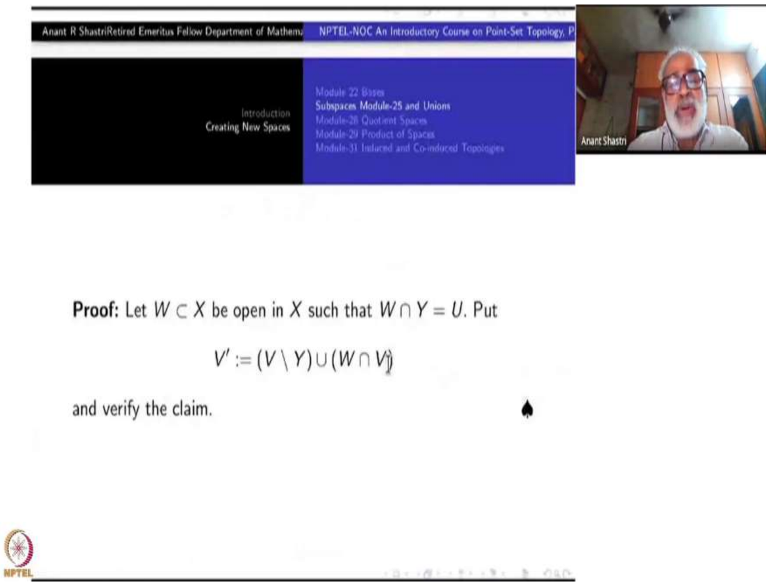

So, what I am taking? Start with W, open in X such that $W \cap Y$ is U. I repeat how do you get it because Y has subspace topology and U is open in Y. Now, put $V' = V \setminus Y$, Y is closed. So, $V \setminus Y$ will be open because V is open. Union with $W \cap V$ ok? This W is there ok? intersect it with V this part ok? Union W intersect with V, and $V \setminus Y$ this part was not there right? So, you take this part union of that part, this part will cover this one and $W \cap V$ will cover whatever is left out. That will take care of whole of A.

So, $(V \setminus Y) \cup (W \cap V)$ ok? This is my V'. W is open inside X, V is open inside X, so intersection is open. It is a union of two open sets that is also open. The only thing you have to verify is: Suppose you intersect it Y, it must be equal to U; when you intersect it with Y this part does not contribute anything because it is in the complement, $V \setminus Y$. It will be only from this part, $W \cap Y$ is U ok, but $W \cap V$ is already part of that one, U is already inside V.

So, this is also the V is already larger subset than U therefore, when intersect it with Y this will be just $W \cap Y$ and that is U ok. So, finally, you have to verify that A is contained inside V'. I will leave it to you. A is contained inside V' is what you have to finally prove. So, that much I will leave it to you.

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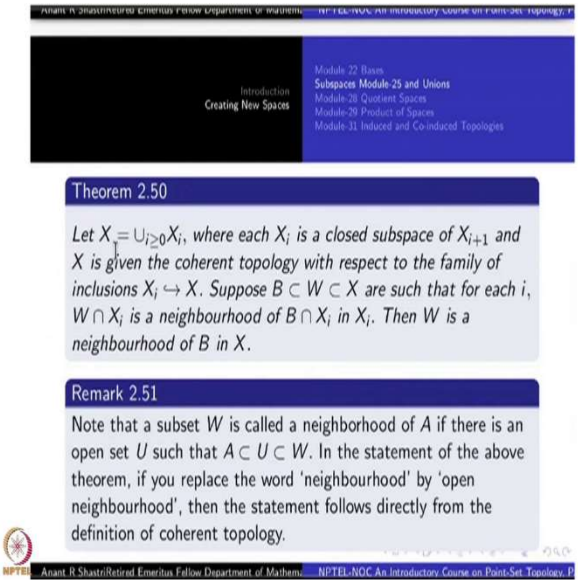

(THEOREM 2.50 IS INCORRECT. A COUNTER EXAMPLE HAS BEEN PRESENTED IN THE LIVE SESSION.)

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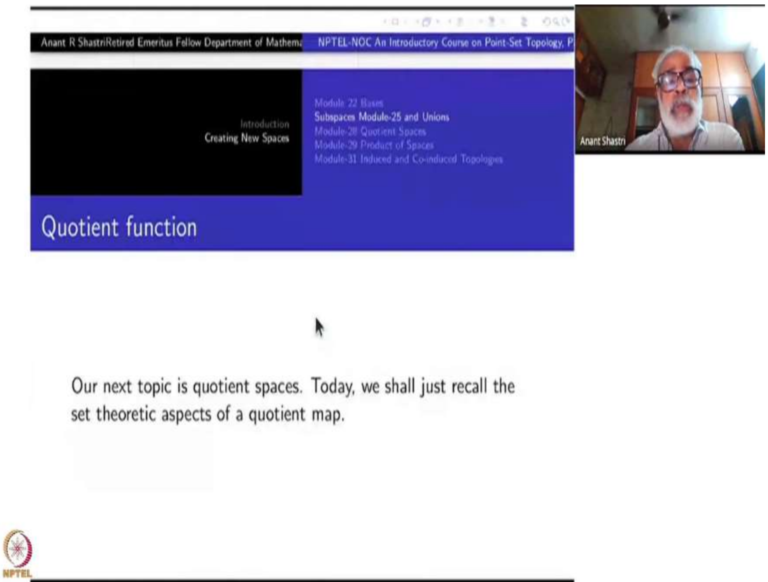

The next topic was about quotient topology I will have to do that one for the rest of the time here. Though that is not exactly the the theme of the of the today's talk as such ok. They all fit into the theme of constructing new spaces. So, today I will not do anything about spaces, but just about sets ok, the next topic is quotient spaces, but today we should just recall the set theoretic aspects of a quotient map. Hope you know it, but let us not assume that you know exactly what I want you to know.

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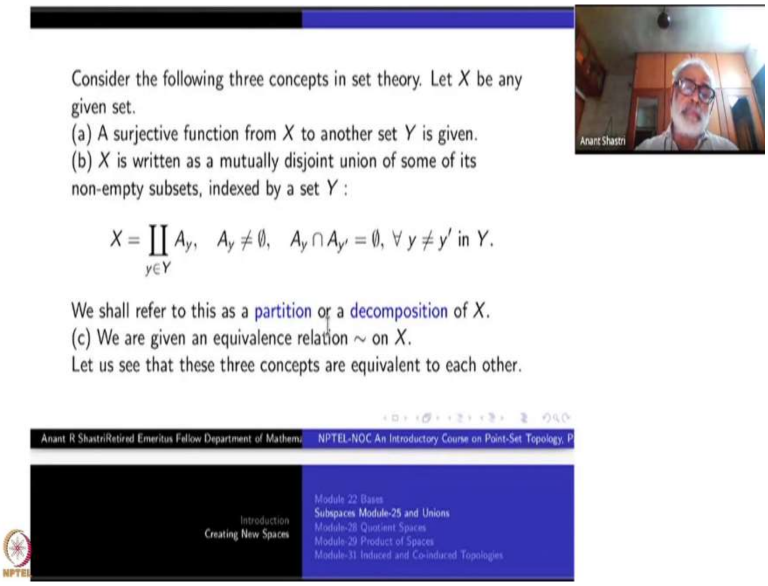

So, I will tell you a few things. Consider the following three concepts in set theory. Start with X, any set ok. First thing is there is a surjective function from X to another set Y ok? The second thing is X is written as a mutually disjoint union of some of its non empty subsets, indexed by a set Y. What do I mean by this? X is disjoint union of A_y, $y \in Y$ that is, indexes are in Y ok? A_y's are non empty, non empty subsets of X, $A_y \cap A_{y'}$ is empty, for $y \neq y'$, that is they are mutually disjoint alright.

So, there is no repetitions and so on here, if there are two repetition then they will be equal, not empty. So, $y \neq y'$, then intersection must be empty. So, this is the meaning of mutual disjoint open sets, sometimes I will just say disjoint union that is what I have started with right? So, here also I am looking at disjoint union, that is the that is the second condition. So, such a disjoint union often is called a partition, partition of a set. Some people call it a decomposition of a set also. So, I will use both the terms. You will get used to both the terms.

The third one is: We are given an equivalence relation on X. So, I have tell you: three different things, surjective function, a decomposition and an equivalence relation. What they have to do with each other? The claim is that these concepts are equivalent to each other. Giving one of them is same thing as giving the other two. What is the meaning of `same thing' I have to explained that to you ok?

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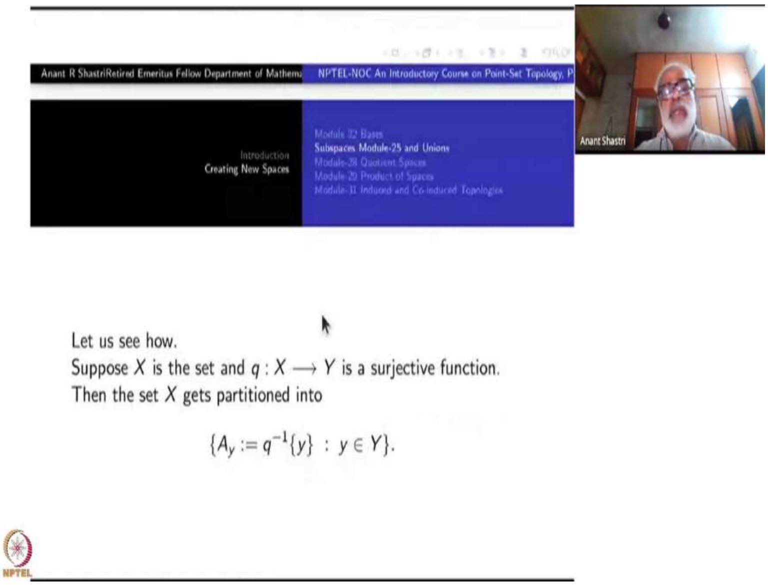

So, let us see how, what is the meaning of that these things are same. To start with suppose X is the set and so the surjective function q that is the first one right, it is (a) here. Take a surjective function. I want to say that it will immediately produce a partition, what is that? The fibers, the fibers of q, what is the meaning of fibers of q? $q^{-1}(\{y\})$ right? That is the fiber over y of q. As y varies over Y, that will cover the whole of X. Because this is a surjective function ok?

That means what? Union of A_y is equal to X, because your function $q^{-1}\{y\} \cap q^{-1}\{y'\}$ will be empty ok? Again because of surjectivity, each $q^{-1}\{y\}$ is non empty. So, this is a partition or a decomposition. So, (a) implies (b). Implies (b) means what? A surjective function gives rise to a partition alright. Let us go further now.

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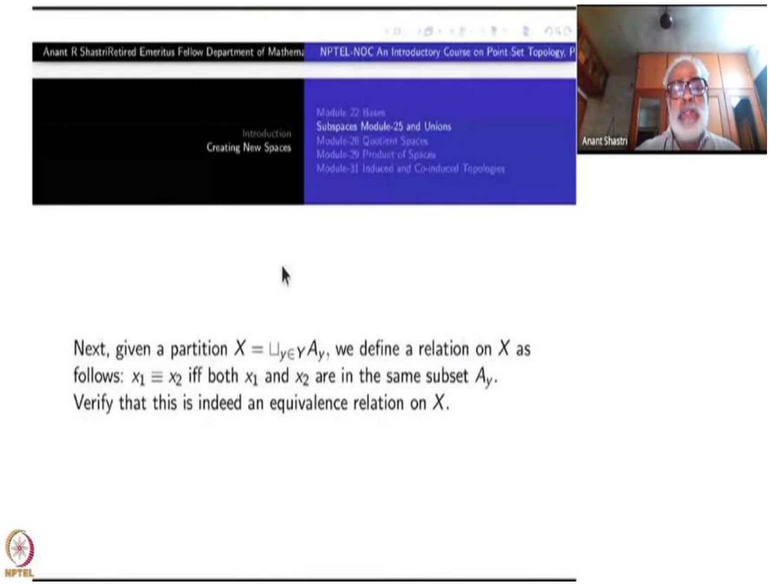

Suppose you are given a partition indexed by a set Y, I want to define an equivalence relation now. Relation I am defining: x_1 is equivalent to x_2, if both x_1 and x_2 are in the same subset A_y. Remember X is union of A_y's. So, given any x it must be inside one of A_y, but if given two of them if both of them are in the same A_y, then I say that they are equivalent, they are related. Why is this an equivalence relation? That is again easy to see. Because if x, y are such that x, y are in same thing and y, z are in the same thing, then x and z must be in the same same thing. That is all, right?

Same set A_y for both of them. So, that is the transitivity. Reflexivity is obvious. Symmetry is obvious because by definition it is symmetric: x and y both belonging to means what? So, it is already symmetric relation. So, this is an equivalence relation. Start with a surjective function, it gives you partition, partition gives you an equivalence relation. You can go back whichever way you like, I can directly go to now the function.

Suppose you have an equivalence relation ok? You want to construct a function from X to some set Y. Y is not given to you nor the surjective function. So, how do you do that? Not through some arbitrary means. That is the function must be something to do with the relation which you have been given, the equivalence relation ok? So take the equivalence classes that is your set Y ok? You have the equivalence relation here. So, deliberately I have written a different notation here ok.

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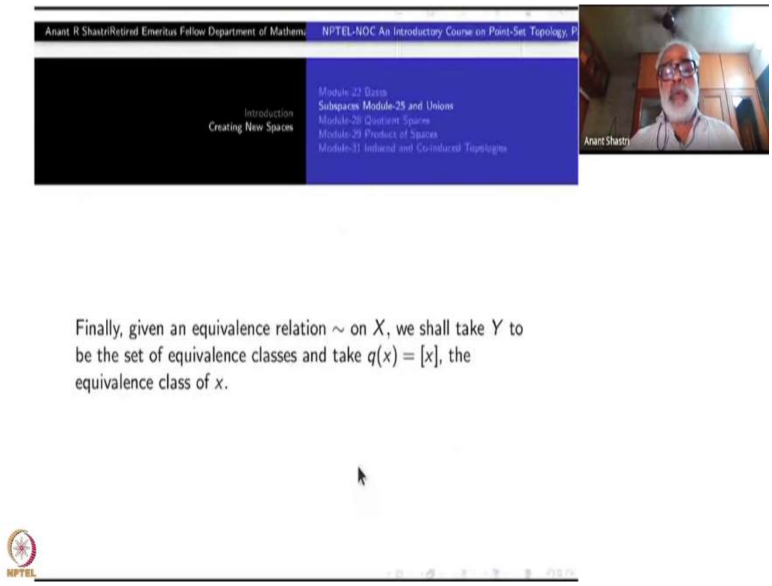

Now, you have some equivalence relation not necessarily coming from a function alright. Some equivalence relation I do not know what function and so on. I am just concentrating on the equivalence relation. Take the equivalence classes, take all the equivalence classes, that is your set Y and take $q(x)$ equal to be the equivalence class of x, that is a function from X to Y. Each equivalence class contains some point. After all it is an equivalence class therefore, this function is surjective. Over. So, you have got a surjective function.

The point is that you can come back all the way and it will give you same thing wherever you started with.

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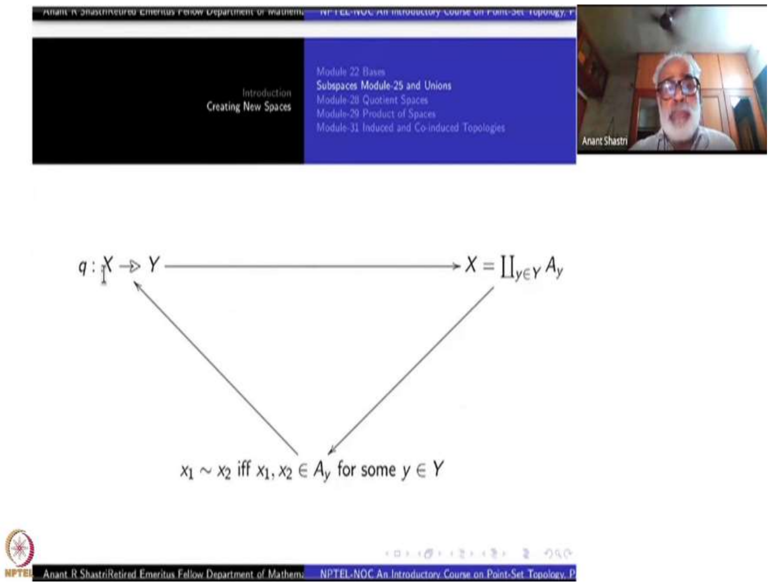

You have 3-cycle here, means cycle consisting of three points, either you can go from here to this what I have written, starting with surjective function, I came to a partition and then to equivalence relation, then back to a surjective function. You can come back directly also or you can go here and come. You will get the same thing which you started with.

Let us just test it just for this one, starting with an equivalence relation I have constructed this map q ok.

So, how do I get back this one? All that I have to do is to define this equivalence relation whenever they are in the same fiber, they are mapped to the same point. The same point is what? Equivalence relation equivalence class. So, I am getting the same equivalence relation. So, you can check the other three also, from here to here here to here so on. So, this is what we mean by, you know there is a bijective correspondence here, you can go from here you come back you get the same thing.

So, these three different pictures of a quotient function has to be kept in mind. The simplest way a surjective function over, but the other two descriptive pictures are also very helpful many times ok. So, keep keep up this one ok. So, there are these words the map q is also called the identification map, quotient map ok? Just a surjective function. I usually the word map only for a continuous function. Identification set is what? The set of equivalence classes.

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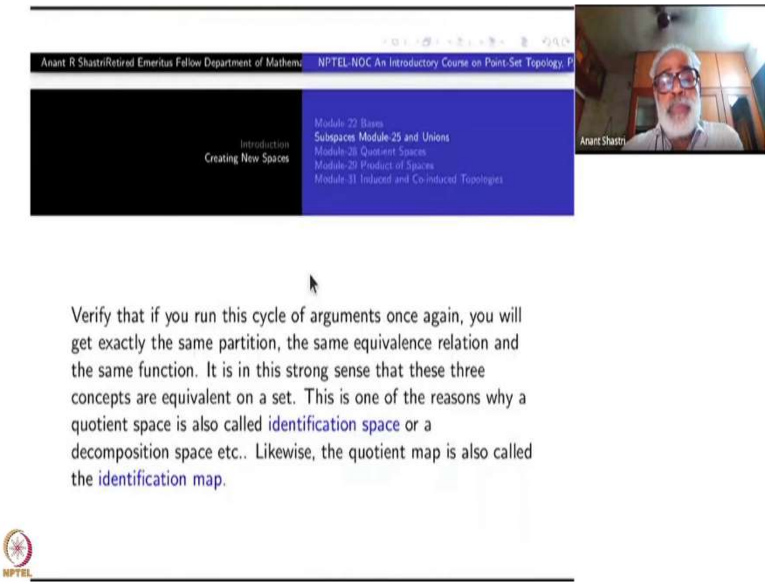

Corresponding to equivalence relation which is which you can say that same thing as is fibers, equivalence classes ok. So, these only set theories so far ok?

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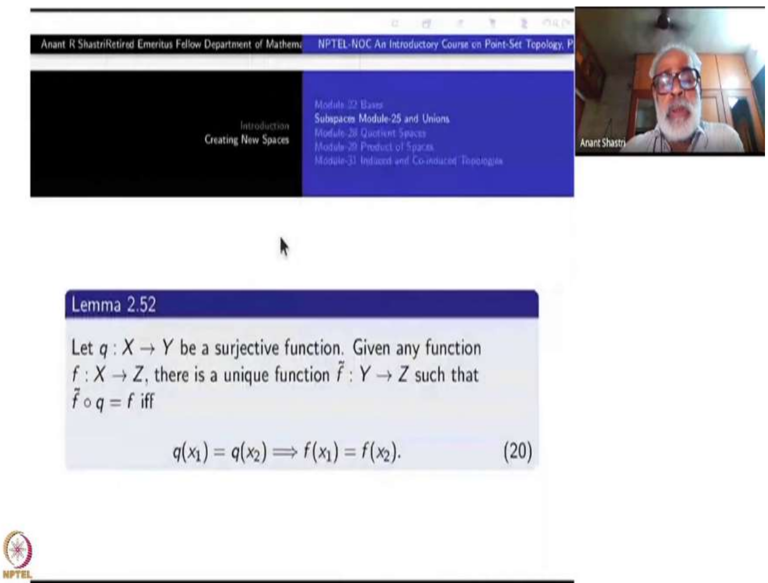

I want to tell you that it has some properties set theoretic function set theoretic property, what is it? Take a surjective function q, just like what what we are done here ok. Now take any function from X to Z, Z is some other set ok? I want to have a function $\hat{f}$ from Y to Z such that this $\hat{f} \circ q$ is f. This will happen if and only if this condition is satisfied:

$q(x_1) = q(x_2)$ implies $f(x_1) = f(x_2)$ ok? It implies $f(x_1) = f(x_2)$, then there will be a function like this. So, this is the lemma. This is just set theory. So, q is from X to Y. It is surjective, f is some map here. Suppose you can fit a function here from Y to Z such that this diagram is commutative. What do I mean by that? q followed by $\hat{f}$, while writing, $\hat{f}$ is first, $\hat{f} \circ q$ must be equal to f.

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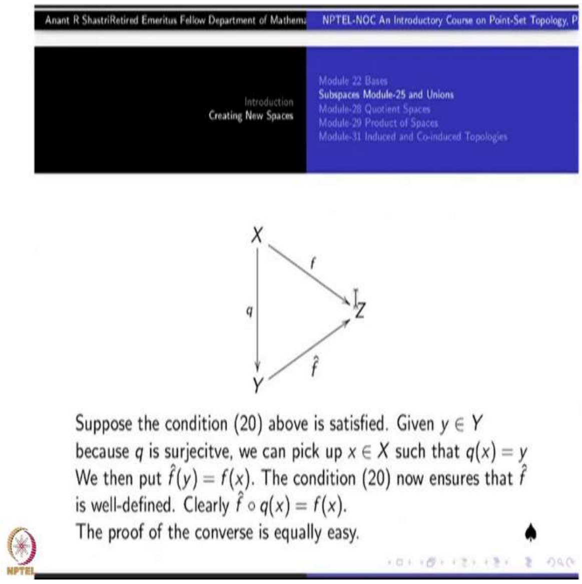

When does it happen? Suppose there is such a function ok? Then I have to verify, that perhaps you know it. If and only if right? I have to verify. Suppose $q(x_1)$ equal to $q(x_2)$. Then what happens? Take $\hat{f}$ of that it will go to same point. But what is the $\hat{f} \circ q$? You know this $\hat{f}$ composite q is f. So, $f(x_1)$ will be $f(x_2)$. So, this condition is easily satisfied. that is `if' part.

Conversely suppose this is true, then I will now define this $\hat{f}$; how? Take a point $y \in Y$ here. How do I take a point here? It is an equivalence or simply, it is q of something here, q of some x in X. Take $f(x)$ and define $\hat{f}([x])$ to be equal to f of that point x which comes to y.

I repeat. We define $\hat{f}(y)$ equal to $f(x)$. What is x? $q(x)$ is equal to y. So, that is a way you have to choose ok.

When you have to chosen, you have a problem. Why this is well defined? Like you may choose some other x' here ok. But now that condition comes here. If there are two points x_1 and x_2 such that $q(x_1)$ equal to $q(x_2)$, then $f(x_1)$ is equal to $f(x_2)$. So, therefore, $\hat{f}$ of this y, whatever x you have chosen, x and x' different points f of those two points should be a same thing.

So, $\hat{f}$ is well defined, but then what is $\hat{f}$ of $q(x)$? If this y is already $q(x)$, by the very definition, $\hat{f}(y)$ is $f(x)$. If y is already $q(x)$ then $\hat{f}$ of that is by definition $f(x)$. Over ok? So, whenever two points here both the same point here, f should also map them to a same point. So, this is used everywhere, whenever there is a quotient map right? In group theory you know, in functional analysis when you want to construct quotient vector spaces, in linear algebra, linear algebra you must have done it already. So, in all these situations, there is this function just a set theoretic function.

Of course, you may have f as linear map then this is also linear map automatically this will be linear map and so on. If this is a normal subgroup and this is quotient here where this is a subgroup then this becomes a group then this will be automatically become a group homomorphism ok. So, that is the relation this is the set theory. So, different disciplines use that same thing, in a correct way.

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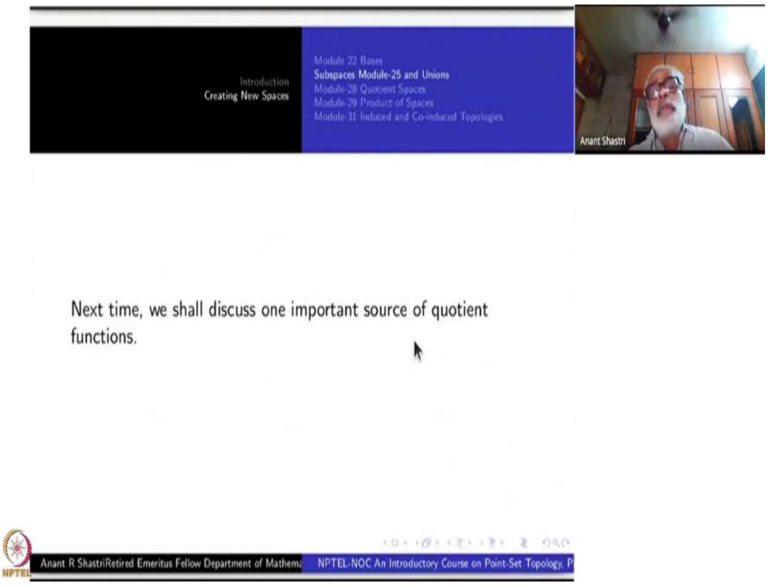

So, now, we are going to use this in topology. So, today I am not going to give start the topology ok. So, that we will do next time. So, this much elementary set theory you must remember. So, tomorrow we will discuss the topological aspect of this one and also some examples before going further.

Thank you.