Introduction to Point Set Topology, (Part I)
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Lecture - 24
Box Topology

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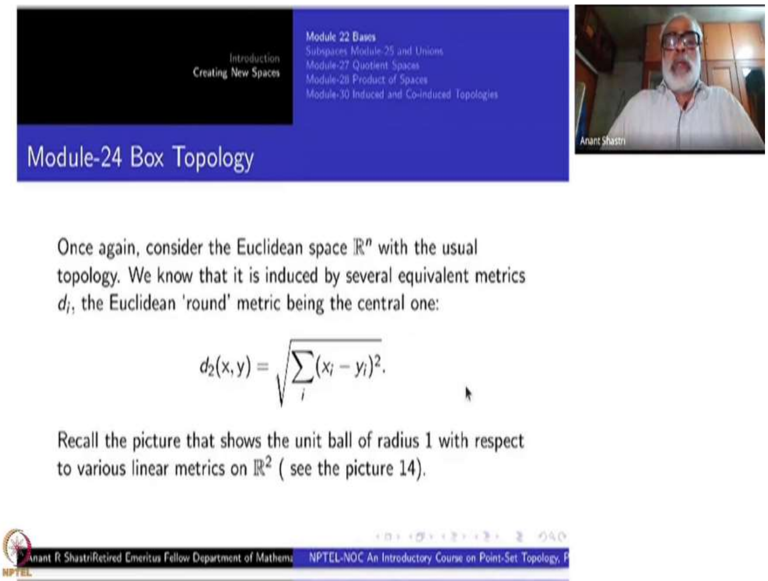

Welcome to module 24 of Point Set Topology Part 1. So, after discussing some kind of a general method of obtaining new topologies with bases and sub bases and several examples of that type we will continue one more very important example again. Consider the Euclidean space $\mathbb{R}^n$ with the usual topology. We know that it is induced by several equivalent norms and thereby, by associated linear metrics which we denote by d_1, d_2 etc, d_∞ ok?

Out of which the so called round metric, the Euclidean round metric given by $d_2(x, y)$ equal to $\left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/2}$; this being the central it is called the usual metric. All of them gives the same topology, is a God's gift for us. Let us have a look at this picture. For either just to recall, or just to feel good about it.

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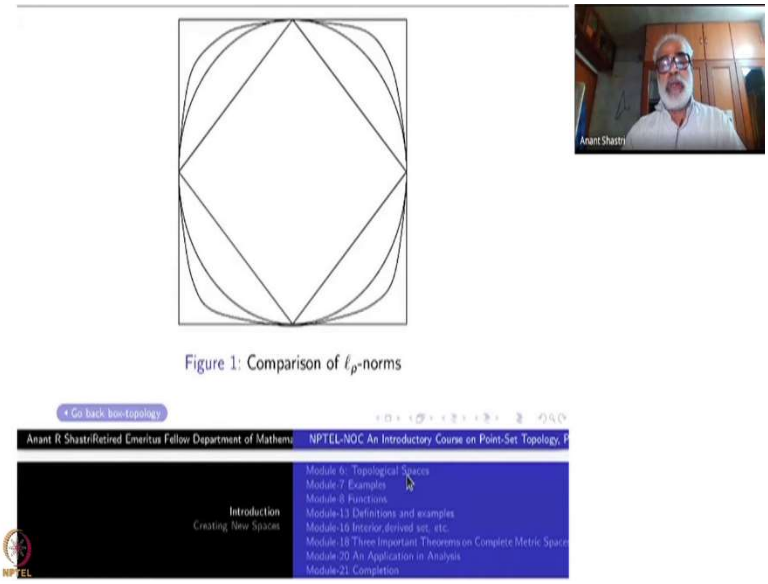

So, this was the the topmost one here, was the d_∞ ball of radius 1. So, these were some d_p's this is the circle was the d_2 ball and this was d_1 ball right. So, like this there was such a relation and we have discuss this one quite in depth. Now, let us concentrate only on the square and the circle; the square corresponding to d infinity and the circle corresponding to the d_2 norm ok.

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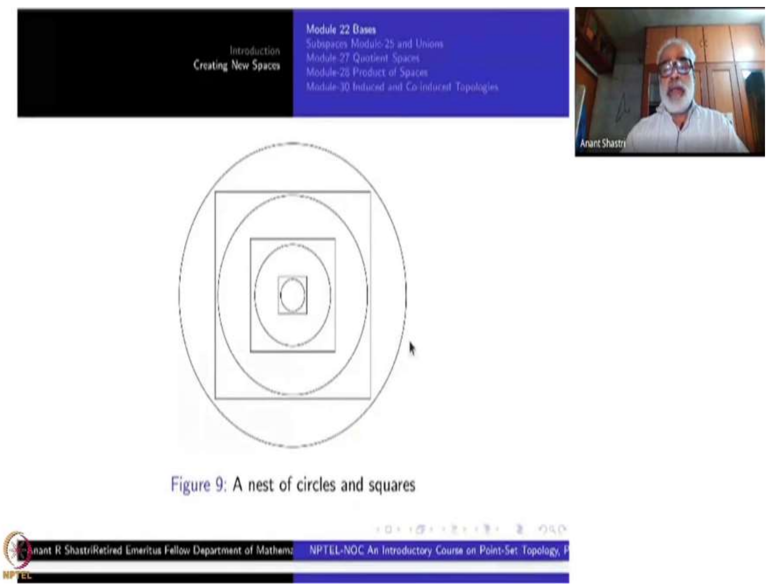

Once again I recall this one namely this picture tells you that inside the circle you have squares, inside the square you have again circles circle square circle square and so on. They are all centered at one single point, but that is not even necessary, you can take any point here. you can put a circle inside the square and then a square inside the circle and so on.

At every point it is true. So what does this mean? If you want to control some phenomena by the circles, control it by the squares automatically it will get controlled by circle. Control it by the circles, it will be controled by squares and so on and vice versa.

So, this is what we have used in several times in elementary mathematics like in complex analysis we said a complex valued sequence is convergent if and only if both the sequences of real parts and imaginary parts converges; that is if and only if. So, this is what is going to happen. The collection of all these open discs centered at various points that forms a base for the usual topology. On the other hand the same thing is true for the collection of all the open squares centered at various points. So, both of them give you the same topology both of them are bases. So, in terms of our modern terminology these things are bases for the same topology ok.

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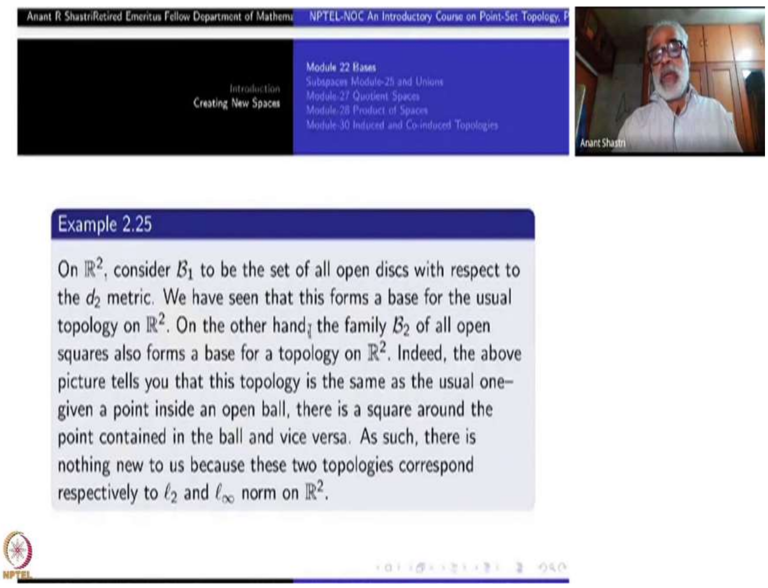

So, I repeat: $\mathcal{B}_1$ be the set of all open discs with respect to the d_2 metric. We have seen that this forms a base for usual topology.

On the other hand the set $\mathcal{B}_2$ of all open squares also forms a base for the same topology. These are we can call them as unit discs in d_∞ metric. Indeed the above picture tells you that the topology is the same. Same means what? Equal to the usual topology, ok.

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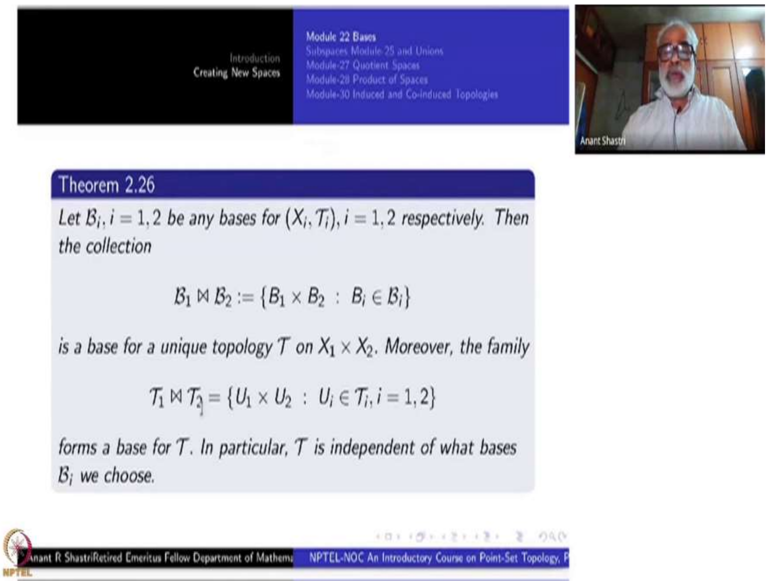

So here is a general theorem. First of all I want to say suppose $\mathcal{B}_i$, i equal to 1 and 2 be any bases for $(X_i, \mathcal{T}_i)$ ok? Then the collection $\mathcal{B}_1 \bowtie \mathcal{B}_2$ read it as $\mathcal{B}_1$ join $\mathcal{B}_2$ collection of all members of sub spaces of $X_1 \times X_2$; I am taking here $\mathcal{B}_1$ first factor is $\mathcal{B}_1$ comes from $\mathcal{B}_1$ and second factor comes from $\mathcal{B}_2$. So, take their Cartesian product, so this will be a subset of subset of $X_1 \times X_2$, ok? Some author just write $\mathcal{B}_1 \times \mathcal{B}_2$ here and that can cause huge confusion ok.

So, I have made this somewhat funny notation you read it as $\mathcal{B}_1 \bowtie \mathcal{B}_2$, ok. This is a base for a unique topology $\mathcal{T}$ on $X_1 \times X_2$. This is a general statement now. Motivated by our observation in Euclidean spaces. Take X_i's to be $(\mathbb{R}, \mathcal{T}_i)$ will be given by open intervals open intervals. Collection of all open intervals cross open intervals is a base for the usual topoloy on $\mathbb{R} \times \mathbb{R}$. Precisely the lesson from here.

So, that is what I am generalizing it here now ok. So, claim is that this will be a base for a topology. So, you have to verify those two conditions (B1) and (B2) ok. Moreover look at the family $\mathcal{T}_1 \bowtie \mathcal{T}_2$, which is same thing as again by definition I am repeating it here $U_1 \times U_2$, where U_i's are in the $\mathcal{T}_i$. This will be also a base for $\mathcal{T}$, the same $\mathcal{T}$. In particular if you choose different bases they do not give you different $\mathcal{T}$'s because this $\mathcal{T}_1 \bowtie \mathcal{T}_2$ is the same ok for all of them this is also a base.

So, this topology tau is independent of what bases you chose because it is equal to the topology generated by this $\mathcal{T}_1 \bowtie \mathcal{T}_2$, ok. So, I have to do two things here. First of all this $\mathcal{B}_1 \bowtie \mathcal{B}_2$, I have to show is a base and then I have to show that $\mathcal{T}_1 \bowtie \mathcal{T}_2$ is also is a base for the same topology. That is all I have to show ok? Two things I have to show.

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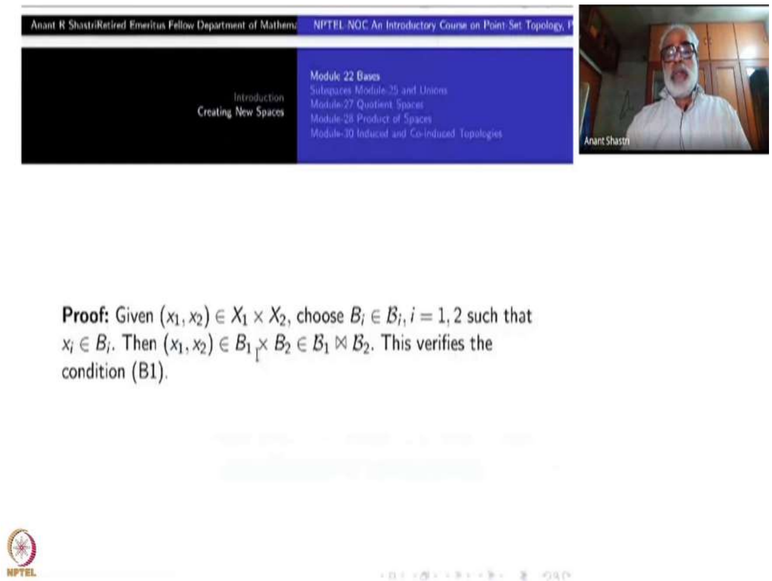

So, given $(x_1, x_2) \in X_1 \times X_2, \mathcal{B}_1$ and $\mathcal{B}_2$ have the property I can choose you know B_i inside $\mathcal{B}_i$ such that x_i in B_i. Then (x_1, x_2) will be inside $\mathcal{B}_1 \times \mathcal{B}_2$ that is an element of $\mathcal{B}_1 \bowtie \mathcal{B}_2$. So, condition (B1) is verified.

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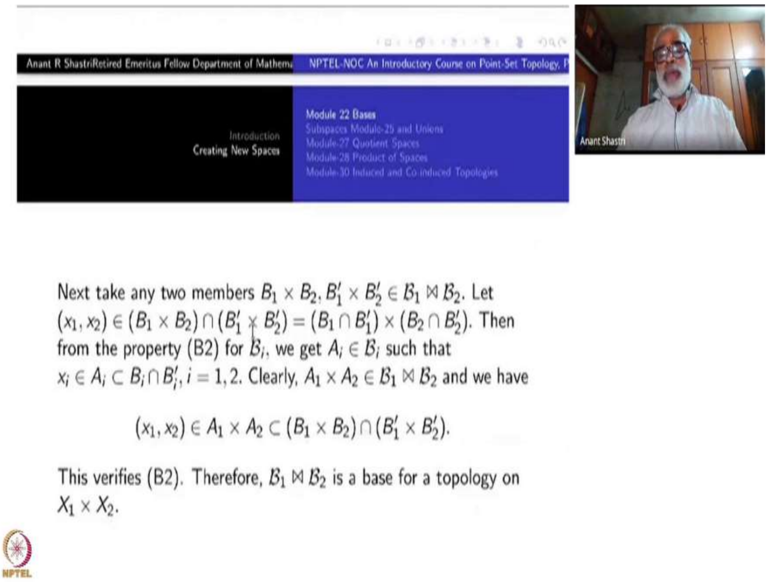

Similarly take two members $B_1 \times B_2, B_1' \times B_2'$ inside $\mathcal{B}_1 \bowtie \mathcal{B}_2$. Suppose this point (x_1, x_2) is in the intersection, what is the intersection of this product? It is nothing but … the first coordinate must be always in B_1 as well as in B_1', the second coordinate must be in B_2 as well as in B_2' which is same thing as $(B_1 \cap B_2') \times (B_2 \cap B_2')$. This is pure set theory ok?

Then from property (B2) for each B_i, we will get some A_i inside B_i such that x_i is inside A_i and A_i is contained in $B_i \cap B_i'$ for $i = 1, 2$. Then $A_1 \times A_2$ will be inside $\mathcal{B}_1 \bowtie \mathcal{B}_2$ because A_1 is in B_1, A_2 is in B_2 right?

So, we have point (x_1, x_2) belonging to $A_1 \times A_2$ contained in the intersection of these two products and that is a member of $\mathcal{B}_1 \bowtie \mathcal{B}_2$. So, this verifies (B2) ok. Therefore, $\mathcal{B}_1 \bowtie \mathcal{B}_2$ is a base for a topology on $X_1 \times X_2$, which we shall denote by $\mathcal{T}$.

Now, I have to show that this tau is the same as the topology generated by $\mathcal{T}_1 \bowtie \mathcal{T}_2$, ok?

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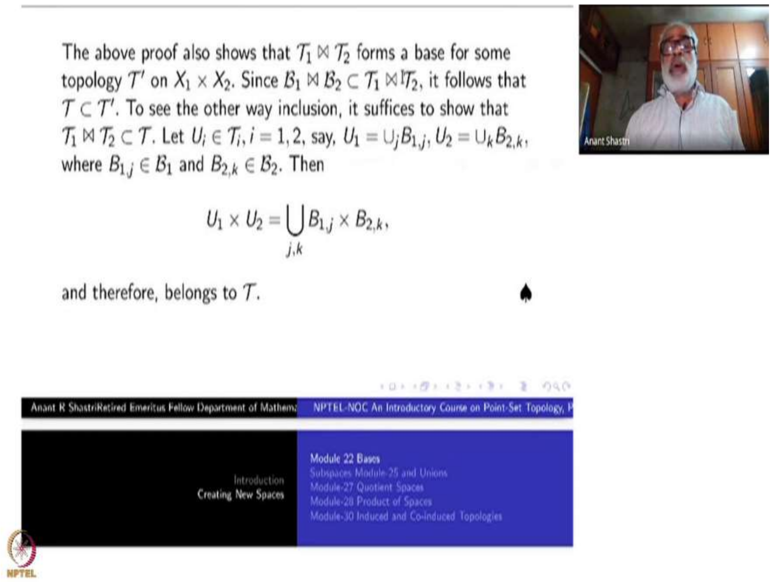

The $\mathcal{T}_1 \bowtie \mathcal{T}_2$ is also a base the proof is exactly the same. Instead of $\mathcal{B}_1 \bowtie \mathcal{B}_2$, we write $\mathcal{T}_1, \mathcal{T}_2$ everywhere. All these things are true for $\mathcal{T}_1 \bowtie \mathcal{T}_2$ also because $\mathcal{T}_1$ and $\mathcal{T}_2$ are also bases for their own topology, their topology is any topology is also a base for itself ok. The same argument which show that it forms a base. So, if you write $\mathcal{T}'$ as the corresponding topology generated by this family ok.

I have to show that this $\mathcal{T}'$ is equal to $\mathcal{T}$, but now the base $\mathcal{B}_1 \bowtie \mathcal{B}_2$ is inside this one therefore, $\mathcal{T}$ will be inside $\mathcal{T}'$ one part is obvious. Now, I have to show that $\mathcal{T}'$ is inside $\mathcal{T}$ ok?

To see the other way inclusion, it suffices to show that this family the base itself is inside $\mathcal{T}$. We have observed that one earlier.

So, I am going to show that $\mathcal{T}_1 \bowtie \mathcal{T}_2$ is contained inside $\mathcal{T}$. It is same thing as taking one open U_1 here and open U_2 here and take the product and show that it is in this topology $\mathcal{T}$. So, let U_1 belong to $\mathcal{T}_1$, U_2 belong to $\mathcal{T}_2$, ok. Means what? U_1 is union of $B_{1,j}$'s where all these $B_{1,j}$'s are in $\mathcal{B}_1$. Similarly, U_2 is union of $B_{2,j}$'s where all these $B_{2,j}$'s are in $\mathcal{B}_2$ because they are the bases for $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively ok. Then $U_1 \times U_2$ it is the product of the unions which is the same thing as union of the products. You have to take product of one element one member here with another member here, member by member, all combinations you have to take, that is summed over both j and k completely freely.

But this is a union and each of them is inside $\mathcal{T}$ right, so union inside $\mathcal{T}$ ok. So, what we have proved so far is that if you fix bases for $\mathcal{T}_1$ and $\mathcal{T}_2$ and then take the corresponding bases you know $\mathcal{B}_1 \bowtie \mathcal{B}_2$, you get a topology and then that topology is independent of what bases you have chosen ok. Moreover we have got a description of how the members of this topology $\mathcal{T}$ will look like in the product space $X_1 \times X_2$.

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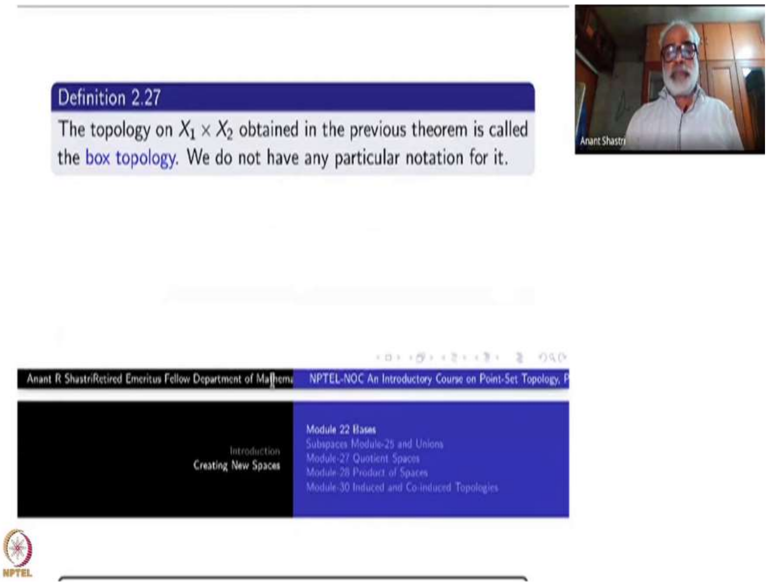

The topology on $X_1 \times X_2$ obtained in the previous theorem this way is called the box topology ok. So, in dimension 3 it actually looks like a box that is why it is called box topology in all higher higher dimensions also ok. Once you have got it for $X_1 \times X_2$, you can imitate it for $X_1 \times X_2 \times X_3$, we can iterate it any finite number of times.

So, for finitely many products, what will be the bases? Take bases for each of them take $U_1 \times U_2 \times U_3 \times U_n$. If you do not have bases, take all open subsets in each of these $\mathcal{T}_i$ and take that one that itself will not be a topology, but it will give you the same box topology as a bases. There are much more open subsets inside the product that is all you have to know ok.

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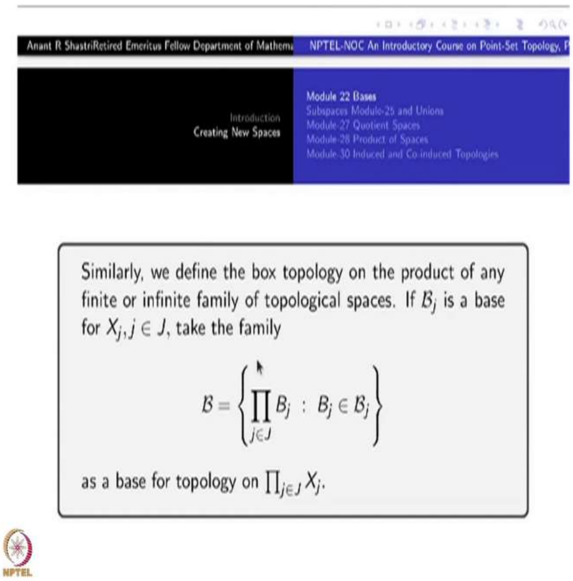

So, you can take product of B_j's where j range on J. This I am doing for now not only finite, but infinite family X_j, j belong to J, each of them is given a topology and then each topology has a bases B_j then you can take these product of B_j's where j range on J and B_j lying in $\mathcal{B}_j$'s. Take all such elements that will be a base for a topology and that topology is box topology on the product. The proofs etc will be the same. There is no set theoretic problem here at all.

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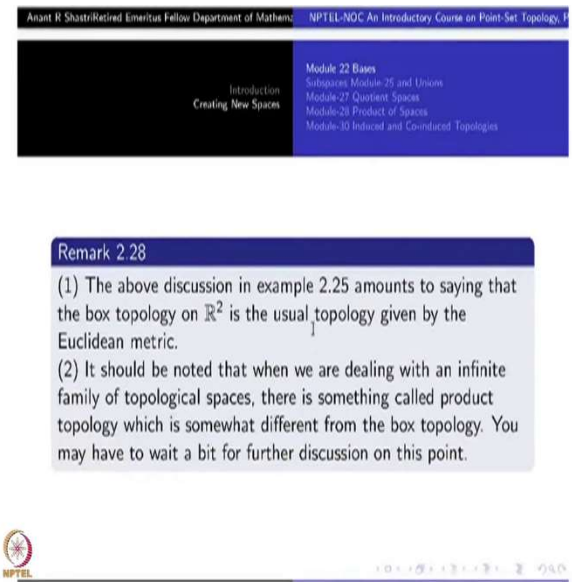

So, I have already told you I have motivated then I am just repeating it here. The above discussion in the example 2.25 amounts to saying that the box topology on $\mathbb{R}^2$ is the same thing as usual topology given by the Euclidean metric.

It should be noted that when we are dealing with at infinite family of topologies there is something called product topology which is somewhat different from box topology, ok? Though the underlying set is Cartesian product, we are careful enough to call this box topology ok? Not product topology.

Later on we will introduce this what is exactly product topology ok. You have to wait a little bit for that, see the two things coincide in the finite case is just a coincidence. We have observed that in $\mathbb{R}^n$, we could pass on to the maximal topology. We then imitated the maximal in the generaal construction. The round thing cannot be imitated unless you have a metric.

In the general cases there is no way of taking square, square roots and summation and so on there is no way ok. The round thing cannot be imitated. So, that is why we are thankful to this ah maximal topology which can be imitated ok.

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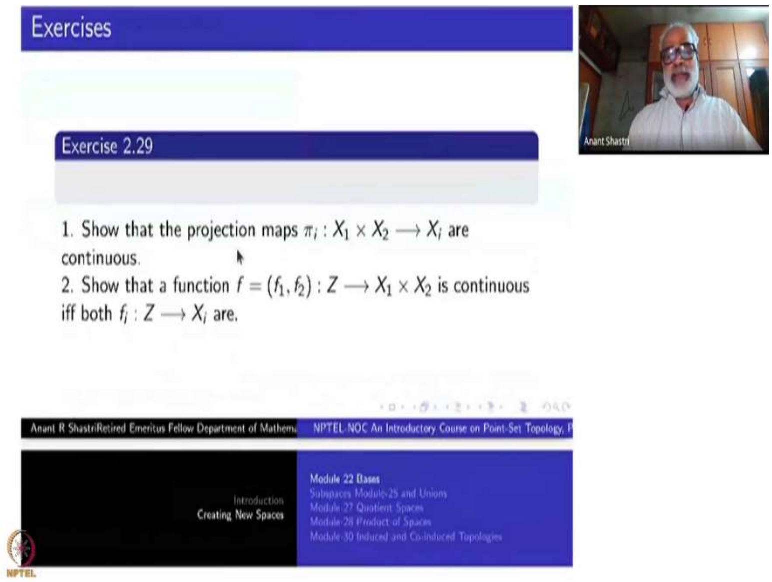

So, that is all for today I have few exercises. I will just go through these things ah not the solutions of course.

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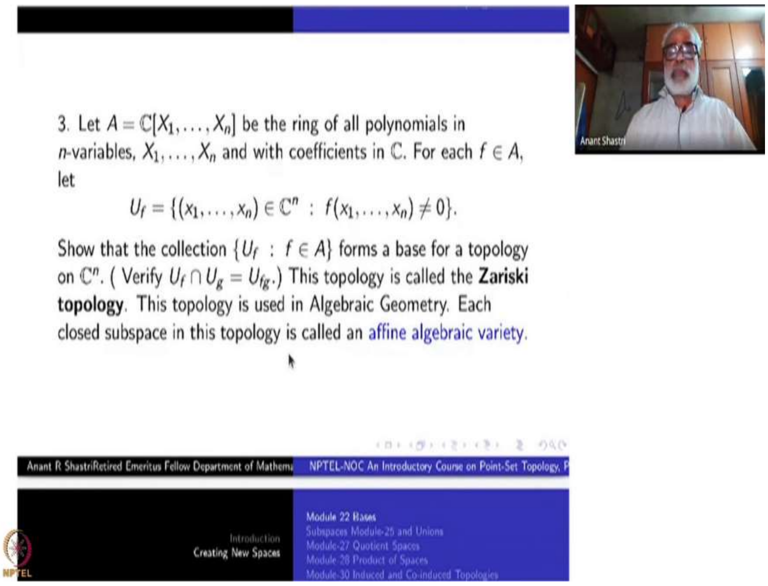

So, the first thing you have to observe is that in this box topology the projection maps $(x_1, x_2) \to x_1$ or $(x_1, x_2) \to x_2$, they are they are denoted by π_1 and π_2, these are continuous. So, I am writing it for product of two spaces, as soon as we verify it for two, it is verified for n of them also, the same proof here works for all the cases, namely infinite product also.

Next thing is again similar to that. Take any space Z and a function into the product ok. A function is completely determined by looking at the various coordinate projections coordinate maps. Namely, what are they? (f_1, f_2); any map f from Z to $X_1 \times X_2$ is given by the two coordinate functions.

So, $(f_1(x), f_2(x))$. So, both of them will be continuous if and only if the function is continuous ok. So, verification of this will be very straightforward. Once you do this exercise, you will get more familiar with what is happening here. Remember to verify that something is continuous you take a base here and check that inverse image of members of that base are open here ok? Because here is the original thing is there. So, the base new base has come here. So, this you have to do directly.

Now here is another example of a topology coming from a base, by declaring a base, this is nothing to do with box topology though. Take the polynomial ring A in n variables over $\mathbb{R}$ or $\mathbb{C}$ whatever. Let us take it $\mathbb{C}$ for a while here ok?

$X_1, X_2, \ldots, X_n$ are variables ok? You can add two polynomials you can multiply the two polynomials you can scalar multiply them and so on, this is actually a vector space as well as a ring. So, such things are called algebras over this field $\mathbb{K}, \mathbb{K}$ is here $\mathbb{C}$ ok. For each f in A, you can assign subset of $\mathbb{C}^n$. What is that? Namely, all those points $(x_1, x_2, \ldots x_n)$ wherein the function f, the polynomial f does not vanish. I have written here U_f. Earlier I had discussed points at which f is actually 0.

So, it is the complement of the zero set say ok? Just now we know that if I take the Euclidean topology or the product topology here because each polynomial is continuous these things are open in the product topology, ok? Euclidean topology, but now we are not going to take the Euclidean topology or the product topology here. What we are going to take is the following: show that this collection of you know these U_f's, where f ranges over all elements of A forms a base for a topology in $\mathbb{C}^n$.

The important thing here is to verify that $U_f \cap U_g$ is again a member here. Remember this just means that this family is closed under finite intersection, it is a strong property. So, once I have told you that you do not there is nothing to verify. You have to quote that theorem that is all to say that this is a base. So, this topology which gives you the topology given by this base is called the Zariski topology ok. So, this is sacrosanct for doing geometry, used in algebraic geometry all the time alright?

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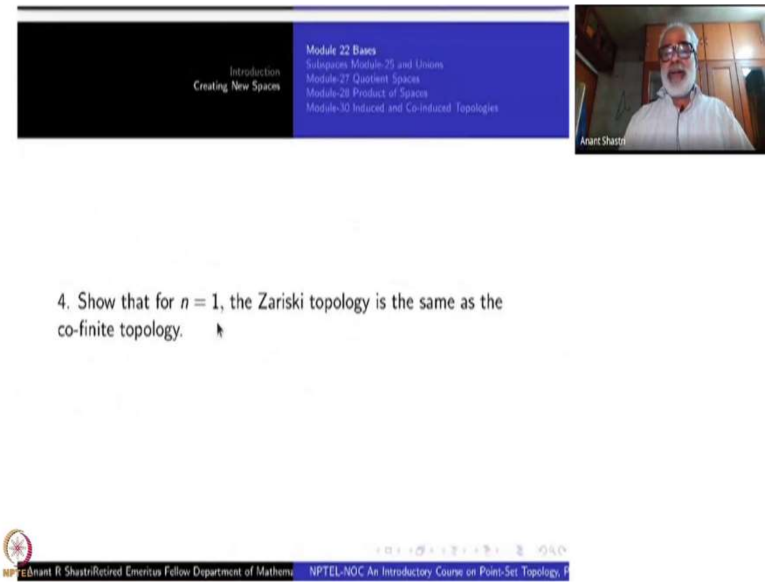

The special case $n = 1$, you know this Zariski topology is the same as the co-finite topology. Remember what is co-finite topology, a set is open if and only if its complement is finite. So, verify this, it is very easy. n equal to 1 case only; n equal to 2, 3 etc it will not be co-finite topology ok.

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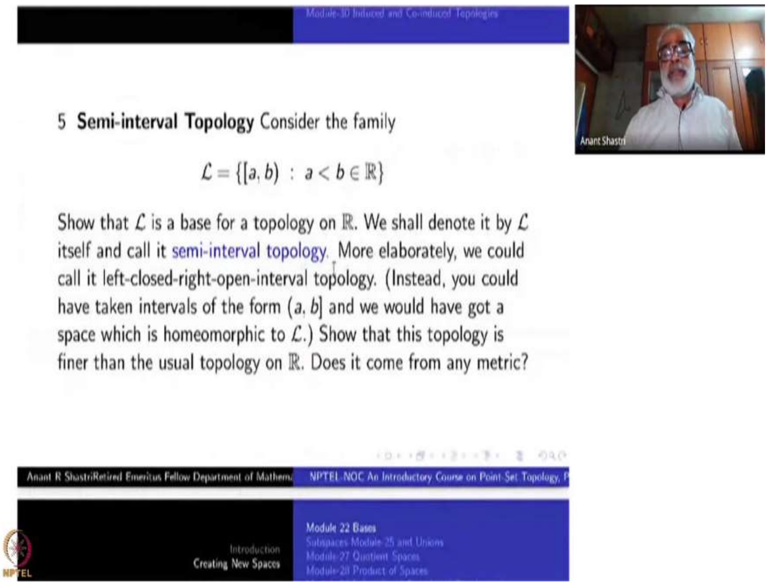

So, here is another interesting topology on $\mathbb{R}$ itself. Look at all half open intervals $[a, b)$, where a is closed. The left side is closed ok. Look at all these. You may call it $\mathcal{L}$, the collection $\mathcal{L}$. This $\mathcal{L}$ is a base for a topology on $\mathbb{R}$. What you have to verify? You have to verify (B1) and (B2) remember that ok. Union of all member is equal to whole of $\mathbb{R}$ is obvious, given any a, I can always take some $b > a$ and take the interval $[a, b)$. So, everything will be covered no problem.

Intersection of two such things you have to see what happens that is it ok? So, we should denote it by $\mathcal{L}$ itself ok. Since we are not going to emphasize on these notations too much because notation may change from author to author. I do not want to float one more notation here ok. So, we will call it semi interval topology, more elaborately we could call it left closed right open interval topology.

If I do this one you will not find it in any literature. In literature it is called semi interval topology. The semi interval topology can make sense for $(a, b]$ also that will be an entirely different topology. If you care you can say that it is similar, but it is different topology on $\mathbb{R}$, open subsets with respect to that will not be open subsets with this and so on ok.

Instead you could have taken intervals of this form also and we would have got a space which is homeomorphic to $\mathcal{L}$, means what? It is a topology here there is a topology there, if you take $x \to -x$ or something of that nature, which is order reversing, a reflection that will take half closed interval this way to half closed interval that way. That is a homeomorphism. But they are not of the same thing.

Show that this topology $\mathcal{L}$ is finer than the usual topology on $\mathbb{R}$. So, it has more open sets or open subsets of $\mathbb{R}$, they are there.

Last question is here, does it come from any metric? Keep trying ok? Before the course ends, we will get a solution, no problem ok?

And then there is another one interesting one here.

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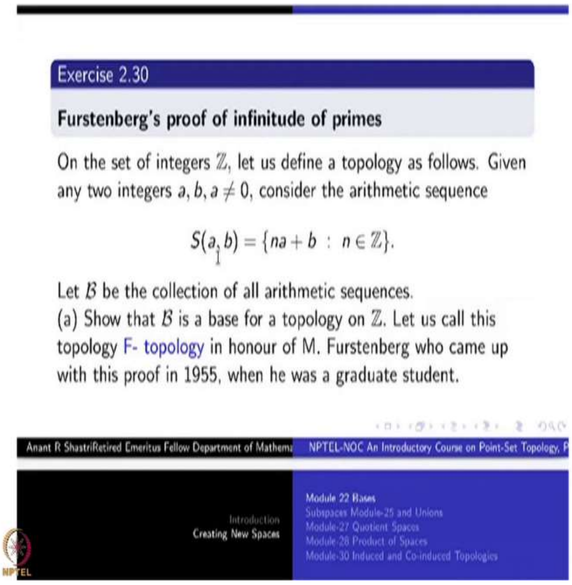

On the set of integers. Let us define a topology as follows. This is not introduced just as another example of a topology. This I am introducing as an entertaining application of very elementary topology to produce a very very ancient theorem in number theory ok, so what is that? This is due to Furstenberg. Furstenberg was just a graduate student when he published this one in 1955 ok.

So, let us introduce a topology here. Given integers $a, b, a \neq 0$, look at the arithmetic sequence: $na + b, n$ ranging over all the integers. So, let us denote it by $\mathcal{S}(a, b)$; this is an arithmetic sequence ok, $na + b, n$ belonging to the integers. Let $\mathcal{B}$ be the collection of all arithmetic sequences $\mathcal{S}(a, b)$.

For each a and b, there is 1 so that $\mathcal{B}$ is the collection. And by the very notation what we are going to do? This $\mathcal{B}$ is a base for a topology. Let us call this topology F-topology in honour of Furstenberg ok.

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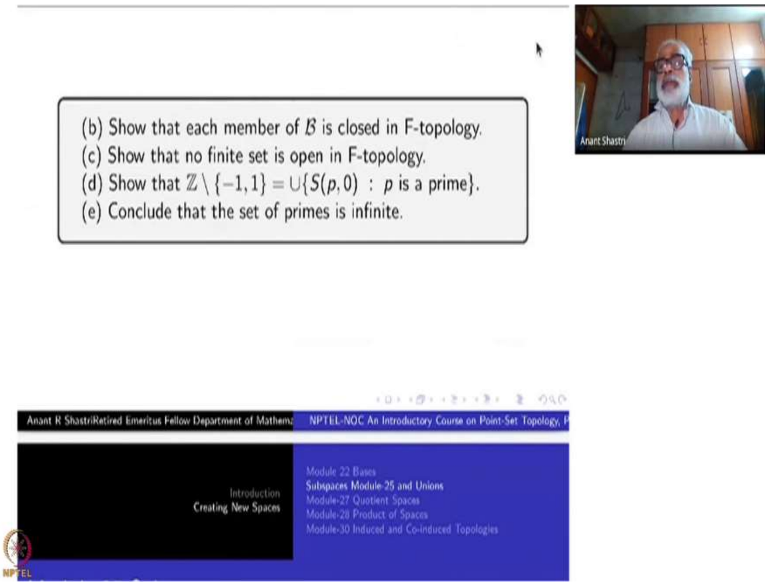

Show that each member of $\mathcal{B}$ is closed in F-topology. This is a strange thing right. Basis elements are open, but here they are closed also ok. I do not have to show that they are open. This is by the very definition, $\mathcal{B}$ is contained inside $\mathcal{T}$.

Show that no finite set is open in F-topology. So this is not so strange. Finite sets are not open inside $\mathbb{R}$ also ok. Now comes the climax.

Consider $\mathbb{Z} \setminus \{-1, 1\}$, throw away these two integers -1 and 1 ok? Show that you can write the entire $\mathbb{Z} \setminus \{-1, 1\}$ that as a union of arithmetic sequences, $\mathcal{S}(p, 0)$, where p is a prime. You do not have to worry about a and b, here $a = p$ is a prime and b is 0. So, a is taken as prime number b is 0, look at only those arithmetic sequences. $\mathbb{Z} \setminus \{-1, 1\}$ is union of these things ok?

This is also an elementary observation. As soon as I have done that you can conclude that the set of primes inside $\mathbb{Z}$ inside actually a natural number is infinite ok?

So, this is quite entertaining. There is nothing deeper here, but for a student who found it out, this is a wonderful thing. So, it appeared in AMS notices. So, let us stop here and do next time. Next time, we will do another important concept namely, subspaces. Thank you.