Introduction to Point Set Topology, (Part I)
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Lecture - 23
Subbases

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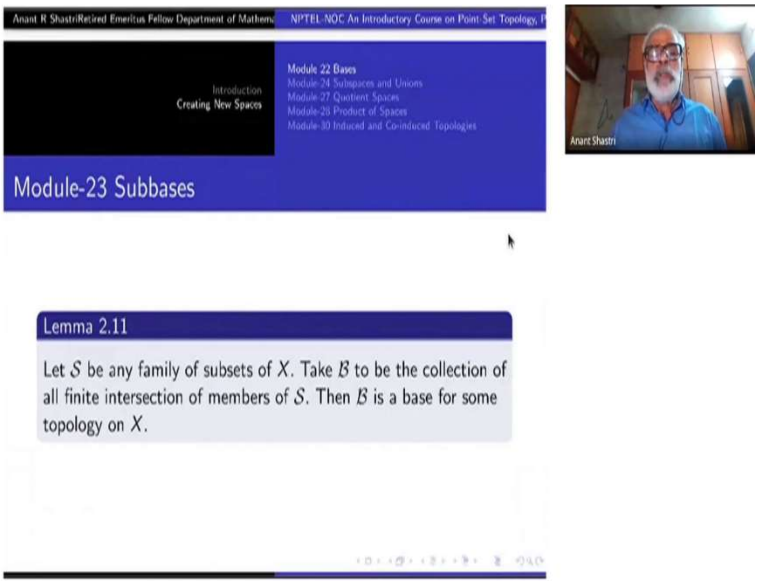

Welcome to module 23 of Point Set Topology Part I course. So, today we will carry on with subbases. We have motivated the definition of subbase last time ok. So, let $\mathcal{S}$ be any family of subsets of X. Take $\mathcal{B}$ to be the collection of all finite intersections of members of $\mathcal{S}$.

First of all $\mathcal{S}$ will be contained in $\mathcal{B}$, all members of $\mathcal{S}$ are be there. Two members you take, their intersection will be there, 3 members intersection will be there. Like that finitely many members, their intersection will be there in $\mathcal{B}$. So, that is $\mathcal{B}$ by definition. $\mathcal{B}$ is that collection nothing more, nothing less.

Then $\mathcal{B}$ is a base for some topology on X. In fact, what is the topology? It will be precisely $\mathcal{T}_{\mathcal{S}}$, which will be $\mathcal{T}_{\mathcal{B}}$ as well. That part we have seen that it has to be $\mathcal{T}_{\mathcal{B}}$, ok? So, I do not have to state that one again. So, what we have to verify? We have to verify the two conditions

(B1) and (B2). I do not have to worry about describing $\mathcal{T}_{\mathcal{B}}$ again. Members of $\mathcal{T}_{\mathcal{B}}$ are already described ok.

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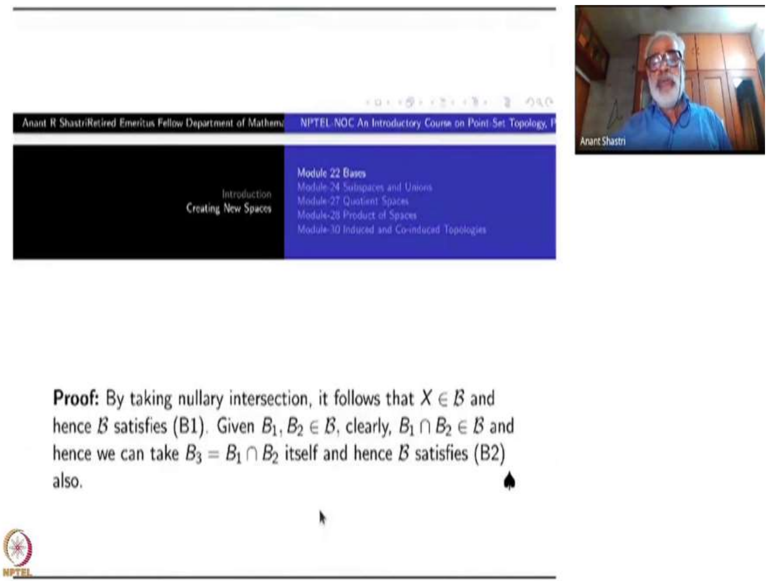

If you take empty intersection it follows that X will be inside $\mathcal{B}$.$\mathcal{S}$ itself may not contain the element X you see, but when you take a finite intersection empty intersection is also allowed here, ok. So, that will give you X inside $\mathcal{B}$. So, (B1) is automatically satisfied in a strong way. See (B1) was that union of members of $\mathcal{B}$ is equal to X, but here I am putting X itself is inside. This is stronger way ok, is stronger than (B1).

Now, given two members of $\mathcal{B}$, their intersection will be in $\mathcal{B}$, because say B_1 is equal to intersection of $B_{1,1}, B_{1,2}, \ldots, B_{1,k}$. And B_2 is equal to intersection $B_{2,1}, B_{2,2}, \ldots, B_{2,\ell}$. When you take $B_1 \cap B_2$, it will be intersection of all of them members of $\mathcal{S}$, a finite intersection. Say here there are 5 and here 6, that will be 11 of them together. So, intersection will be also inside $\mathcal{B}$, ok. So, I can take B_3 itself as $B_1 \cap B_2$ to get the condition, condition (B2) right? What is condition (B2)? Given any point in the intersection there must be a third one which contains that point and contained in the intersection. I can take to the $B_1 \cap B_2$ itself because that is a member here.

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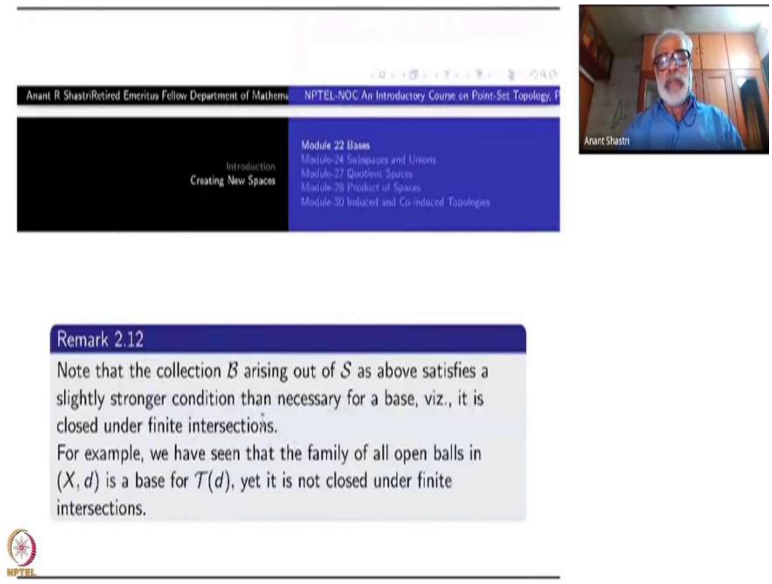

So, both (B1) and (B2) are satisfied in a strong way. In any case this is a base for some topology and that topology is nothing but $\mathcal{T}_{\mathcal{S}}$. Note that, the collection $\mathcal{B}$ arising out of $\mathcal{S}$ as above satisfies a slightly stronger condition than necessary. So, it is closed under finite intersections right? So, for example, we have seen that the family of all open balls in (X, d), is a base for $\mathcal{T}(d)$. It does not satisfy this strong condition, because intersection of two balls may not be a ball. So, in order to give proper recognition to this result, lemma 2.11, we will make a formal definition.

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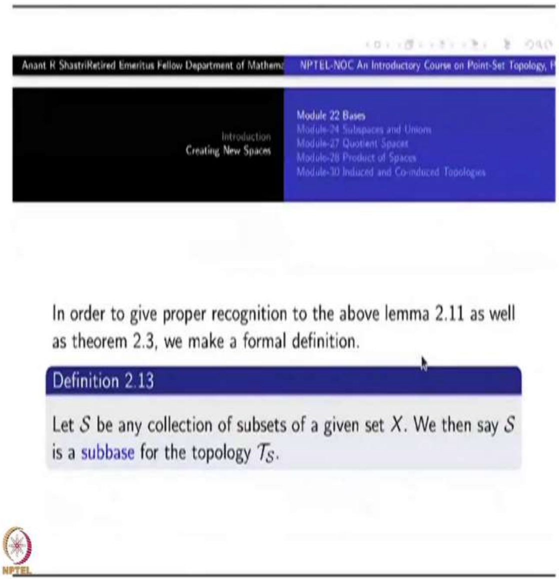

Let $\mathcal{S}$ be any collection of subsets of a set X, ok? Here X is a set. We say $\mathcal{S}$ is a subbase for the topology $\mathcal{T}_{\mathcal{S}}$ ok. So, this definition may look funny, but it is what it is. It is just like a nomenclature. We do not put any condition on $\mathcal{S}$. So, any subset is a subbase for a particular topology what is it? $\mathcal{T}_{\mathcal{S}}$ the unique topology ok, which is the smallest topology containing $\mathcal{S}$ ok.

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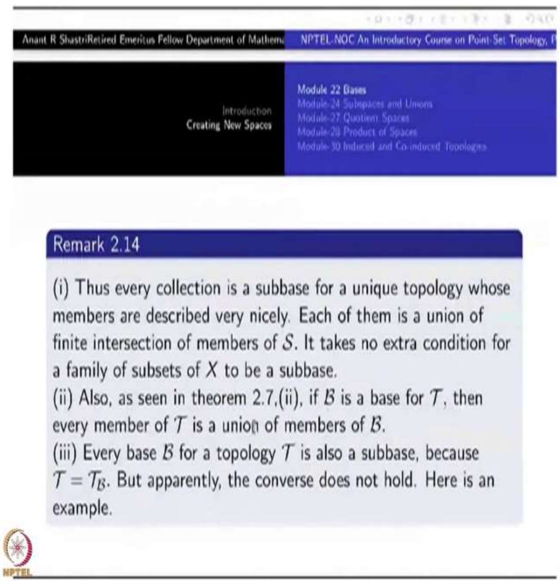

So, I repeat: thus every collection of subsets of X is a subbase for a unique topology whose members are described very nicely. What are they? First look at the base, the base given by $\mathcal{S}$ namely, take finite intersections of members of $\mathcal{S}$ ok. So, each of member of $\mathcal{T}_{\mathcal{S}}$ will be an arbitrary union of members of $\mathcal{B}$, which are just finite intersections of members of $\mathcal{S}$, ok?

So, it takes no extra condition for the family $\mathcal{S}$ to be a subbase for a topology as seen in 2.7 part (ii). If $\mathcal{B}$ is a base for $\mathcal{T}$, then every member of $\mathcal{T}$ is a union of members of $\mathcal{B}$. So, every base $\mathcal{B}$ for a topology is also a subbase, because $\mathcal{T}_{\mathcal{B}}$ is the smallest topology $\mathcal{B}$ ok. So, $\mathcal{T}_{\mathcal{S}}$ equal to $\mathcal{T}_{\mathcal{B}}$ means $\mathcal{B}$ is a subbase also. So, $\mathcal{B}$ is a base is something more stringent, but $\mathcal{T}_{\mathcal{B}}$ equal to $\mathcal{T}_{\mathcal{S}}$ immediately implies that $\mathcal{B}$ is a subbase, there is no problem ok.

The converse does not hold always, not all subbases are bases because condition (B2) may not be satisfied. Even (B1) may not be satisfied ok.

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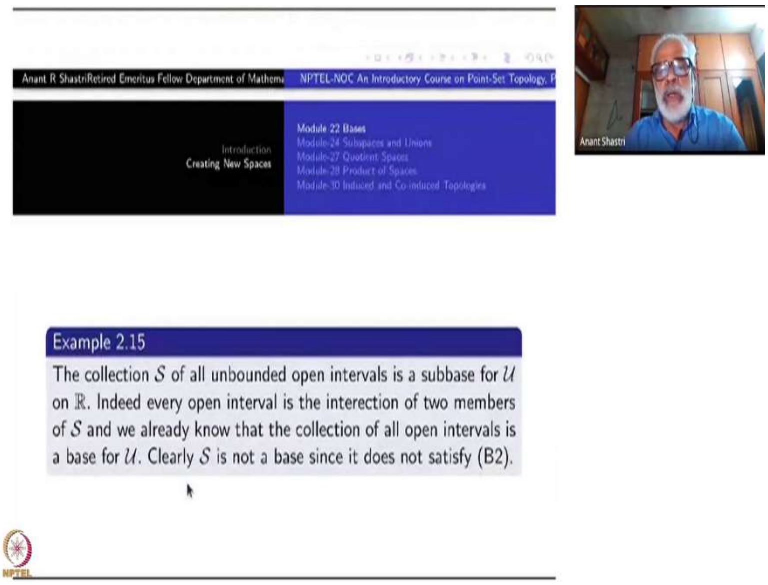

Look at this example. The collection consists of all unbounded intervals inside $\mathbb{R}$. What are they? (a, ∞) or $(-\infty, a)$. Look at all such rays. The collection is a subbase for a topology. This is not a base for any topology neither it is a topology ok. It is subbase for a topology and what is that topology? Because if you take two members like this ok. With a less than b, then take $(a, \infty) \cap (-\infty, b)$, that will be the open interval (a, b).

That topology is the same as the usual topology why? Because if you take two members like this ok, with a less than b, then take $(a, \infty) \cap (-\infty, b)$, that will be the open interval (a, b). So, all open intervals are finite intersections of members of $\mathcal{S}$ and then we know that all open intervals form a base. So, therefore, $\mathcal{S}$ is a what? $\mathcal{S}$ is subbase for the usual topology. Clearly it is not a base ok.

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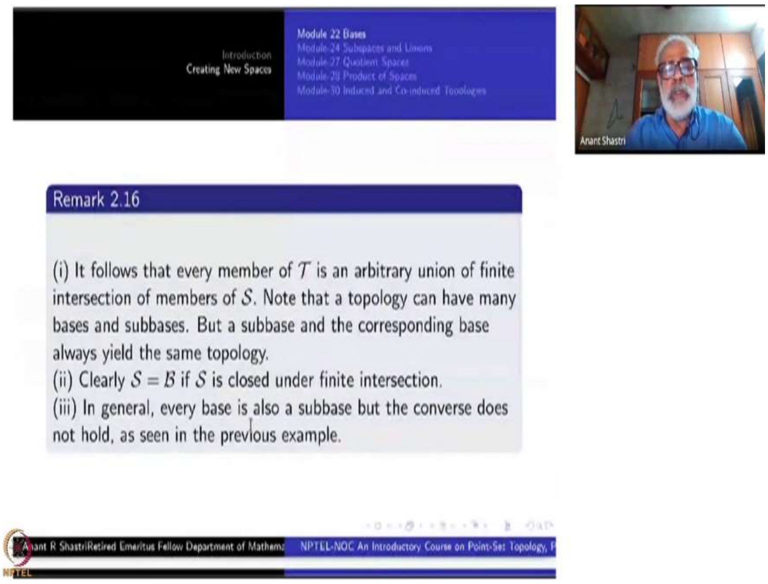

So, it follows that every member of $\mathcal{T}_{\mathcal{S}}$, an arbitrary union of finite intersection of members of $\mathcal{S}$. I repeat this one, this is what you have to by heart maybe. Note that a topology can have many bases and subbases, but a subbase and the corresponding base always give the same topology, ok?

So, clearly $\mathcal{S}$ equal to $\mathcal{B}$, if $\mathcal{S}$ is already closed under finite intersection. Then if you take further finite intersections, you do not get any new members, they are already members of $\mathcal{S}$. In general, every base is also a subbase, but the converse does not hold as in the previous example. So, I am just repeating all these things in this part, have not said anything new.

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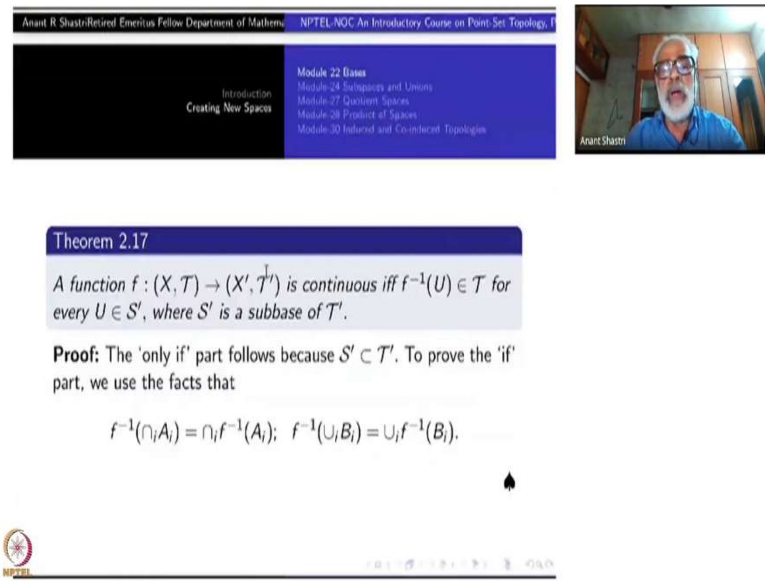

Now, I will state a theorem here ok, which will show you the utility of bases and subbases. A function f from $(X, \mathcal{T})$ to $(X, \mathcal{T}')$ is continuous if and only if $f^{-1}(U)$ belongs to $\mathcal{T}$ for every member U in S', where S' is a subbase for $\mathcal{T}'$.

If it is continuous take a member U in S', it will be already in $\mathcal{T}'$. Therefore, by definition of a continuity $f^{-1}(U)$ is inside $\mathcal{T}$: inverse image of an open set is open that is what we have shown. So, conversely we have only partial condition: inverse image of an open set is open holds only for elements of S'.

But that is enough for continuity. Because once this is true for members of S'; it will be true for intersection of two of them. It will be true for intersection of finitely many of them; that means, it is true for members of $\mathcal{B}$ now ok? But then if you take inverse image of the union is also union of the inverse images. So, this is purely set theoretic fact you have to use. Inverse image of the intersection is intersection of inverse images. Inverse image of the unions is union of the inverse images. So, first use this one then use this one to get all the members of tau prime, their inverse images are open, therefore, f is continuous.

So, what is the role of this theorem? It reduces the study of a continuous function, checking the continuity of a function. You have to do it for all members of tau prime no. You have to just do it for membes of a chosen subbase that is enough, ok.

For example, you can apply this theorem here in this example ok? You have some function $\mathbb{R}$ to $\mathbb{R}$. To check that it is continuous you have to just show that inverse image of unbounded intervals are open, both $(-\infty, a)$ as well as some (a, ∞). For all of them if inverse images are open that is enough. You do not have to worry about all open sets ok? So, we will come back to this phenomena again.

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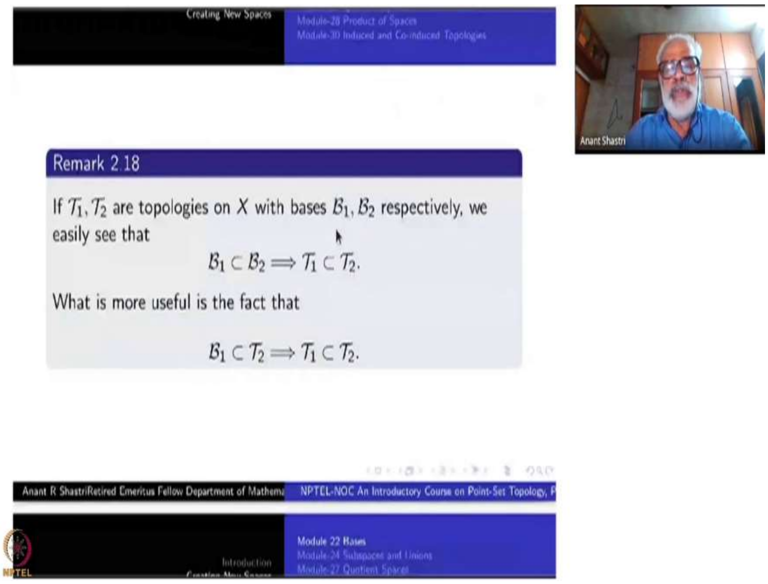

So, a few more comments about bases and subbases; if $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies, $\mathcal{B}_1$ and $\mathcal{B}_2$ are respectively their bases ok. You have chosen some bases. If $\mathcal{B}_1$ is contained in $\mathcal{B}_2$ automatically will be $\mathcal{T}_1$ is contained inside $\mathcal{T}_2$. Why? Because members of $\mathcal{T}_1$ are unions of members of $\mathcal{B}_1$ therefore, their member union members of $\mathcal{B}_2$ also therefore, they are in $\mathcal{T}_2$ that is it very easy right ok.

What is more useful is you do not need $\mathcal{B}_1$ to be contained $\mathcal{B}_2$. If $\mathcal{B}_1$ is contained in $\mathcal{T}_2$ then $\mathcal{T}_1$ is contained in $\mathcal{T}_2$ because $\mathcal{T}_1$ is the smallest topology containing $\mathcal{B}_1$ right. So, this is quite useful exactly same thing is true if you replace $\mathcal{B}_1$, $\mathcal{B}_2$ by $\mathcal{S}_1, \mathcal{S}_2$ which are subbases of a same reason, I have not written down at this one here, but that is obvious.

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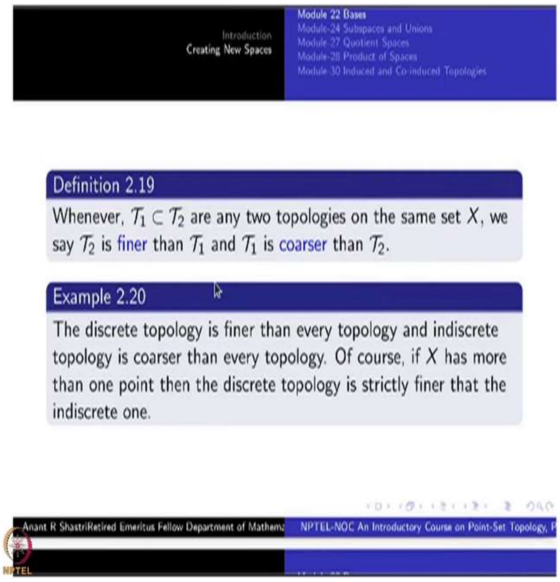

So, once again, for later use I will introduce these two definitions here $\mathcal{T}_1$ contained in $\mathcal{T}_2$ ok, any families of subsets of a set X (not only for topology). See here I have defined for topologies; just means that $\mathcal{T}_2$ is finer than $\mathcal{T}_1$. So, this is the definition of the word finer here. At the same time $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$. It is just like saying that $\mathcal{T}_1$ is less than $\mathcal{T}_2$ or $r_1 \leq r_2$ implies $r_2 \geq r_1$, that is all.

So, coarser and finer. So, you have to get used to these terms. This can be used for any families of subsets of X, same set X. So, in particular, I have given this definition for two topologies no problem, here are the examples. Discrete topology on a given set is finer than every topology on that set. Exactly same way indiscrete topology is coarser than every other topology on that set ok. If X has more than one point then indiscrete topology is strictly finer sorry, discrete topologies strictly finer than the indiscrete topology. So, strict inequality also holds that is no problem ok.

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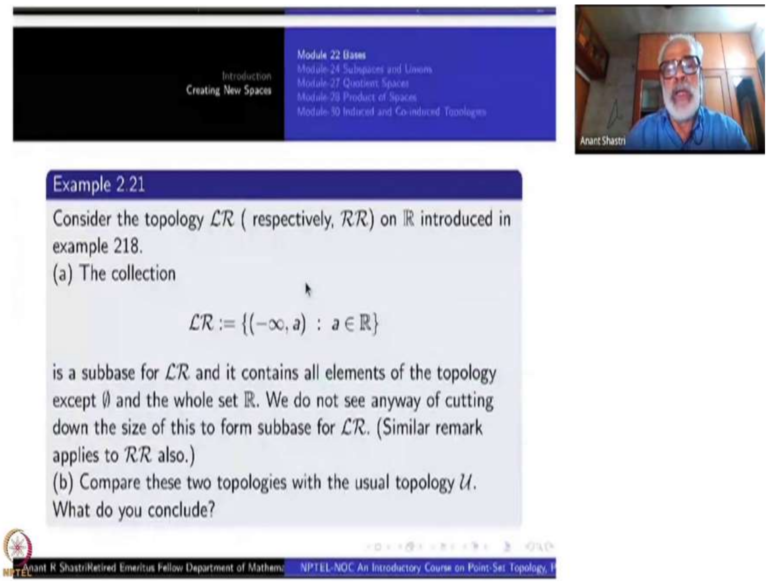

So, I come back to this example again here. The left ray topology and the right ray topology ok, introduced in example 1.115 (3). The collection $\mathcal{LR}$, recall namely, $(-\infty, a), a$ belonging to $\mathbb{R}$ is a subbase ok. For $\mathcal{LR}$ not for usual topology, for usual topology you have to combine both of them. You take $\mathcal{LR}$ itself only $\mathcal{LR}$'s that will be a topology and that topology we have taken it is a subbase for $\mathcal{LR}$ and it contains all elements of the topology except empty set and the whole set.

This is a strangest thing why because look at this one. Take intersection of $(-\infty, a)$ with $(-\infty, b)$, what it will it be? Depends upon whether a is smaller or b is smaller ok. If a is smaller intersection will be $(-\infty, a)$. So, it is a member. So, this family is closed under finite intersection except empty intersection; it is also closed under finite union. So, what is missing here to be a topology? What is missing here? Empty set is not there. The whole $\mathbb{R}$ is not there that is all.

So, empty set and the whole set are not there otherwise it is already a topology right? So, such a base or such a subbase it is actually subbase right, because if the union is not the whole of X it is not a base. So, this is not even a base. It is not a topology, but it is a subbase, it is a very strange kind of example. similar remark holds for $\mathcal{RR}$ also ok.

You can compare these two topologies with the usual topology ok. What do you conclude? I do not want to tell you. Conclude yourself. This is this is obvious, but you have to keep doing this kind of thing. That is why I have put it that is all ok. Now, let us make a definition and illustrate why these things are important ok.

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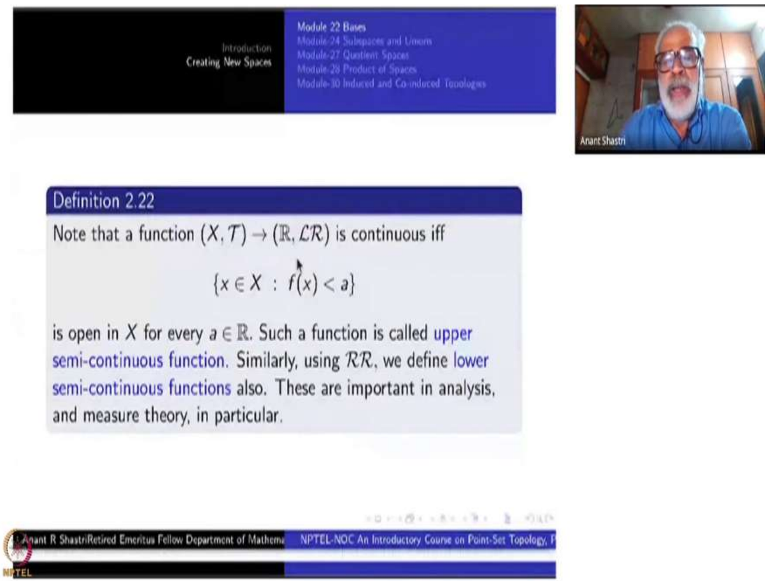

Take a function from $(X, \mathcal{T})$ to $(\mathbb{R}, \mathcal{LR})$. This is left rays right? It is continuous if and only if the set of all points x belonging to X such that $f(x)$ is less than a is open in X, for all $a \in R.X$ is any topological space, here I have taken $\mathbb{R}$ with $\mathcal{LR}$ ok. So, what is the condition for a function to be continuous? Inverse image of a basic or sub basic open set is open that is enough. Sub basic open set is all rays $(-\infty, a)$.

Inverse image will be just all points x belonging to X such that $f(x)$ is less than a. If a function satisfies this property it is called upper semi continuous function. So, this terminology is taken from analysis. Similarly if you replace this one by $\mathcal{RR}$ ok, then what will be the condition for continuity; x such that $f(x)$ bigger than a must be open ok? Exactly similar. If that happens we call it lower semi continuous function. So, these are important in analysis especially in measure theory.

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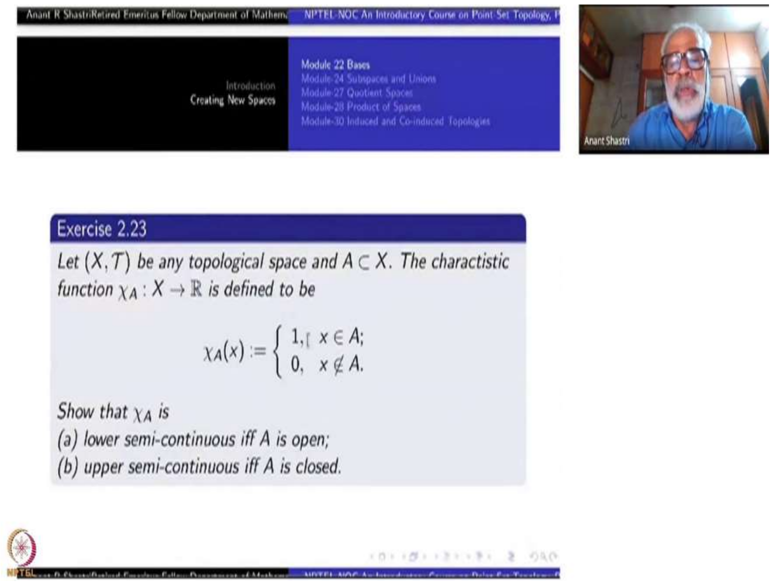
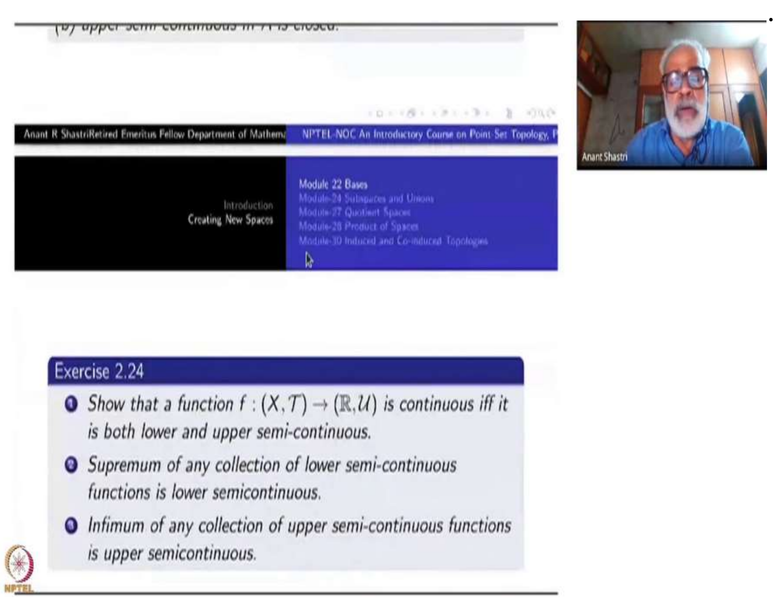


So, I have included a few as exercise their examples. A few properties of this lower and upper semi continuous functions ok.

So, go through them they are not all difficult. So, exercise 1, gives you an example namely this is called characteristic function of a set. $\chi_A(x)$ equal to 1 if x is in A and equal to 0, if x is not in A ok? So, characteristic functions are lower semi continuous if and only if A is open. Upper semi continuous function if and only if A is closed. (Refer Slide Time: 19:39)



These are highly discontinuous functions, but they have some continuity property. So, that is very important in measure theory. Your starting point of defining measure and all that ok? So, here are more. Supremum of any collection of lower semi continuous functions is lower semi continuous. Infimum of any collection of upper semi continuous function is upper semi continuous. If something is continuous then it will be both lower semi continuous and upper semi continuous and conversely ok.

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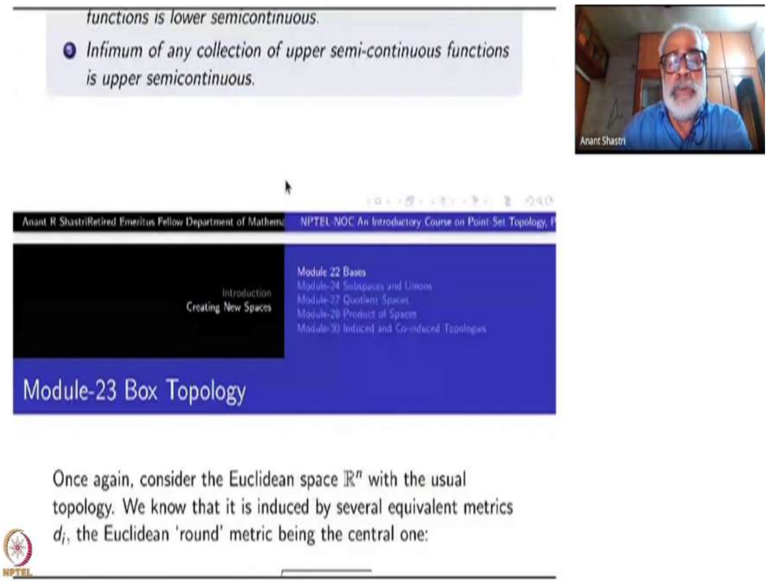

These are all elementary exercises. So, take some time to do them, do not just leave them. Because if you ignore them, soon these things will ignore you, you give them proper attention and time and then they will become friends to you ok? So thank you, we will ah meet next time again.