Introduction to Point Set Topology, (Part-I)
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Lecture - 26
Union of Spaces

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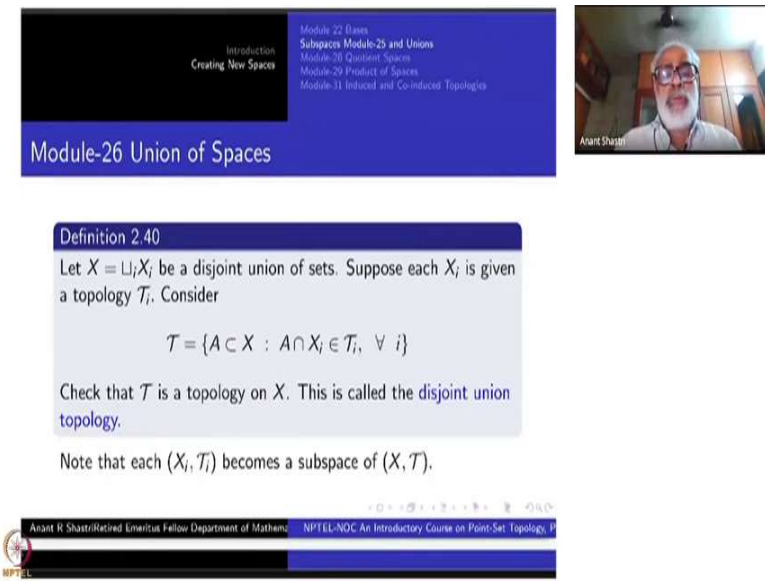

Welcome to module 26, today our topic is discussion of Union of Spaces. To start with: Let X be a set which is written as the disjoint union of sets X_i, ok. This is one special case of what we want to discuss. Suppose each X_i is given a topology $\mathcal{T}_i$, ok? Consider $\mathcal{T}$ equal to all subsets A of X such that $A \cap X_i$ is in $\mathcal{T}_i$ for every i.

See A being a subset of the disjoint union, it is a disjoint union of subsets from each X_i right? So, those are $A \cap X_i$. So, all of them must be open in the corresponding X_i that is the meaning of $A \cap X_i$ is in $\mathcal{T}_i$. Take all such A, put them together in one single family, that family $\mathcal{T}$ is a topology on X. So, this is called the disjoint topology. I start with a family of subsets here, all mutually disjoint, that is the meaning of this symbol $\sqcup$, I am always using this for disjoint union ok.

Verification that $\mathcal{T}$ is a topology is easy: (T) The set X belongs to $\mathcal{T}$ because each X_i belongs to $\mathcal{T}_i$. So, if you take A equal to X, then $A \cap X_i$ will be X_i so, they are there in $\mathcal{T}_i$. (FI) If A_1 and A_2 are there then $A_1 \cap X_i$ and $A_2 \cap X_i$ are there in $\mathcal{T}_i$. So, $A_1 \cap A_2 \cap X_i$ will be there for each i. So, intersection is there and so simialrly, verification of (AU) is also straightforward.

What is important here is that $(X_i, \mathcal{T}_i)$ becomes the subspace of $(X, \mathcal{T})$. Why? Because what is an open subset here? If something is an open subset here in $(X_i, \mathcal{T}_i)$, I can just take that open set whatever it is and do not put anything from any other X_j at all, its intersection with X_i will be the same set and its intersection with any other X_j will be empty. Therefore, an open subset of X_i here is inside $\mathcal{T}$ automatically.

So, each A_i belonging to $\mathcal{T}_i$ is already inside $\mathcal{T}$, ok? So, $(X_i, \mathcal{T}_i)$ becomes a subspace of $(X, \mathcal{T})$, because once A_i is there because $A_i \cap X_i$ will be just A_i alright. Not only that, each X_i will be now open inside X. And therefore, it is also closed inside X because each X_i is the complement of the union of other X_j's, ok?

So, this is the easiest picture: when you take disjoint union ok ?This is also an interesting one, but more interesting cases will be when things are not disjoint ok? That is where the concept of patching up topological spaces comes into picture. However, I will not use the word patching up because patching up has different meanings for different people ok.

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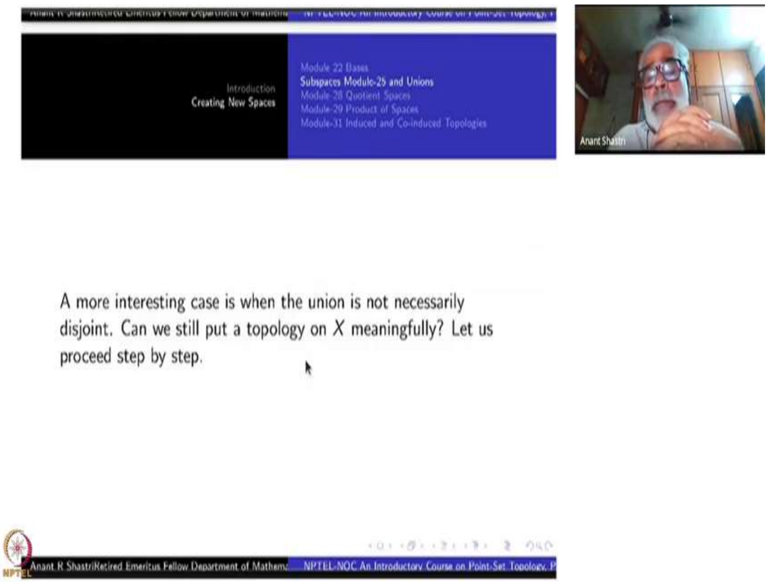

So, what I want to do? The more interesting case is when union is not necessarily disjoint. Can we still put a topology on X meaningfully? Of course, you can always put a topology indiscrete or discrete and so on. But it has something to do with each $\mathcal{T}_i$ since there is already a topology for each of these subsets X_i, ok? So, let us go step by step.

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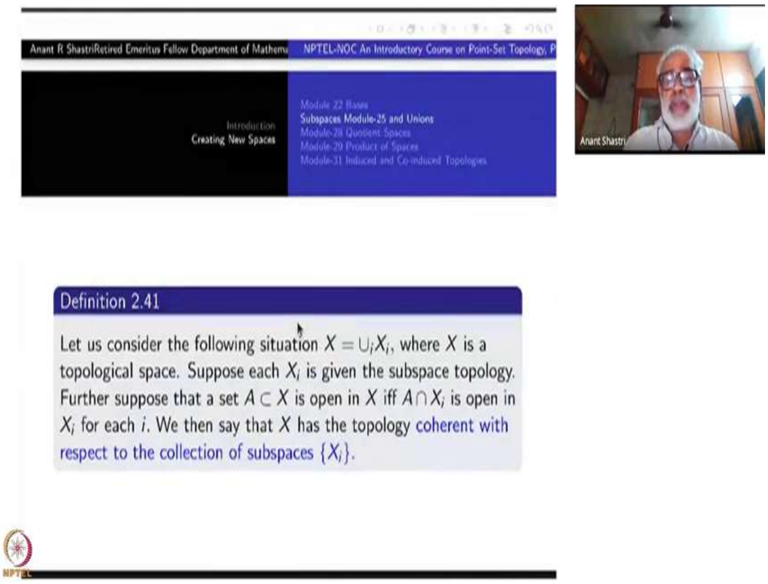

Let us consider the following situation, where X is the union of X_i. Now I put $\sqcup$ here so, this not necessarily disjoint. Now, suppose X is topological space, and each X_i is given the subspace topology. X is a topological space these are subsets so, we know what is the meaning of subspace topology.

Further, suppose that A contained inside X is open in X if and only if $A \cap X_i$ is open in X_i for each i. So, this is an extra assumption I am making. Start with a topological space X which is written as union of X_i's and give the subspace topology on each X_i. And then assume this condition, ok? It is not clear why this condition should be true at all ok?

So, this property I have copied from from whatever we observed in the disjoint union case. So, that is why I am trying to put that condition here and see what happens. So, it is all in the data now. It is better to call it by a name so, that we do not have to keep on telling all these things every time. So, in such a situation we say X has the topology coherent with respect to the collection of subspaces X_i. So, this is the definition of coherent topology; coherent with the collection of subspaces means this much ok?

Maybe later on, we will improve upon this definition but right now this is the definition.

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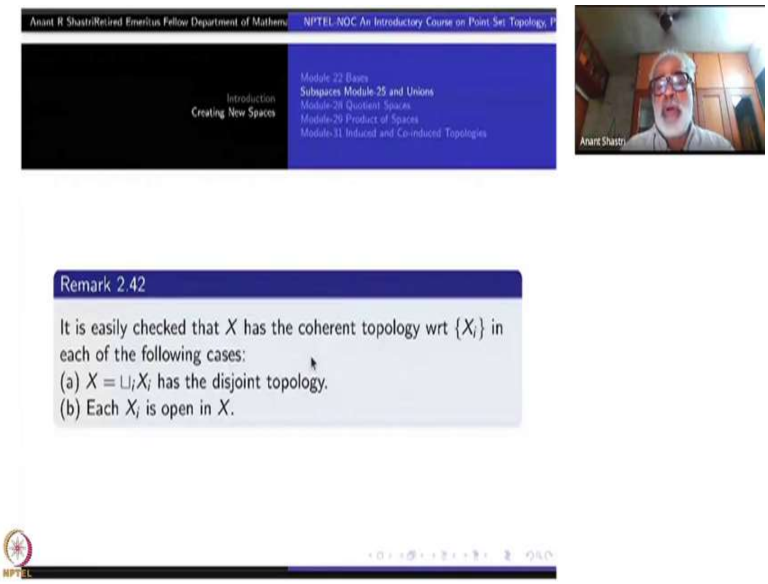

It is easily checked that X has coherent topology with respect to X_i in the following cases, namely, the first case when X is the disjoint union of X_i's. We have actually copied that property of the disjoint union topology here.

Another easy example is wherein each X_i is an open subset of X. So, I recall this. Take X equal to the union of X_i's, X_i's are some subspaces union is the whole thing X, but now I am assuming that each X_i is open in X ok? Then what happen? Suppose $A \cap X_i$ is open in X_i for each i. Automatically it will be open in X also that is what we have seen. So, each $A \cap X_i$ is open in X, their union will be A, so that is also open. So, the condition of coherency whatever is satisfied here. We have verified it here ok?

So, these are two examples, easy examples of coherent topology alright.

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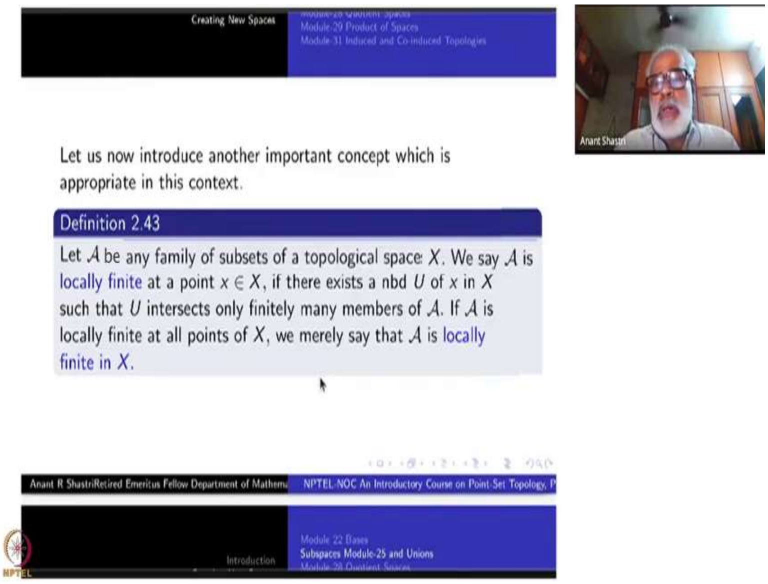

Let us introduce another important concept here ok, which is relevant in this context, appropriate in this context and that will allow us to have more examples. Let $\mathcal{A}$ be any family of subsets of a topological space X. We say $\mathcal{A}$ is locally finite at a point x, (it is like continuity definition it is pointwise, I am defining it first at a single point), if there exist a neighborhood U of x such that U intersects only finitely many members of $\mathcal{A}$; if this happens at all points $x \in X$, then we say the family is locally finite.

At each point you are given a neighbourhood, the neighborhood obviously depends upon the point ok. The number of members which intersect also depends upon the point as well as the neighborhood. Once there is a neighborhood all smaller neighborhoods will also satisfy that property ok?

So, that is why I can put just a neighborhood then you can make it an open neighborhood also because every neighborhood contains an open neighborhood. So, there is no harm in putting neighborhood ok? If this happens at every point of X then you call the family locally finite. That is why, I have indicated the similarity between this definition and the definition of continuity; continuity at a point and then continuity at all the points is the definition of continuity of a function. So, similar local finiteness at each point and then local finiteness on the whole of X.

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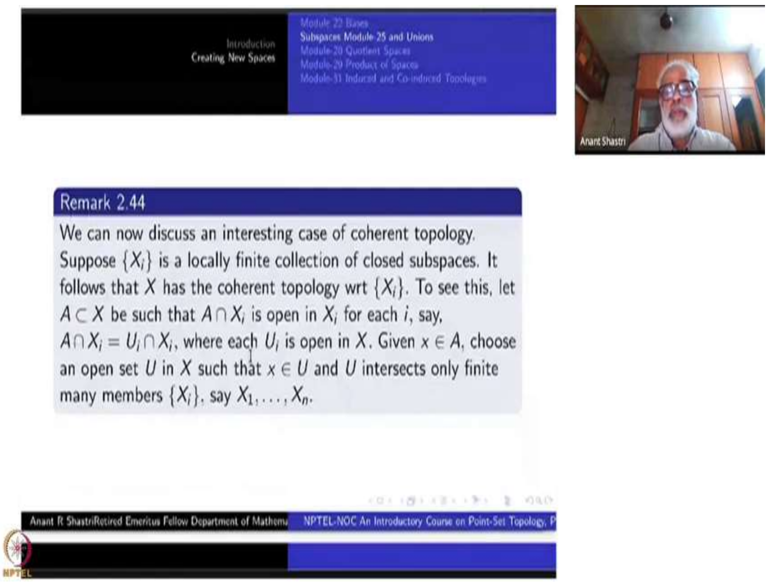

So, now I will give you another interesting example of coherent topology the third example you may say. Now, suppose X_i is a locally finite collection of closed subspaces of X. So, this is the extra condition I am assuming. And X is equal to union of X_i. That is given already. So, this is the extra condition ok? namely, each X_i is closed in X and the family is locally fineness. So two extra conditions. Then I claim that X has the coherent topology with respect to X_i's.

So, let us check this one. I have to check this pointwise, right? That is what I have to do right uh. Sorry, I have prove the coherence of the topology. I have to prove that if A is such that $A \cap X_i$ is open in X_i for all i, then A is open in X. Pointwise consideration comes later.

We start with a subset A of X such that $A \cap X_i$ is open in X_i for all i. This just means that $A \cap X_i$ is equal to $U_i \cap X_i$, where U_i is open in X. So, this is the definition of subspace topology, X_i's are subspaces of X ok? Now, to show that A is open in X, given x belonging to A, ok? So now I have to use local finiteness to produce an open subset V of X such that x is in V and V is contained in A. That is what I have to do right?

So, use the local finiteness and choose an open set U in X such that x is in U and U intersects only finitely many members of this family. We can just label these members as $X_1, X_2, \ldots, X_n$ alright.

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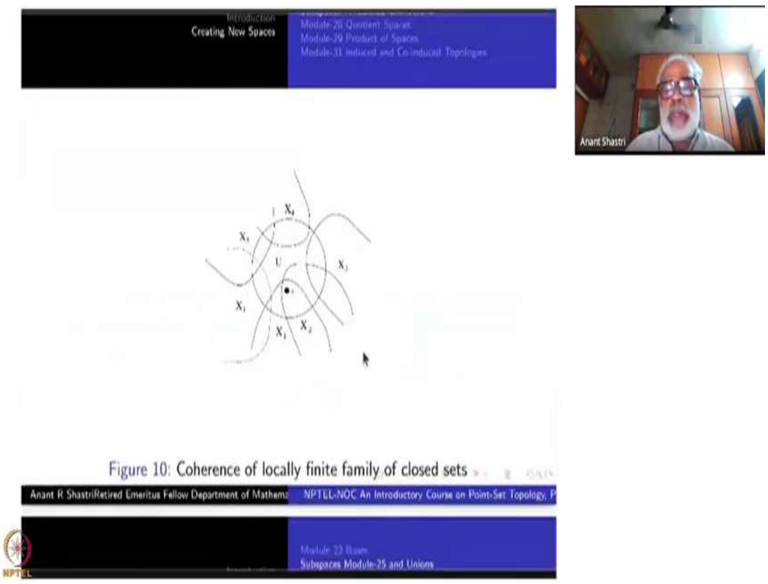

So, this is the picture. This x is in A, I have found an open set U around x, this U intersects only finitely many of X_i's all other X_i's are far away. Out of this list, look at those X_i's for which x does not belong. In this picture, x belongs to X_1 also X_2, but it does not belong to X_3, X_4, X_5 etc ok. So, that is important for me.

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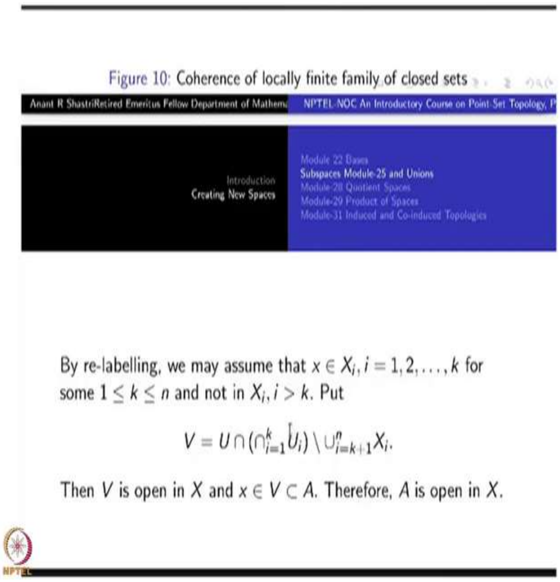

So, what I do? Start with an open set U around x, such that this intersects the only finitely many members, among them, by relabeling $X_1, X_2, \ldots, X_n$ if necessary, you may assume that x is inside $X_i, i = 1, 2, 3, \ldots, k$ and not in other X_i's for $i > k$.

It may happen that it is inside all of the X_i,that does not matter. It must be in one of them after all ok? Because it is inside U anyway and X is union of all X_i's of which we have taken all those which intersect U, so it must be inside at least one of them ok. So, some of them will be there this part is non empty this part may be empty, but does not matter.

So, now we put V equal to our U whatever we have chosen intersected with finitely many of these U_i's. What are U_i's? U_i's are open subsets of X such that $X_i \cap U_i$ is the given open set that is what A intersection X_i.

Student: Excuse me sir. x also must belong to all this U_i for $i = 1, \ldots k$. That follows?

Professor: That follows you see. What is $A \cap X_i$? x is an element of A first of all. If this is in X_i also, then it is in $A \cap X_i$ and hence in U_i.

Student: But x is not in all $U_i \cap X_i$ only but.

Professor: No, no. x must be inside X_i only for those i.

Student: Ok.

Professor: It will be in U_i.

Student: Alright.

Professor: This is equality yeah.

Student: Yes.

Professor: So, whether you demand that x is in X_i or U_i, whichever way, it is same thing.

Student: Ok, correct thank you.

So, first of all it must be there in $X_1, X_2, \ldots, X_k$, k is at least greater than equal to 1 that is important. otherwise this will be empty you will not get any neighborhood ok? $U \cap U_i, i = 1, \ldots, k$, and throw away all these X_i's, i from $k+1$ to n. Throw away means what we are taking intersection with the complements. Complements are open subset because each X_i is closed in X, ok?

So, the complement is also an open subset that is why you are intersecting with that which is same thing as that V is an open subset now. Since x is not here and x is here therefore x is in V and V is contained inside A now. Because what is U? U is union of U intersection all these $X_1, X_2, \ldots, X_n$'s, it is contained in $X_1, X_2, \ldots, X_n$ itself and it does not intersect this part. So, it will be inside this one. So, this V will be inside A. So, it is true for every x inside A so, V is a subset of A and is open.

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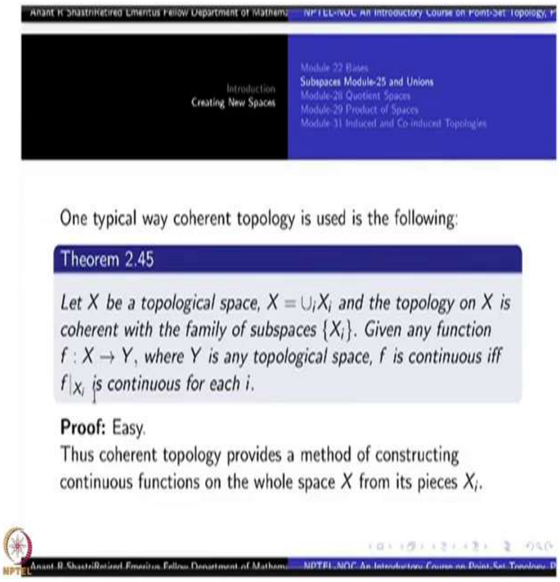

A typical way coherent topology is used is the following namely, patching up continuous functions which you have been using in analysis. In one variable calculus you usually use the other one namely a continuous function is given first in one closed intervals and in the second closed interval, the two closed intervals intersect only at one point and there the function is just defined that is all, but then automatically it will be continuous on the whole thing.

Only the function should be defined properly means in the intersection the value should be the same coming from two different definitions. So, there you are taking finite intersections of closed sets, but usually if you have an arbitrary union of open sets then also this is possible. This will be used only when you go to $\mathbb{R}^n$. I am talking about what you do in analysis, ok? In one variable analysis you do not meet this one quite often.

Let X be a topological space X which is the union of X_i's and the topology on X is coherent with the family of subspaces of X_i. Now, I am not making any assumption whether X is are open or closed and so on. Those three examples were there disjointness openness or closedness with local finiteness. Under these three conditions, we know it is coherent.

But suppose we have any families and the topology is coherent. Then given any function X to Y, a set theoretic function ok? And Y is some a topological space, f is continuous if and only if restricted to each X_i, it is continuous. This is one line proof. How do you prove? By definition of continuity, take an open subset U of Y, $f^{-1}(U)$ must be open inside X.

How do you prove X is open? Use coherent topology, intersect with each X_i and show that it is open. What is the intersection of $f^{-1}(U)$ with X_i? It is nothing but f restricted to X_i take the function take inverse of U under that. So, suppose you call this restriction f_i, then f_i inverse of U is nothing but $f^{-1}(U)$ intersected with X_i. Because f_i is nothing but f itself, but restricted to X_i, ok. So, we have the coherence topology it has property that something is open if and only if intersection with these X_i is open in X_i.

Thus coherent topology provides a method of constructing continuous functions on the whole space from continuous functions given on each piece ok? So, this is what you have to remember.

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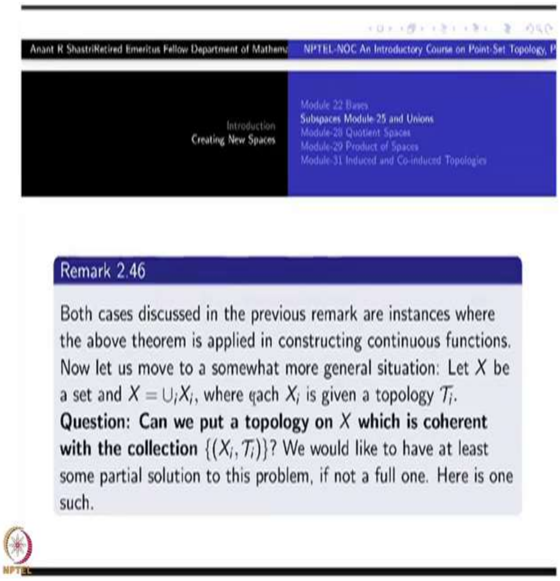

Actually out of three examples we have discussed, the last two cases are more common. Disjoint union was introduced as a motivating example ok? So, both cases discussed in the previous remark are instances where the above theorem is applied in constructing continuous functions. Even you can include the disjoint case, which is obvious any way. So, nobody mentions that one, but that is the starting point of all this discussion.

Now, let us move to somewhat more general situations. Let X be a set which is written as a union of subsets how many I do not know, whether they are closed or open, I do not know. Just X is union of X_i's, but each X_i has a topology $\mathcal{T}_i$. Now, you see what I have done in case of coherent topology. We start with a topology on X. Here I am just given topologies on each X_i, that is all ok? Can we put a topology on X which is coherent with the collection $(X_i, \mathcal{T}_i)$?

I repeat this one what is the meaning of this question. You must have topology on X such that the given topologies $\mathcal{T}_i$ on each X_i, must be the subspace topology. After that coherence condition is there, namely, something is open in $\mathcal{T}$ if and only if intersection with each X_i is open in X_i. So we can do all this. There is no topology given on X. Can we find such $\mathcal{T}$ is the question, ok?

So, this is like a fundamental question, but it does not seem to have a `if and only if' answer ok? But that is lucky actually. It may give you, we may have theorems which will be useful. Suppose, this happen then this is true suppose this happen then that is true. That kind of theorems are more useful than `if and only if' theorems are, quite often.

So, we would like to have at least some partial solutions of this problem if not a complete one ok.

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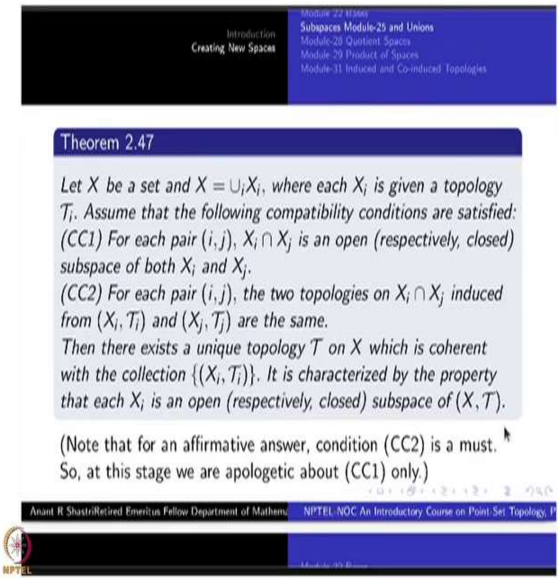

Here is one such. There are two conditions here actually. So, I have clubbed them together. So, do not get afraid because it is going to be too long. So, let us go through it slowly. Indeed, most of the things are repeated here. X is union the of X_i, each X_i has a topology ok. So, that much is the underlying hypothesis.

Now, assume that the following compatibility conditions are satisfied. There are two conditions here each of them has two of them ok? One of them appears in brackets. They are two different sets of condition that is like that.

So, for each pair $(i, j), X_i \cap X_j$ is open. This is one condition. This word open, you replace by closed that gives another set of conditions. So, we can call it (CC1) and (CC1') if you like ok. For each pair $(i, j), X_i \cap X_j$ is open in both X_i and X_j. So, this is the first condition.

The second condition is for each pair (i, j), the two topologies on $X_i \cap X_j$ induced from $(X_i, \mathcal{T}_i)$ and $(X_j, \mathcal{T}_j)$j must be the same. There are two topologies on the intersection, they are the same ok?

See, $X_i \cap X_j$ is a subspace of X_i, it is also a subspace of X_j, ok? If there is a topology on X, they will be the subspace topologies from the same topology on X. Then by the transitivity property of subspaces that we have proved, automatically, the two topologies would have been the same. I want to put a topology on X. Then this condition is a must.

So, that is why I have put this condition. You understand because of the transitivity of the operation of taking subspace topology on the intersection, you can come from X_j or X_i, two different ways they must be the same ok? Right now there is no topology on X. I do not have that one, but if I want to put one, this condition is a must.

So, better start with this condition of course, this condition may not guarantee the final conclusion. So, I have some more conditions also here namely, the first one ok. So, the conclusion is now that there exist a unique topology $\mathcal{T}$ on X which is coherent with the collection $(X_i, \mathcal{T}_i)$. So, answer is positive.

So, it is characterized by the property that each X_i is open inside X (or in the other part when you have taken closed sets, it will be closed subspace of X). So, as I have told you the condition (CC2) is a must, but this one is not a must we do not want everything to be open or closed on something we just want them to be you know subspaces and coherent topologies that is all that I want. X_i's are some spaces, we want them to be subspaces of a common topological space X.

In the very first case, namely disjoint union case, we had such a thing right? So, that other two cases I am taking if all of them are such that intersections are open in both of them or intersection of closed in both of them then it is possible. So, this is the next case that we have ok.

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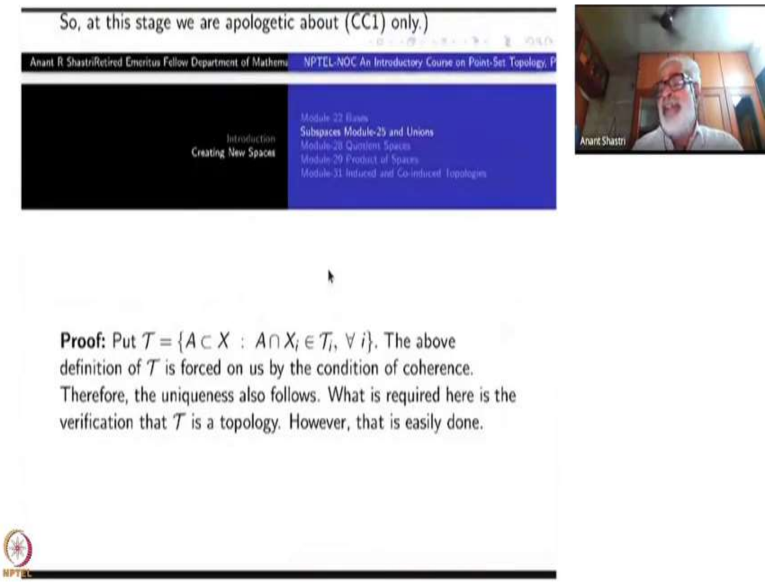

The proof is very easy. Only the statement is long. All that I do is exactly same way I defined for the disjoint union I am going to define $\mathcal{T}$; $\mathcal{T}$ is the family of all A contained inside X such that intersection with each X_i is inside $\mathcal{T}_i$ alright.

When you verified that $\mathcal{T}$ is a topology, in the disjoint case, that disjointedness was never used for the verification. It is just the phenomena that if A is written as a union of intersections then it is the same thing as intersections of the unions and so on that is what that mattered ok. So, this is a topology is no problem. The above definition of tau is forced on us by the condition of coherence because if a subset A is open here, $A \cap X_i$ ok must be inside $\mathcal{T}_i$, for every i that is coherency ok?

So, I have to put that and I have put that much only. That is enough that is the point ok? Therefore, the uniqueness follows. The topology tau must be like if it is coherent ok? What is required here is that verification of $\mathcal{T}$ is a topology which I have already done. Whatever condition we have put is forced on us and we have put just that much. That gives you coherence automatically.

But why collection is a topology? That comes easily. That is the general phenomena ok? So, sometimes easy proofs will stun you. So, better spend a little more, half a minute more on that so that we are not making any obvious mistakes here ok? So, there are two other things which I have to prove. Suppose, each X_i is such that $X_i \cap X_j$ is open inside both X_i and X_j. Then I have to show that X_i itself is open. (Similarly for closed.)

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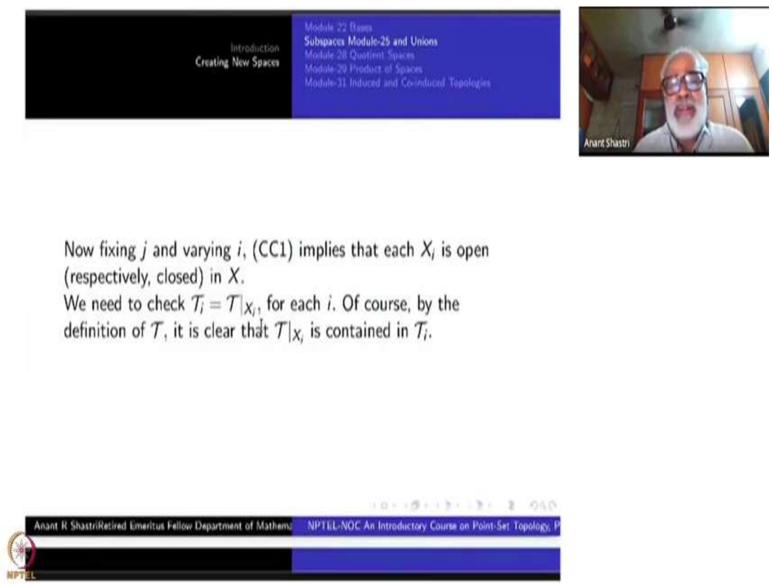

But that is very easy again. For fix i, I want to show that X_i is open in X right? So, what is the coherency condition? Intersection with each X_j must be open in X_j, j equal to i is also allowed. $X_i \cap X_i$ is X_i itself that is open X_i; intersection X_j is open in X_j is the condition given in (CC1) that is all. Similarly, you replace `open' by `closed' that will give you (CC1) the for the closeness ok.

Finally, one more thing I have to check. Why the original topology $\mathcal{T}_i$ is the subspace topology. The definition of subspace topology is what? A subset is open if and only if it can be written as intersection with some open set in X. So, how do we check this one? Look at that the definition of $\mathcal{T}$ itself ok. So, what is $\mathcal{T}$? Take any member A of $\mathcal{T}$, $A \cap X_i$ is in $\mathcal{T}_i$; that is the condition. So, when you restrict it it is already inside $\mathcal{T}_i$. You have to show that there are no more elements in $\mathcal{T}_i$, right?

So, take an open set inside $\mathcal{T}_i$; that means, something belonged to $\mathcal{T}_i$ then I am not sure that it is $B \cap X_i$, where this B is an open subset of X so right?

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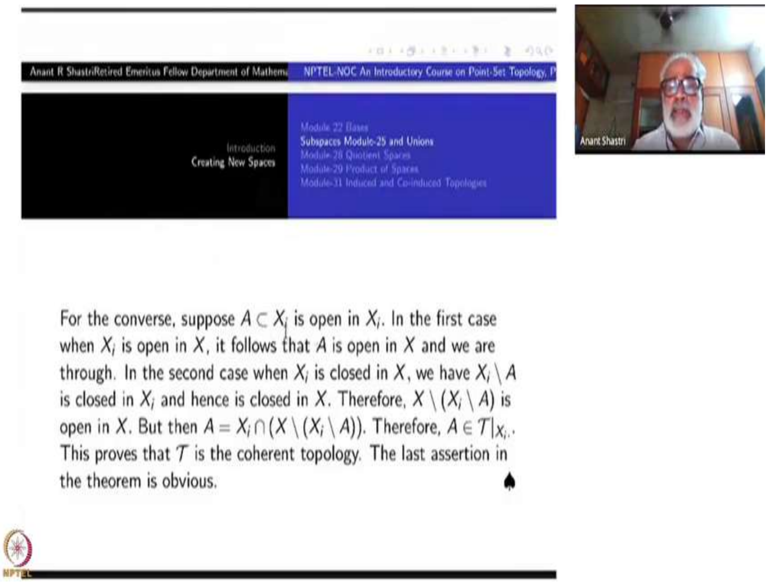

So, that is the converse part. Suppose A contained in X_i is open in X_i. We have two different cases separately. In the first case we have shown that X_i is open in $X.A$ is open in X_i, X_i is open in X so A is open in X_i, ok? It is over. So, it follows that each $\mathcal{T}_i$ is contained inside $\mathcal{T}$ restricted to X_i and conversely ok.

In the second case X_i is closed in X right? Use De Morgan law. $X_i \setminus A$ is closed in X_i because I start with A open inside X_i. So, $X_i \setminus A$ is closed in X_i and X_i is closed in X, so this is closed in X. Therefore, its complement in $X, X \setminus (X_i \setminus A)$ is open in X. But what is this set? You have to see that this set intersection with X_i is precisely equal to A. Begin with A open, this one is a closed subset and this is open subset. Now, intersection of this with X_i will be precisely A, ok?

This is purely set theory. A is a subset of X_i, we started with. It is contained in this $X \setminus (X_i \setminus A)$. You are throwing away A first and then the throwing away $X_i \setminus A$ itself, so A will come back ok. When you intersect that with X_i, only elements of A will remain. So, this intersection is precisely equal to A. Therefore, A belongs to $\mathcal{T}$ restricted to X_i, ok?

So, that proves the coherency of this topology completely. There are three things we have to verify. First of all $\mathcal{T}$ is a topology. Then restricted to each X_i, it gives you the original topology $\mathcal{T}_i$ ok? Then depending upon the closeness and openness we have to show that each X_i is open or each X_i is closed, two different cases.

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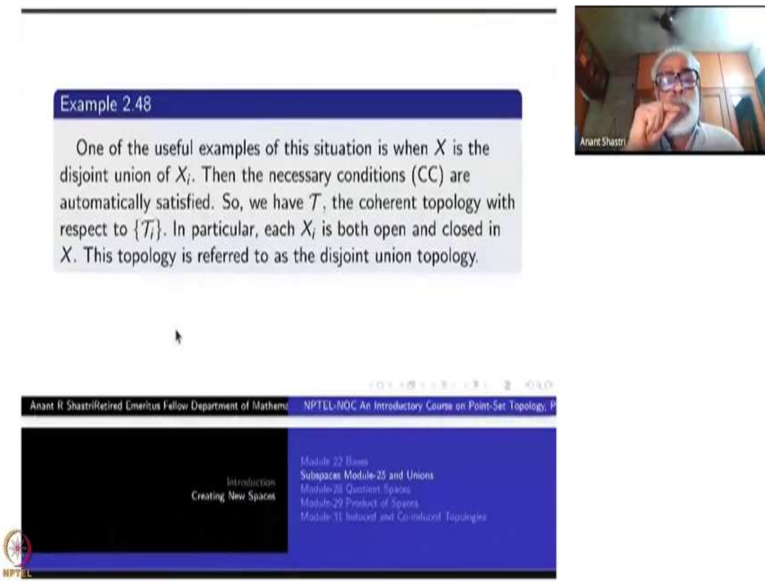

So, here is an example one of the useful examples. In the situation when X is the disjoint union ok, then the necessary condition (CC1) and (CC2) are all automatically satisfied. So, we have tau the coherent topology with respect to the family $\mathcal{T}_i$. In particular each X is both open and closed ok. So, I have already indicated I am just repeating this. Here in this case, wherein each X_i is both open and close will come because intersections are all empty which are both open and closed ok. So, let us stop here today.