Introduction to Point Set Topology, (Part I)
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Lecture - 25
Subspaces

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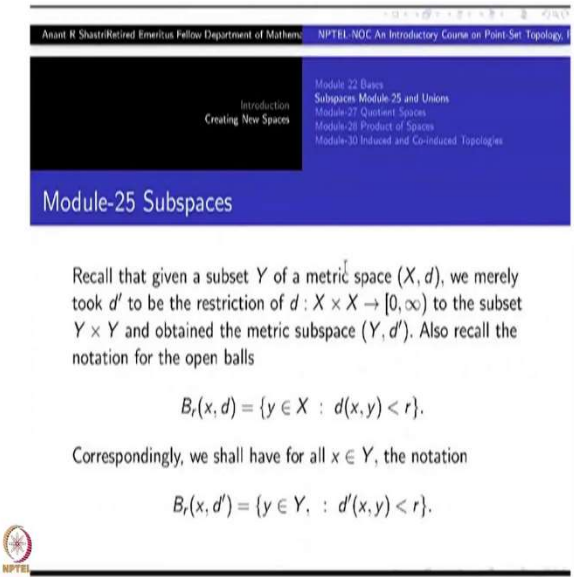

Welcome to module 25 Subspaces. So, starting with a topological space we want to define what is the meaning of subspaces of this topological space. Once again we go back to the metric spaces for motivation. Given a subset Y of a metric space (X, d), what did we do? We merely took d' to be the restriction of d on $Y \times Y$ to $\mathbb{R}$; that was a metric on Y. You know d restricted to $Y \times Y$ becomes a metric on Y.

Recall, even before that, we had a normed linear space, took a vector subspace this time and then restricted the norm to the subspace to get normed linear subspace. Here what do we want to do? There are no functions here to restrict. On the other hand go back to definition of the topology, given by a metric. The basis members were open balls ok? Then the open balls with respect to d' they will be inside the subspace Y, but the definition will be exact exactly same. And therefore, what happens is this $B_r(x, d)$ set of all $y \in Y$ such that $d'(x, y) \leq r$.

Same thing as $d(x, y) \leq r$ right except; x and y are points of Y, that is the difference. But then this is nothing but $B_r(x, d) \cap Y$ ok? (with x and y inside Y here alright.) One of the easy way of start would be to take a basic element here in the larger space X, intersect it with the subset Y, together all of them will form a basis for a topology Y, in both the vector subspace case and metric subspace topology. If that works fine then that should be the definition in the general case as well.

And we will check that that is precisely what works properly and that gives you a definition here ok? So, $B_r(y, d')$ is $B_r(y, d) \cap Y$ for every point inside Y.

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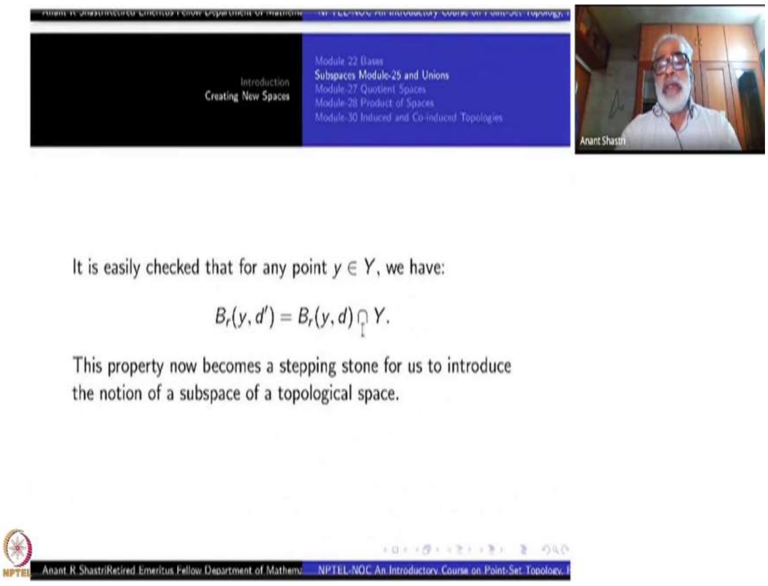

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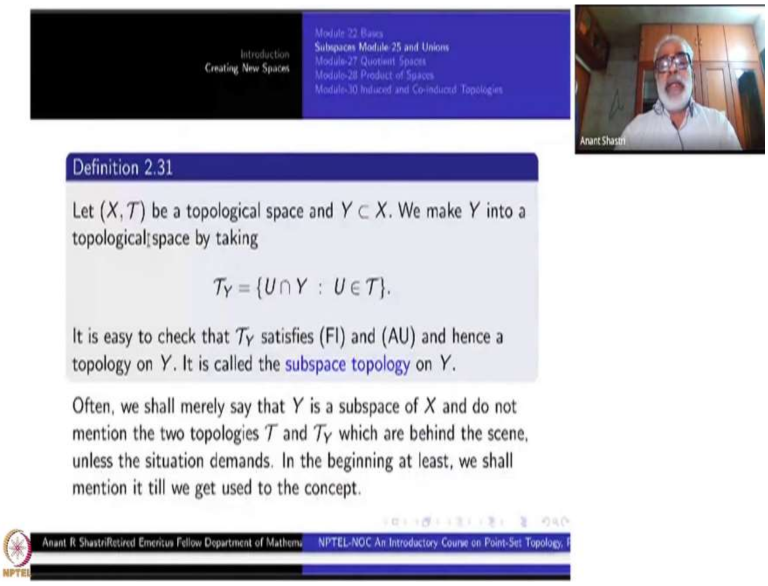

So, take a topological space X and a subset Y of X. We make Y into a topological space by taking, $\mathcal{T}_Y$ equal to all members of $\mathcal{T}$ and intersect with Y. So, I am directly defining this way. And I verify that this $\mathcal{T}_Y$ itself verifies conditions for topology on Y. Clearly members of $\mathcal{T}_Y$ we will cover the whole of Y, they would not cover the whole of X because they are in all inside Y ok. And empty set will belong to that, those things are easy even if you do not check it. (FI) and (AU) if you check that is enough. Finite intersection of say $U_1 \cap Y, U_2 \cap Y, \ldots, U_n \cap Y$ then their intersection is nothing but $(U_1 \cap U_2 \cap \cdots \cap U_n) \cap Y$.

But intersection of $U_1, U_2, \ldots, U_n$ is inside $\mathcal{T}$ therefore, this will be inside $\mathcal{T}_Y$. Similarly for the (AU) here ok. So, checking that this is a topology is just elementary set theory.

So, this is called the subspace topology on Y ok. So, we will have a lot of instances, wherein we have to use subspace topology. Each time what we do is take open subset in Y that can be expressed as $U \cap Y$ where U is open inside X.

So, do you have to keep on writing this one ok. So, we will not decorate $\mathcal{T}$ and $\mathcal{T}_Y$ and so on we just say Y is a subspace of X and do not mention the two topologies $\mathcal{T}$ and $\mathcal{T}_Y$ which are behind the scene. But if there are different topologies then we have to mention this, ok. In the beginning, at least we shall mention it till we get used to this concept.

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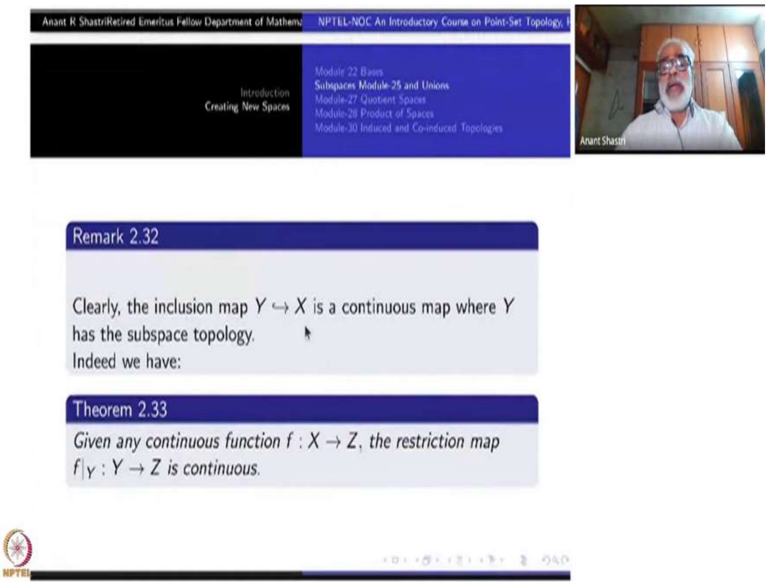

The inclusion map is one of the fundamental maps here from a subset to the given set right? Luckily this is a continuous map now. Indeed more generally, take any continuous function from X to Z, ok? Restricted to Y is it continuous? Original function is continuous should imply the restricted function is continuous. Here X to X identity map is continuous restriction map is what? Inclusion map ok. Conversely, suppose you have proved the inclusion map is continuous, inclusion from Y to X followed by f is nothing but the restriction of f, right? Composite of two continuous functions is continuous. So, that will verify this one. If you verify this one continuous verification of this one follows, if you have verified this one that follows right. The verification of any one of them is as easy as anything else.

Start with an open set here, take the inverse image that is an open set. To come to Y, what is f inverse of an open subset here when you take the restriction map? It is just the full inverse image intersected with the subset Y that is all ok? So, that verifies this one.

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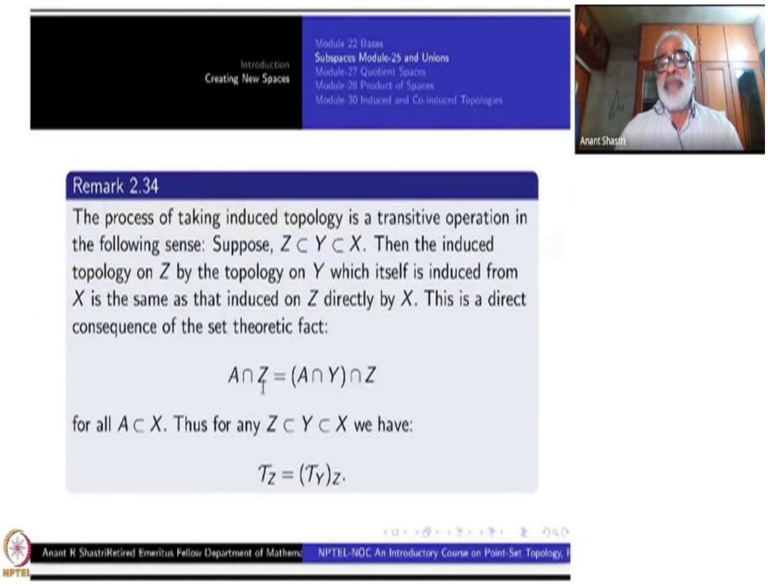

Now, the process of taking subspace topology ok? It is also called induced topology, is a transitive operation in the following sense. Take a subset Y and another subset of Y, that is say $Z \subseteq Y$, then you can have an induced topology on Z ok; as a subspace of Y. Also, you can have an induced topology of Z directly as a subset of X, ok. So, Y is a subspace of X, Z is subspace of Y. So, that is like this is related to this this is related to this that is a process ok. The two are the same. That is the meaning of transitivity, ok. The induced topology on Z by the topology on Y which itself is induced from X is the same as that induced directly from X.

This is a direct consequence of the following: I have to finally take a subset of X here A, intersect it with Z, which is the same thing as first step taking $A \cap Y$ and then taking $(A \cap Y) \cap Z$, ok. This is true for all subsets of X, in particular when you have an open subset you get the subspace topology. Thus for any $Z \subseteq Y \subseteq X$ what we have is, the $\mathcal{T}_Z$ is $\mathcal{T}_Y \cap Z$. So, $\mathcal{T}$ restricted to Z is the same thing as $\mathcal{T}_Y$ restricted to Z.

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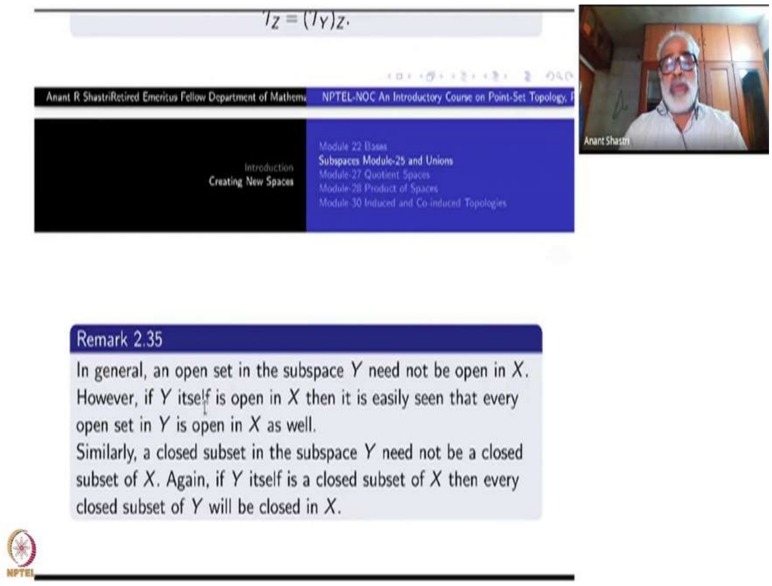

In general, an open set in the subspace Y need not be open in X; however, if Y itself is open in X then its easily seen that every open set in Y is open in X as well. Why because every open set in Y by definition is $U \cap Y$ where U is open in X, but you started with Y open, so $U \cap Y$ will be open in X. So, that is a simple reason ok? Similarly a closed subset in the subspace Y need not be closed subset of X. Again if Y itself is closed in X then by taking complementations every closed subset of Y will be closed in X also, ok?

So, closed-closed will be closed that a transitivity, open-open that is a open, that is also transitivity. So, subspace taking subspace is a very strongly transitive relation alright.

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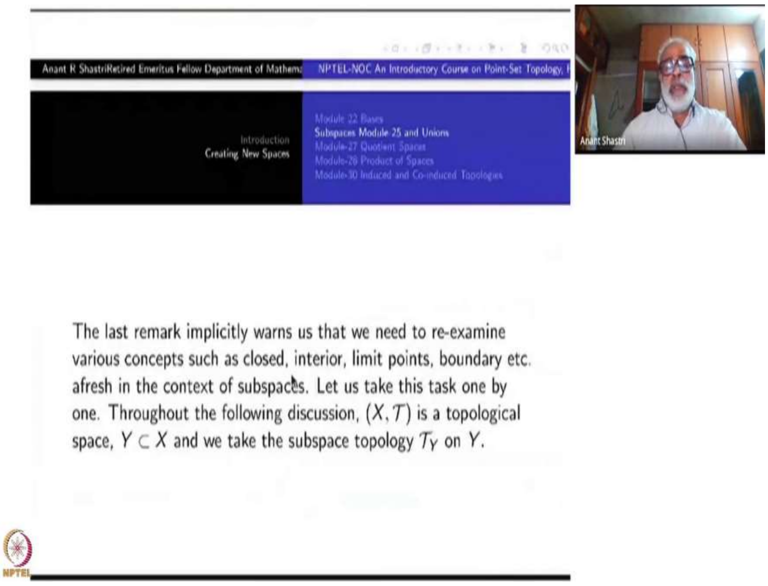

The last remark implicitly warns us that we need to re-examine various concepts such as closed, interior, limit points, boundary points afresh in the context of subspaces. What is the last remark? A subset which is open inside Y, in general, may not be open inside X. If you want to make it open always, then you have to take Y itself open, but then closed subsets of Y may not be closed in X. Now suppose Y is open inside X, but now you take a closed subset of Y that may not be open inside X. So, every time you are taking interior, limits, boundary and so on, you have to be extremely cautious about what is going on, ok. So, you have to freshly check whether the old definition for Y itself whether something is working or not alright.

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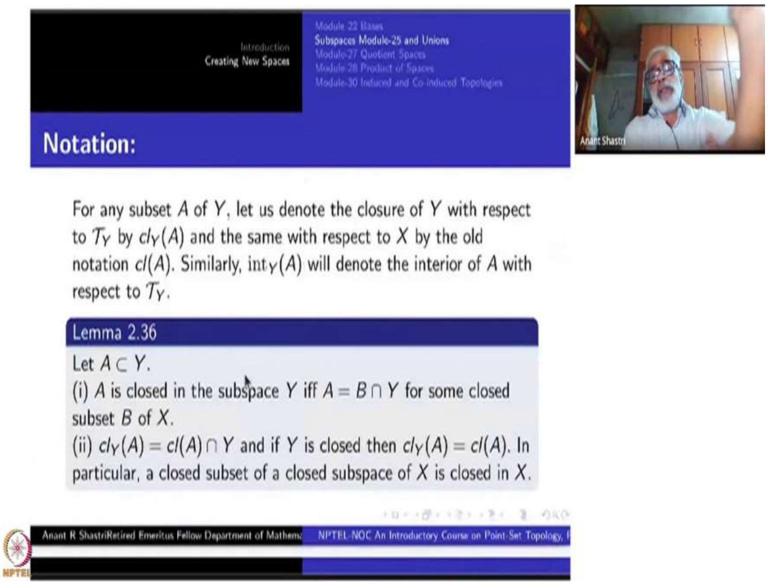

So, let us have some elaborate notation here, and study the closure properties and interior properties and so on for a while, in the light of this definition of subspace topology. For any subset A of Y, let us denote the closure of Y with respect to $\mathcal{T}_Y$ everything happening inside $\mathcal{T}_Y$, with $cl_Y(A)$ ok. All that because we are worried that, we are having an ambient space here, ok. The language of the larger space, of the larger country has to be respected here. It is like X is the country and Y is a state inside that ok. So, closure of A inside Y is by definition with respect to everything happening inside Y, with respect to $\mathcal{T}_Y$. And the same with respect to X, namely by the old notation $cl(A)$, this will be with respect to X, now the whole space. So, there we are using the simpler notation of course, we could have a closure of A with respect to X also here.

If there is one more subset Z in between, and so on.. then we may have to write that Z also here and so on right? For the original ambient space X, for the mother space X, we will keep the old simplest notation. Similarly interior of A in Y will be denoted by $int_Y(A)$; interior of A with respect to the topology $\mathcal{T}_Y$. If you do not write this Y, then it will be the interior of this one as a subset of X in the topology of X. Is the notation clear?

This notation will be used only when there is a scope for confusion. When there are two or more different subspaces involved interior of what and where are we taking it and so on ok.

This is the lemma: Start with a subset A of Y ok? All of them are inside X, that is the assumption already. A is closed in the subspace Y if and only if $A = B \cap Y$ for some closed subset B of X. This was not the definition for us ok? Something is a closed subspace of Y if and only if the compliment of A in Y is open in Y. From that I want to get this thing namely, directly coming from X. What I have to do? Take a closed subset of X intersect it with Y, just like you would have defined the open sets you could have defined closed sets also. So, that is the gist of this property one.

The second property says, the $cl_Y(A)$ is just the $cl(A) \cap Y$, if Y is closed ok? Sorry, this is always true. If Y is closed then the $cl_Y(A)$ is actually $cl(A)$ in X; there is no need to take the intersection. In particular a closed subset of a closed subspace is closed. Why? I am talking, in particular? Suppose Y is closed then, when you take A is closed inside of Y, this closure of A in Y will be A itself.

That means A is equal to $cl(A)$; that means, that A is closed in X. So, this part follows. So, we have to show this part namely, $cl_Y(A) = cl(A) \cap Y$, if Y is closed then this is equal to that. Let us prove these things.

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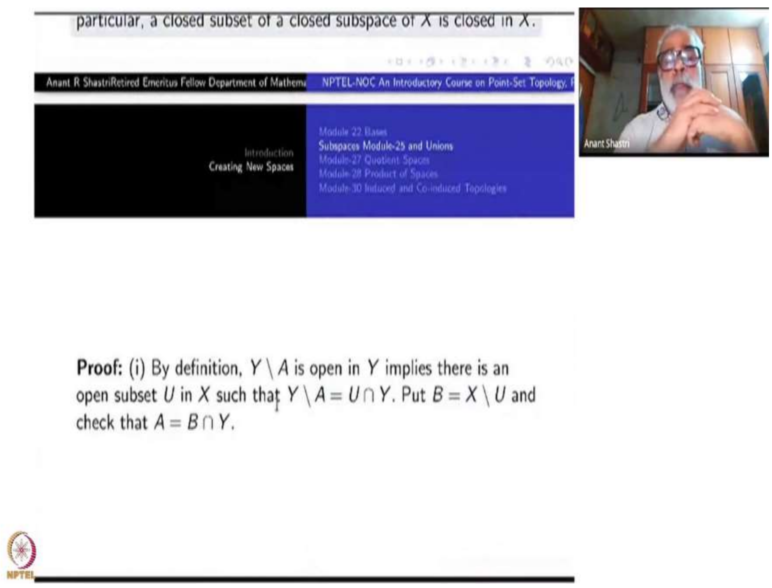

By definition, A is closed in Y if and only if $Y \setminus A$ is open in Y, right. That implies again by definition that, there is an open set U in X such that $Y \setminus A$ is $U \cap Y$.

Now, put B equal to $X \setminus U . X \setminus U$ is a closed subset of X. And all that you have to check is now A is $B \cap Y$ ok? So, set theoretic thing you have to verify alright. B is $X \setminus U$, ok.

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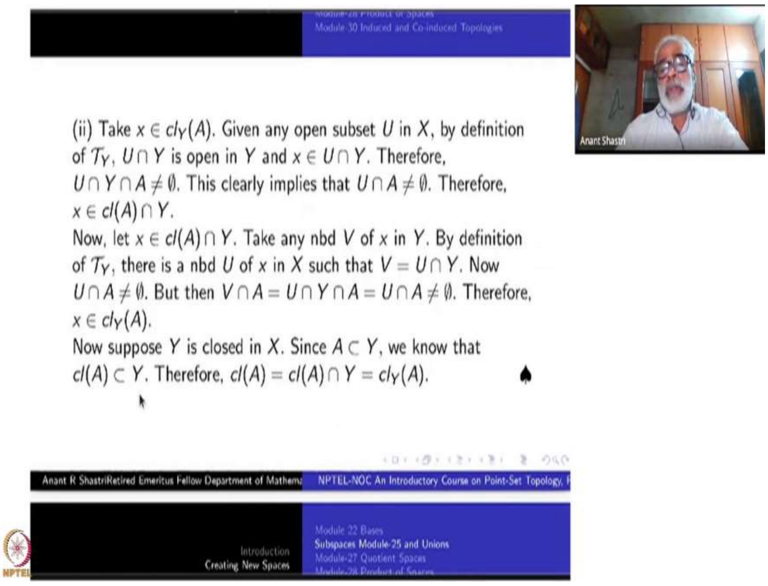

Second part, take x as a closure point of A in the topology Y. You know given an open subset U of X by definition of $\mathcal{T}_Y$, such that $U \cap Y$ is open in Y. Now what is the closure point? Closure point means, now x will be inside $U \cap Y$, ok first of all. So, $U \cap Y$ is a neighbourhood of x inside Y right? Why x is in $U \cap Y$? Because x is a point closure of A in the topology $\mathcal{T}_Y$. So, x is already inside Y ok. So, take an open set U such that x is inside U, it will be automatically inside $U \cap Y$.

Therefore, $U \cap Y \cap A$ is non empty ok. Why because x is in the closure of A, every open subset containing x will intersect A, that is the meaning of this one. But then this intersection is non empty means $U \cap A$ is non empty alright. Therefore, x is the closure of A. I put Y because I have started with x as a point of Y. So, $cl_Y(A)$ is contained inside $cl(A) \cap Y$.

Now, I take the other way inclusion: Start with a point here $cl(A) \cap Y$. Take any neighbourhood V of x in Y, ok. Something is a neighbourhood means what? By the definition of $\mathcal{T}_Y$, we have a neighbourhood U of $x \in X$ such that, V is $U \cap Y$, alright. Now x is a closure point in the original topology X therefore, $U \cap A$ is non empty.

But what is $V \cap A$? It is $U \cap Y \cap A$ right. But that is same thing as $U \cap A$, right. Because A is a subspace of Y. So, this $Y \cap A$ is just superfluous which is same thing as A. Therefore, x is in the closure of A. So, we have proved the equality all that you have to do is take the closure of A inside the larger space and then take only points of Y that is intersect with Y. That will give you the relative closure closure of A with respect to the smaller space Y.

Now, suppose Y is closed in X. A is already inside Y by the way, you know that right. So, we know that closure of A will be contained in Y, right. Because closure of A is contained in closure of Y, but closure of Y is Y. Therefore, closure of A is equal to closure of A intersection Y, because its already inside Y. But by definitely we now just shown by part one is equal to the closure of A with respect to Y, ok.

This is this easier part. Here we have to completely go through the definitions on both ways ok. So, the whole idea is if you have done it now, then you will remember it. You do not have to do it every time whenever you need. Similarly for interiors ok?

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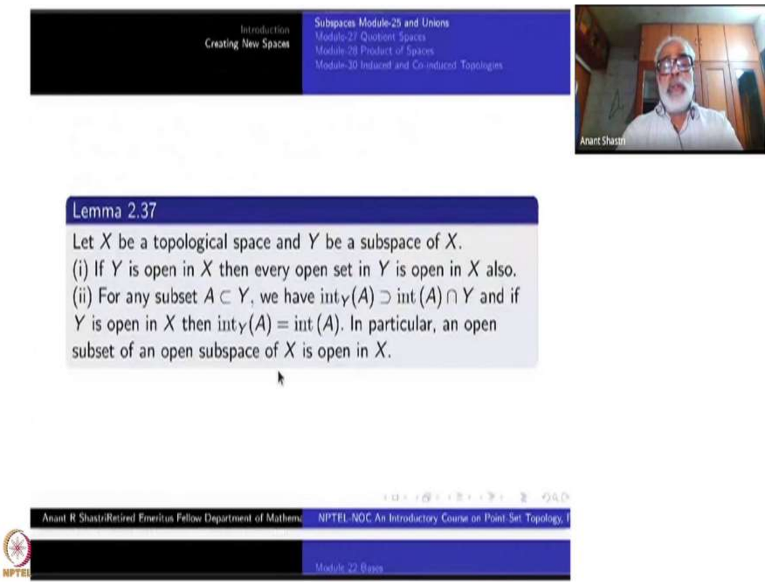

Let X be a topological space, Y be a subspace of X. If Y is open in X, then every open set in Y is open in X also.

This we have already commented. So, I have put it here as a reference. Second part is more important. For any subset A of Y, the interior of A in Y contains the intersection of interior of A with Y. So, here there is no equality. Say, if Y is open then the two are equal ok. There is no need to take intersection. In particular an open subset of an open subspace is open ok. So, we have to verifying this, namely, interior of A with respect to Y contains interior of A intersection Y, ok?

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Actually many times this part the right hand part may be empty, but this part may not be empty lots of examples are there like that ok. So, let us come to this one by the very definition, A contained inside Y is open in Y if there is an open subset U in X such that, A is equal to $U \cap Y$. If Y is open the intersection is open this is the first part, I have repeated it.

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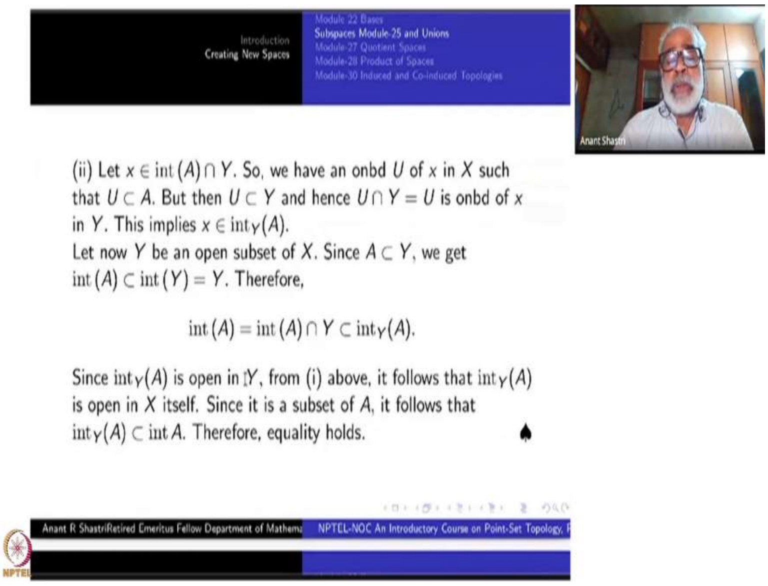

The second part: start with a point x in the interior of A, but this point is already inside Y. So, intersection with Y, I have taken. So, we have an open neighbourhood U of $x \in X$ such that U is contained in A. Indeed, we could have directly taken U equal to the whole of interior of A. Anyway, since A is contained in Y, U will be also contained in Y ok? A is subset of Y that is the assumption, ok. If an open subset of X is already contained in Y that will be automatically open in Y. Because when I intersect it with Y, I get the same set $U; U$ intersection Y is U, hence U is an open neighbourhood of x in Y. If you have found an open neighbourhood of of a point that point must be in the interior right.

So, it is in the interior of A with respect to Y. So, the first part we have proved namely the interior intersection with Y is contained in the interior A with respect to Y ok. There is no equality here.

Now we go to the second part namely, suppose Y is an open subset of $X. A$ is contained inside of Y that is given ok. So, interior of A is contained inside interior of Y, because the interior operator has his property. But interior of Y is Y itself, right?

So, all these things happening inside X. Therefore, interior of A is contained in interior of A intersection Y, but this we have proved $int(A)$ is contained in interior of A with respect to Y ok. Interior of A with respect to Y is open in Y. So, one way to prove this is this part is contained inside here interior of A is open in Y, right. Every interior A with respect to any topology is open in that topology, ok?

Therefore it follows that Y-interior of A is open in X itself, because Y is open in X. So, it is the first part. Since it is a subset of A it follows that interior A with respect to Y is contained in the interior of A, right. After all, interior is a subset of the set always. This is subset of A. Now we have what? this is an open subset therefore, it is contained inside the interior of A. Because interior of A is the maximal open subset, largest open subset contained in A. Remember that ok.

So, therefore, there is an equality here, this way equality and this is the other way you know inclusion alright. So, I have already told you how equality may fail in general. So, we will have lots of such examples.

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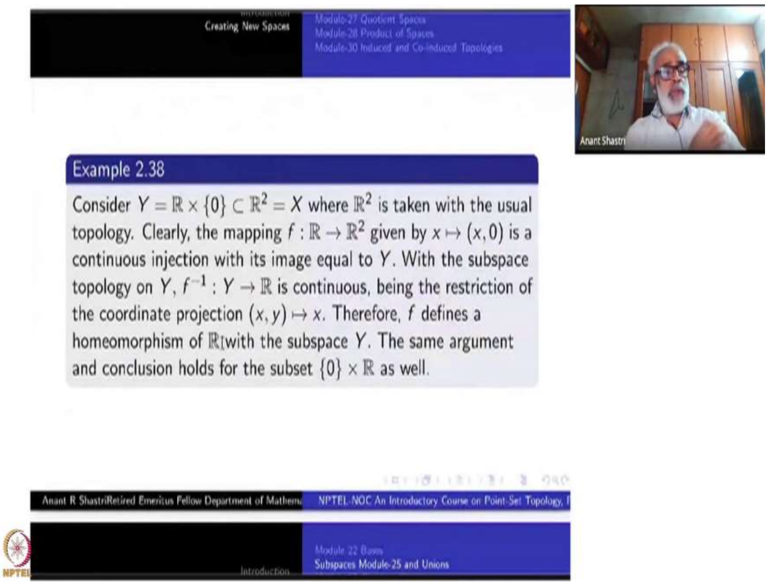

Let us look at $\mathbb{R} \times 0$ contained inside $\mathbb{R}^2$. So, let us take $\mathbb{R}^2$ as X, and $\mathbb{R} \times 0$ as Y, ok. Now take the usual topology in $\mathbb{R}^2$.

Now, the mapping f from $\mathbb{R}$ to $\mathbb{R}^2$ given by $x \to (x, 0)$. It is a continuous injection right? it is a continuous injection with its image, what is the image of this map? It is Y namely $\mathbb{R} \times 0$ ok? So, this how we think of the real line $\mathbb{R}$ as a subspace of $\mathbb{R}^2$, this is one standard way right. So, the image is Y. With the subspace topology on Y a subspace topology, what is subspace topology? Everything you have to take open subsets in $\mathbb{R}^2$ and intersect with $\mathbb{R} \times 0$.

Then f inverse from Y to $\mathbb{R}$ is also continuous why? Because, this map is nothing, but the restriction of the coordinate function (x, y) going to x. So, that is the inverse image of this x going to $(x, 0)$. It comes back to x only on $(x, 0)$. So, if we have taken the whole space the projection map is continuous.

Being a restriction of a continuous function, this function f inverse is continuous. Therefore, f defines a homeomorphism of $\mathbb{R}$ with Y, ok? The same argument and conclusion hold for $0 \times \mathbb{R}$ also. There is nothing very special about the first coordinate being 0 or the second coordinate being 0; conclusions are identical, arguments are identical. All that we have to is to interchange the coordinates x and y, and the appeal to the fact that (x, y) going to (y, x) is a homeomorphism.

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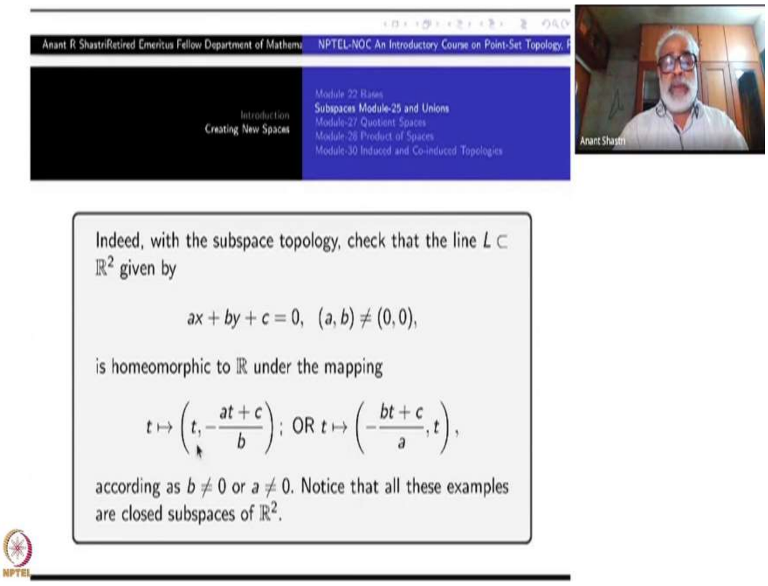

More generally, you can take any line given by an equation $ax + by + c = 0$. To make it a nice line as such, and not the whole space, all that I have to assume is that this vector $(a, b) \neq (0, 0)$. The line so defined is homeomorphic to $\mathbb{R}$ under a mapping which depends upon which coordinate a or b is not 0. $(a, b) \neq (0, 0)$ means a is not equal to 0 or b not equal to 0; may be both of them are not equal to 0 that is also allowed.

But you do not know which one is not 0. So, accordingly you can choose two different maps both of them may work if a and b are not 0. What is it? It is the graph of this function, $y = (at + c)/b$. t going to t comma this is something y equal to $f(t)$, you have to solve for y in this equation that is all.

Or you solve for x in terms of y here. So, I am instead of y, I am writing t here. Therefore, the t goes to t here second coordinate is y; this is your x coordinate ok. So, this is elementary you know, tenth standard under stuff. You are solving for y in terms of x or x in terms of y ofcourse, that needs either a is not equal to 0 or b not equal to 0 that condition has to be there, according as b not equal to 0 or a not equal to 0, you have to choose the function.

Notice that all these examples are closed subspaces of $\mathbb{R}^2$ ok. So, therefore, if you take a closed subset of $\mathbb{R}$, then their image inside this line as a subspace of $\mathbb{R}^2$, it will be closed. But open? No chance ok.

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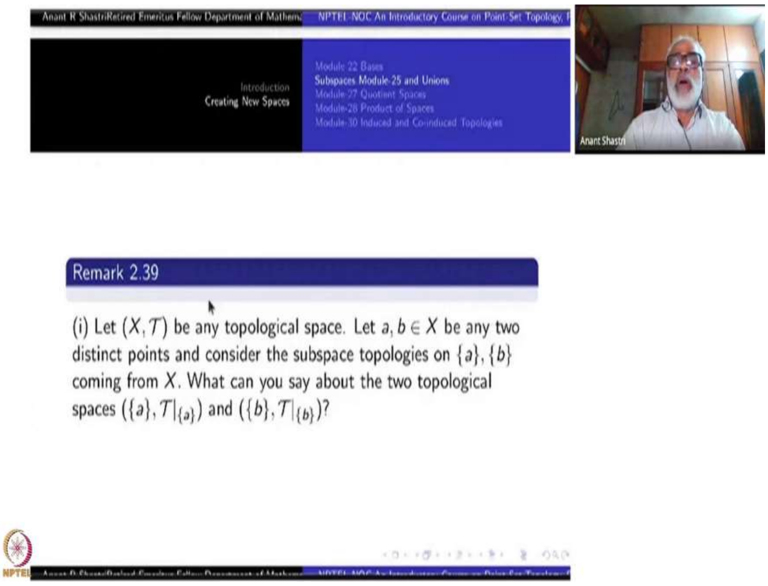

I am asking you somewhat vague question here, so, that I want you to think about this. So, whole idea here is I have put you know subspaces with the occur with the familiar spaces. See this is $\mathbb{R} \times 0$, this is $0 \times \mathbb{R}$, this is any line they are all homeomorphic to $\mathbb{R}$, ok they are all homeomorphic to $\mathbb{R}$. So, this is the first example.

So, now come to second example. Take $(X, \mathcal{T})$ to be any topological space. Take any two points a and b distinct points. Look at the subspace topology on singleton a and singleton b. X is a topological space, ok? So, singleton a is a subset. So, there is a topology on it, singleton b there is a topology on it as well. Question is what can you say about these two topological spaces? Ok, separately and together also what can you say ok?

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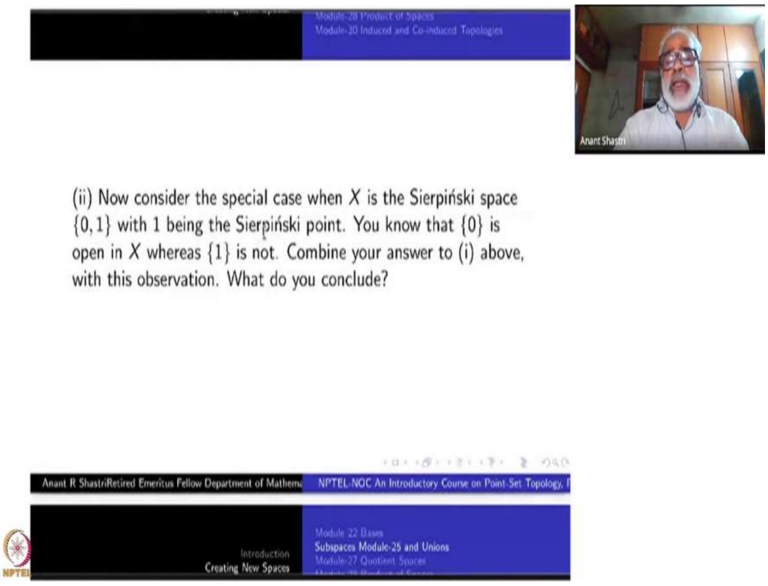

The second, now consider special case when X is the Sierpinski space with $\{0, 1\}$ ok? Any $\{a, b\}$ will also do; any two distinct points, but we have denoted them 0 and 1, with one being the Sierpinski point. This just means that the only open set containing 1 is the whole space 0 1. Singleton 0 is open therefore, singleton 1 is closed, but singleton 1 is not open that is the meaning of Sierpinski point 1. You know that 0 is open in X whereas, 1 is not open ok.

See here I did not get they are all closed subspaces, though they are all homeomorphic. So, I want to give an easy example of this.

One is a closed subspace another is not closed, or one is open and other is not open. What can you say about the two topologies was the first question. Here, whatever you have said there, combine it with this special case here and what do you conclude? So, come up with some answer and then check it with your tutors ok? That is the game you have to do. So, let us stop here there will be many examples many more new things coming up and so on next time.

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Next time we will have to do union of spaces from smaller things to larger thing ok.

Thank you.