Introduction to Point Set Topology, (Part I)
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Lecture - 22
Bases and Subbases

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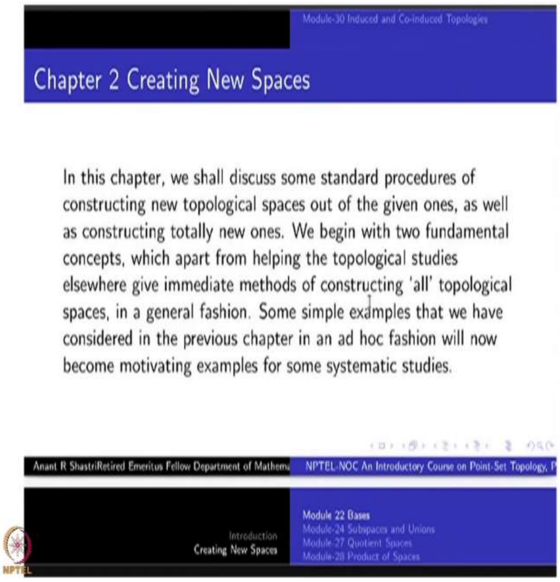

So, welcome to chapter 2 of the course, I have named it as Creating New Spaces. We shall discuss some standard procedures of constructing new topological spaces out of the given ones, as well as constructing totally new ones also. Of course, we shall not just construct them, we shall also put them in proper perspective, study them a little more. That is all this chapter is about.

We begin with two fundamental concepts. These concepts, apart from helping the topological study elsewhere, give immediate methods of constructing all topological spaces, in a general fashion. Some simple examples that we have considered in the previous chapter in an ad hoc fashion will now become motivating examples for some systematic studies. I will point out to them, whenever we come across those things.

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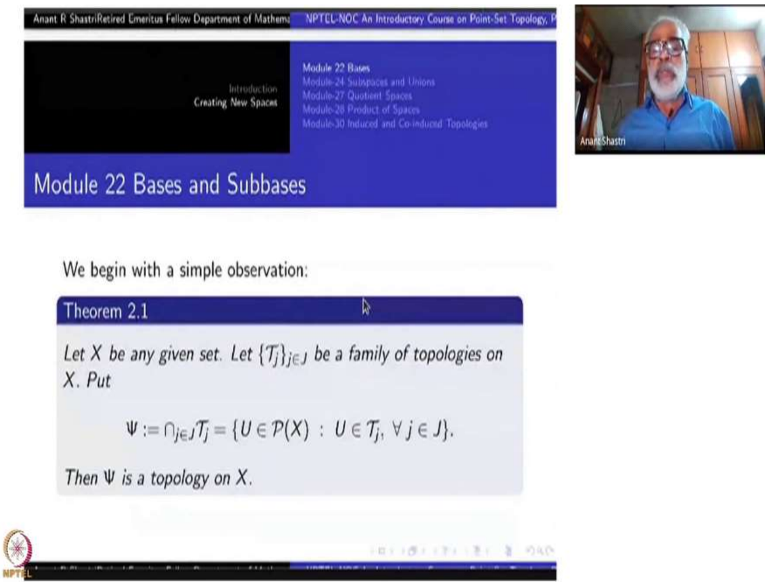

So, welcome to Module 22 : Bases and Subbases. You may as well take one of them today we will see, but I am already laying foundation for the second one also. So, today maybe we will just discuss bases because of lack of time. So, the first observation is that start with any set and a family of topologies on X, ok? a family of topologies on X. Take the intersection of all these families. Which means what? All members of $\mathcal{P}(X)$, which means subsets of X, which are in every member of this family $\mathcal{T}_j$'s. So that is the intersection, that family will become a topology on X.

So, this is what I meant by saying that creating new topologies out of the old topology. For example, this one comes into that category, right; $\mathcal{T}_j$'s are a families of topologies. Now, I am taking the intersection of them. Perhaps this one is a new one, not always if you take say $\mathcal{T}_1$ contains $\mathcal{T}_2$, then this intersection will be just $\mathcal{T}_2$. So, it is not always a new one ok? That is why this is a general method, alright.

So, this ψ is a topology after all, is very important concept for us. The proof is very easy. Let me just see. What we have to do? Empty set and the whole set are inside ψ. Why? Because they are there inside every topology. If you take U_1 and U_2 inside of ψ, they will be there in each $\mathcal{T}_j$, so their intersection will be in each $\mathcal{T}_j$. So, intersection is here.

Similarly, if you have a family of open sets say, U_α ok? Members of ψ; that means, what each $\mathcal{T}_j$ contains all of U_α. Therefore, the union of all U_α is inside $\mathcal{T}_j$ for every j. So; that means, that union is a member of ψ. So, that completes the proof of this is a topology ok?

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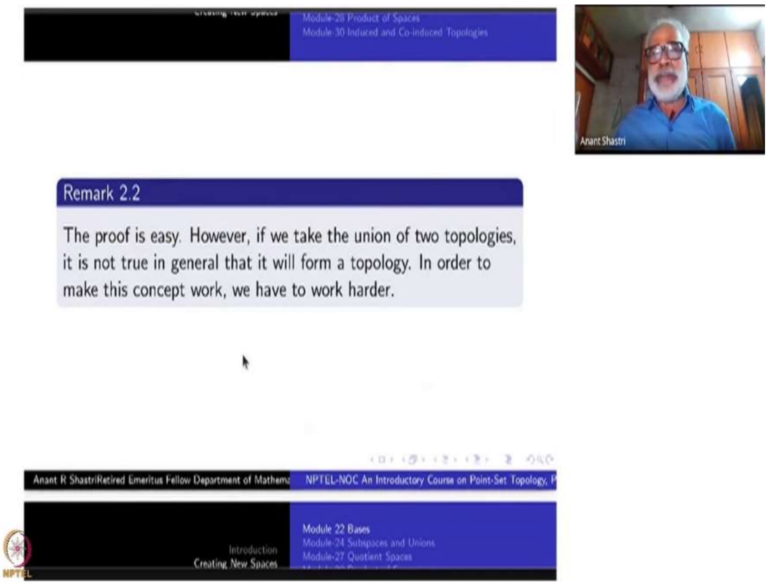

This is just like what happens with vector spaces. If you have a family of vector subspaces, then the intersection is a vector space. Similarly, if you have a family of subgroups of a group then the intersection is a subgroup. However, in either of these cases the union utterly fails to be a subgroup here, vector subspace there the same thing is happening here.

If you take two topologies, the union may not be a topology. Of course, if one is contained in the other then the union is the other one so, so that is a very special case. In general, union may not be a topology, but we do not want to give up just like that. So, you will work harder to get topologies out of these things also. So, what do we do? So, here is another general method which is actually an application of the previous theorem that you have, ok.

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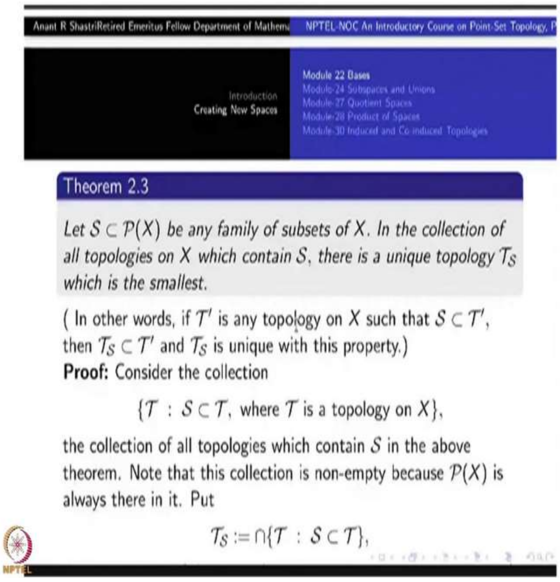

So, this theorem this is the foundation it is like it keeps helping us again and again. So, start with any family of subsets ok? instead of taking topologies and so on just any family $\mathcal{S}$ of subsets of X, ok?

In the collection of all topologies on X which contain this $\mathcal{S}$, there is a unique topology $\mathcal{T}_{\mathcal{S}}$ which is the smallest. First of all the discrete topology, the entire $\mathcal{P}(X)$ contains this one. So, this family of all topologies containing $\mathcal{S}$ is non empty ok. Logically, it is not necessary, but we can fix it up that it is non empty. Now, I am taking intersection of all these topologies which contain $\mathcal{S}$. This theorem says that such intersection is a topology right? And since all of them contain $\mathcal{S}$ this will contain $\mathcal{S}$ also, ok?

So, one part is fine, namely, we have got a topology which contains $\mathcal{S}$ now, ok? But this is the smallest one. Why? Because it is contained in the each of them because this being the intersection of all of them.

So, that is all, so one step application of this previous theorem, one gives you this result ok, namely, given any set there is a smallest topology that is the unique smallest topology containing $\mathcal{S}$, it is intersection of all topologies containing that $\mathcal{S}$. Why unique one? Because if there are two of them their intersection must be equal to both of them right? Yeah.

So, we can make a definition ok. What is that?

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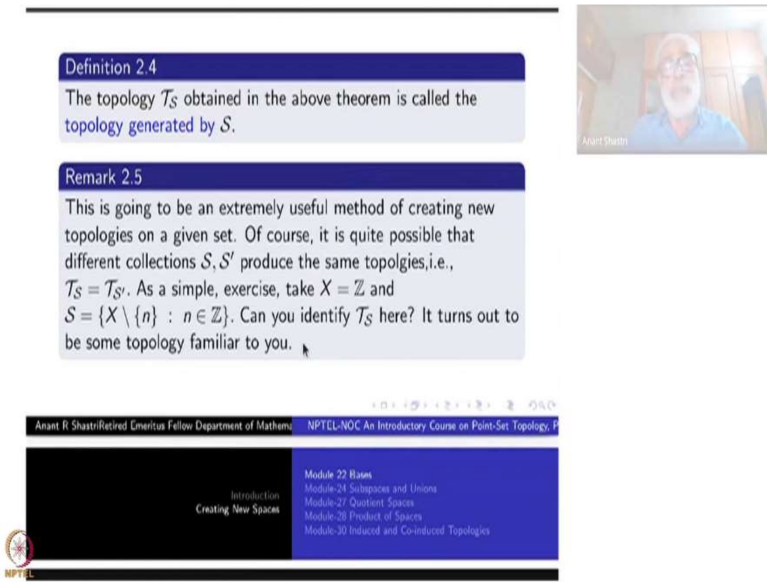

The topology $\mathcal{T}_{\mathcal{S}}$ namely the smallest topology containing $\mathcal{S}$ is called the topology generated by $\mathcal{S}$. If $\mathcal{S}$ itself is a topology what happens? $\mathcal{T}_{\mathcal{S}}$ will be equal to $\mathcal{S}$ because $\mathcal{S}$ is already a topology, right? So, everything else contains it so, intersection will be $\mathcal{S}$. So, if $\mathcal{S}$ is already a topology no problem, but if $\mathcal{S}$ is not a topology then this is a new one.

So, this is again similar to what you do in linear algebra; you have a vector space, you have a set of vectors there ok? How do you create a vector subspace which will contain all these points all these vectors? So, that is the vector space generated by this ok? So the same word we have taken here, same terminology ok?

So, this one-step-consequence of the our first theorem this is going to be extremely useful method of creating new topologies because I may start with any subset of the power set of X, any collection $\mathcal{S}$ of subsets of X. That will give you a unique topology, the smallest one that contains $\mathcal{S}$, right.

Of course, what may happen is two different subsets of $\mathcal{P}(X)$ may give you same topology that is possible ok? $\mathcal{T}_{\mathcal{S}}$ may be equal to $\mathcal{T}_{\mathcal{S}'}$ ok?

So, here is a simple exercise. Take X equal to Z or any countable infinite set ok? And take $\mathcal{S}$ equal to X minus one single point belonging to X, one point is missing from each subset here. Look at that collection, ok? What is the $\mathcal{T}_S$ corresponding to this $\mathcal{S}$? What is the topology generated by this family on the set of integers? It turns out to be something familiar to you. Can you see what is happening, ok? I do not mind telling you. This is precisely my point namely we need to work harder to identify the topology generated by a set $\mathcal{S}$.

Like in the case of vector spaces, you give a collection of vectors collection of elements ok, what are all the elements in the vector space generated by that? They will be all finite linear combinations of elements from the set. So, that is a description. So, we would like to have a similar description here ok. So, let us try that.

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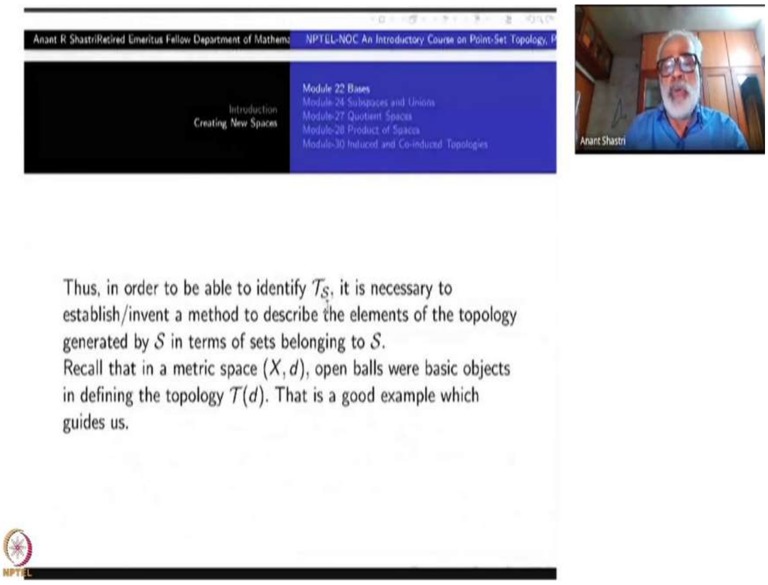

So, in order to be able to identify members of $\mathcal{T}_S$, it is necessary to establish or invent a method to describe the elements of the topology generated by $\mathcal{S}$ in terms of the sets inside $\mathcal{S}$, ok? Here $\mathcal{S}$ is given ok. So, here again you see the motivation comes from metric spaces. So, that is why certain things there we have done just in an ad hoc fashion perhaps. Whatever motivated them there, but now they will motivate in turn, the new constructions here.

Recall that in a metric space (X, d), open balls were basic objects in defining the topology for tau; this $\mathcal{T}(d)$ right, open balls right? That is a good example that will guide you, ok. So, how was the $\mathcal{T}(d)$ defined then? Arbitrary union of open balls, alright.

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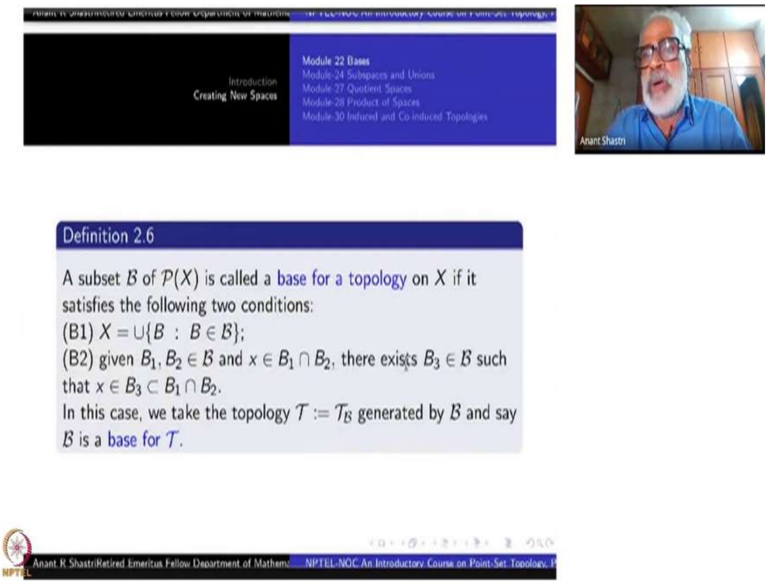

So, based on that we will do something now, namely we will define another a term here base for a topology. So, that was the title of the today's topic after all. A subset $\mathcal{B}$ of $\mathcal{P}(X)$; that means, the collection of subsets of X is called a base for a topology on X if the following conditions are satisfied. First condition is that if you take union of members of $\mathcal{B}$ that should cover the whole of X.

Second condition is that; given two members of $\mathcal{B}$ and a point x inside the intersection you must have a third member in $\mathcal{B}$ ok, such that x is inside B_3 contained in $B_1 \cap B_2$. When I say third member, I am not saying that they are all distinct no, but the point is that $B_1 \cap B_2$ itself may not be there in $\mathcal{B}$, that is the point ok. If B_1 is contained in B_2, then intersection will be just B_1 and so on. So, those are easy cases. So, you may have two different subsets they may not intersect, then also I do not have any botheration. But if they intersect for every point inside $B_1 \cap B_2$, I must have a member of B_3 such that that member is contained in the intersection and contains the point ok. So, pay attention to this one this may take a little more time, but look at the set of open balls in a metric space.

This is precisely, what we had proved there. If you take intersection of two open balls, it need not be an open ball, but every point inside it is contained in an open ball contained in the intersection, around each point there is an open ball contained in the intersection ok. So, that is why we have put this condition. So, that is the motivation from open balls in a metric space. Once condition (B1) and (B2) are satisfied, look at the topology generated by $\mathcal{B}$ now $\mathcal{T}_{\mathcal{B}}$ that is what the least topology which contains $\mathcal{B}$, then we say that this $\mathcal{B}$ is a base for $\mathcal{T}_{\mathcal{B}}$ ok.

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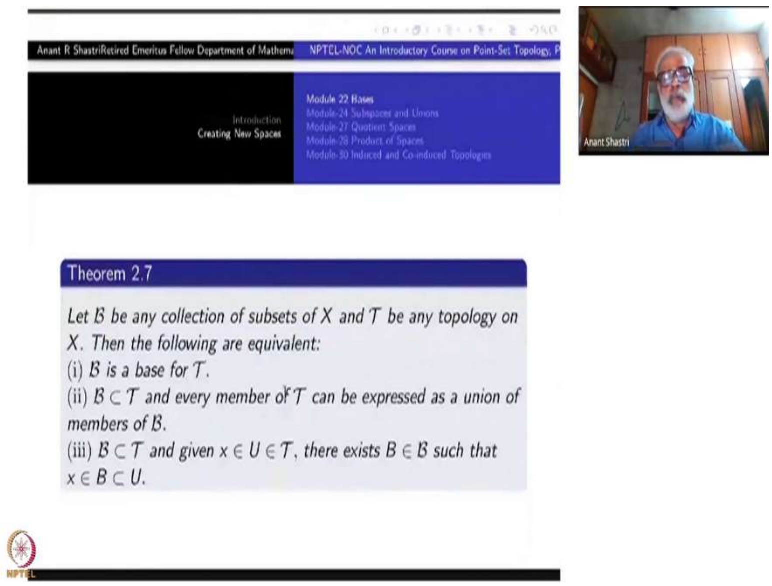

Now, here is a theorem first of all, to characterize what kind of collections will be a base for some topology, ok. So, start with a collection $\mathcal{B}$ and let $\mathcal{T}$ be any topology. Will $\mathcal{B}$ be a base for $\mathcal{T}$? This is the question we want to address now. So, this theorem says that the following three conditions are equivalent. The first condition is that $\mathcal{B}$ is a base for $\mathcal{T}$ that is our final goal.

The second condition is that first of all $\mathcal{B}$ is contained in $\mathcal{T}$, which is obviously necessary, if $\mathcal{B}$ is going to be a base for $\mathcal{T}$. And every member of $\mathcal{T}$ can be expressed as a union of members of $\mathcal{B}$. So, this is precisely the result that we proved for open balls. Sorry, In fact, it was the definition that we had for the topology from the open balls ok, for $\mathcal{T}(d)$ was precisely defined this way, right. So, that condition comes here.

Third condition is of course, $\mathcal{B}$ is contained inside $\mathcal{T}$ is common; and given any x inside U, where U is inside $\mathcal{T}$. $\mathcal{T}$ is what? $\mathcal{T}$ is the given topology. So, U is an open subset in this topology, I must find a member B inside $\mathcal{B}$ such that x belongs to B contained in U. These three conditions are equivalent. That is the statement of this theorem.

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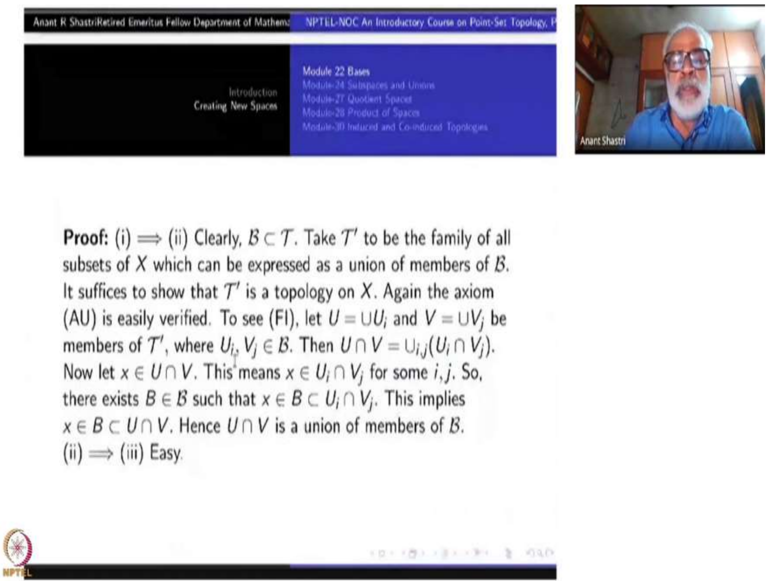

Let us go through this proof: (i) implies (ii) ok? If $\mathcal{B}$ is a base for $\mathcal{T}$, ok? By the definition, $\mathcal{T}$ has to contain $\mathcal{B}$. So, that is by definition. The second part is what? That every member of $\mathcal{T}$ can be expressed as a union of members of $\mathcal{B}$, ok? So, take $\mathcal{T}'$ to be the family of all subsets of X which can be expressed as a union of members of $\mathcal{B}$, ok?

Let this be the family; then I want to show that this $\mathcal{T}'$ itself is a topology. The proof is exactly similar to what we did for the open balls. There we had to work harder, here we have made whatever that work into a definition here; namely, the condition ok. So let us see $\mathcal{T}'$ be the family of all subsets of X which can be expressed a union of members of $\mathcal{B}$, ok.

So, why is $\mathcal{T}'$ a topology? The axiom (AU), (AU) is what; arbitrary union is easy, because union of union; union over the families of union is again another union. To see (FI), the finite intersection, suppose you verified it for two of them then again the intersection of unions will be unions of intersections; therefore, it is enough to verify it for only two of them anyway.

I have written complete proof here: $U = \cup_i U_i, V = \cup_j V_j$, ok? let them be members of $\mathcal{T}'$ that is by definition. What are U_i, V_j's? They are inside $\mathcal{B}$. Now, we will look at intersection, it is U_{ij}, ok intersection taken over all i and $j, U_i \cap V_j$. Now, suppose x is inside the intersection $U \cap V$ on the left hand side, then it must be inside one of these ok; this means $U_i \cap V_j$ contains x for some i, j.

So, there exists a B belong to $\mathcal{B}$, this is the part of the definition for $\mathcal{B}$ is a base, So, x belongs to B contained in $U_i \cap V_j$, it is a part of the definition for $\mathcal{B}$ to be a base.

So, that is what I am using here. So, this implies that x is in B and B is inside $U \cap V$, ok, because this is the union of $U_i \cap V_j$'s. Therefore, $U \cap V$ is a union of members of $\mathcal{B}$, every point is inside some B. So, it will be union of members of $\mathcal{B}$. So, therefore, it is an element of $\mathcal{T}'$.

Once this is a family is a topology,$\mathcal{T}_{\mathcal{B}}$ which is the smallest one will be contained inside $\mathcal{T}'$. So, that is every member of $\mathcal{T}$ can be expressed as a union of members of $\mathcal{B}$. Once it is contained inside $\mathcal{T}'$, what is $\mathcal{T}'$? Every member of it is expressed as a union of members of $\mathcal{B}$ ok? So, this proves (i) implies (ii).

Now, (ii) implies (iii). That is very straightforward because every member expressed as union of members of $\mathcal{B}$ implies given any x belonging to U, since U a union of members of $\mathcal{B}$, x must be inside some member of $\mathcal{B}$. And anyway $\mathcal{B}$ is contained inside $\mathcal{T}$ already. So, this part is very easy.

Once again (iii) implies (i) will be the hardest if at all. Even that is not difficult. So, let us go through that.

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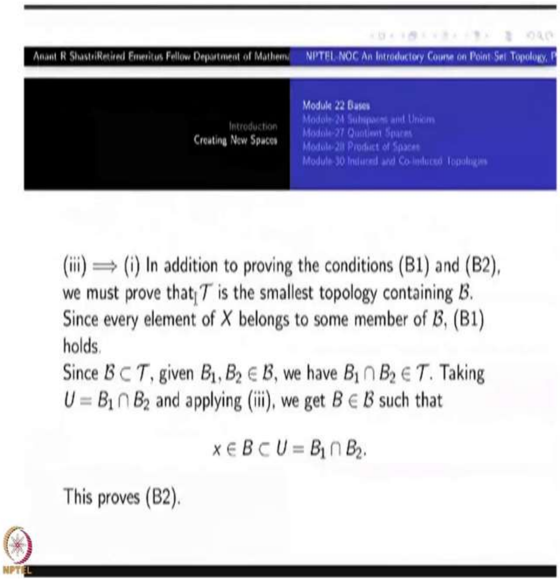

In addition to proving conditions (B1) and (B2), we must also prove that $\mathcal{T}$ is the smallest topology containing $\mathcal{B}$. So, this is the meaning of $\mathcal{B}$ is a base for $\mathcal{T}$. First of all, I prove that conditions (B1) and (B2) satisfied. Then we we prove the smallest topology containing $\mathcal{B}$ must be equal to $\mathcal{T}$.

So, do these two things. I have to show actually three things I have to show ok.

Since, every element of X belongs to some member of $\mathcal{B}$, ok? (B1) holds. Since $\mathcal{B}$ belongs to $\mathcal{T}$, given any x in X, there must be a member B of $\mathcal{B}$ such that x in B. This give (B1).

Secondly, $\mathcal{B}$ is contained inside $\mathcal{T}$ that is given right in (iii) that is given ok. $\mathcal{B}$ is contained inside $\mathcal{T}$, therefore, given B_1 and B_2 belonging to $\mathcal{B}$, their intersection will be a member of $\mathcal{T}$, because each of them is a member of $\mathcal{T}$. Now you take $U = B_1 \cap B_2$ and apply the condition (iii) for $U = B_1 \cap B_2$, you will get a B_3 such that x belongs to B_3 contained in $B_1 \cap B_2$, ok? So, (B2) also holds.

Finally, I have to show that $\mathcal{T}_{\mathcal{B}}$ is equal to the given $\mathcal{T}$ here, ok? $\mathcal{B}$ is contained inside $\mathcal{T}$. Therefore, $\mathcal{T}_{\mathcal{B}}$ is contained inside $\mathcal{T}$, because $\mathcal{T}_{\mathcal{B}}$ is the smallest topology. I have to show that $\mathcal{T}$ is contained inside $\mathcal{T}_{\mathcal{B}}$, ok? Only that one remains to be proved now here, ok?

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So, we want to show that $\mathcal{T}'$ be any topology on X such that $\mathcal{B}$ is contained inside $\mathcal{T}'$, ok? Then we want to show that this $\mathcal{T}$ is contained inside $\mathcal{T}'$. That will show that $\mathcal{T}$ is the smallest topology containing $\mathcal{B}$, that is $\mathcal{T}_{\mathcal{B}}$ and will be equal to $\mathcal{T}$ then, right?

So, given any element U in $\mathcal{T}$, for each x inside U, we have some B_x in $\mathcal{B}$ such that x belonging to B_x such that B_x is contained inside U. This is given to us by (iii). Therefore, U is the union of B_x, x belongs to U right? Note that, each B_x belongs to $\mathcal{B}$, but they are all members of $\mathcal{T}'$ as well. And $\mathcal{T}'$ is a topology. Therefore, the union of all B_x must be in $\mathcal{T}'$. So, starting with U which in an element of $\mathcal{T}$, I have shown that it is in $\mathcal{T}'$, that is all ok? So, $\mathcal{T}$ is the smallest topology containing $\mathcal{B}$; that means $\mathcal{T}_{\mathcal{B}}$ is equal to $\mathcal{T}$.

So, equivalence of (i), (ii) and (iii) is established.

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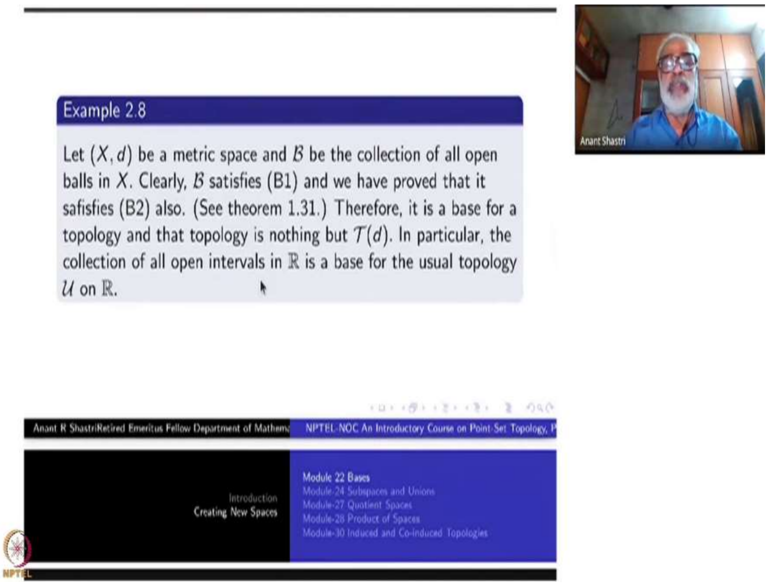

So, that is what I am telling here again, which I have told you already. If (X, d) is a metric space and $\mathcal{B}$ is the collection of all open balls in X, clearly, this $\mathcal{B}$ satisfies (B1) and we have proved that it satisfies (B2) also in that theorem ok, in chapter 1.

Therefore, it is a base for a topology and that topology is nothing but $\mathcal{T}(d)$. This much we had seen anyway. In particular, the collection of all open intervals in $\mathbb{R}$ is a base for the usual topology. I just want to recall all these things. They were the basic things, they were the motivating examples for us, ok?

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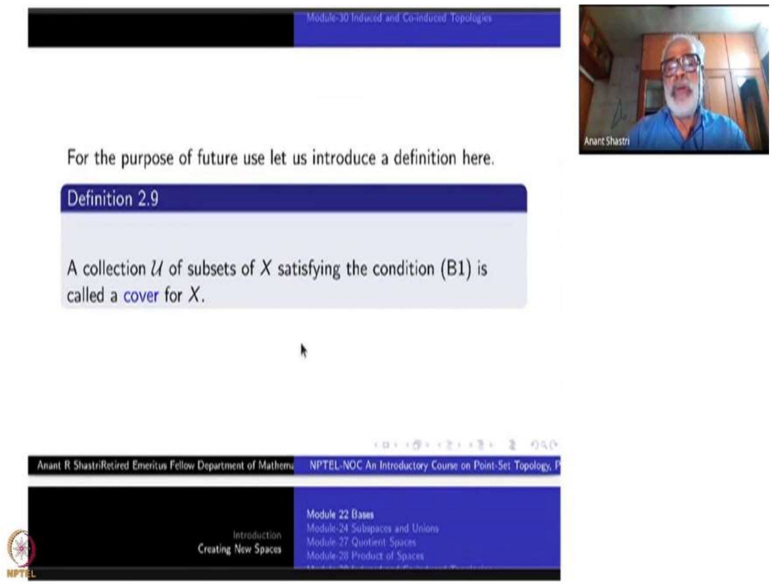

For the future purpose, only for future right? Now we do not need itright now. Pay attention to this condition (B1). We can make that as a separate definition, namely, any collection $\mathcal{U}$ of subsets of X satisfying the condition (B1) is called a cover for X, ok? Union of members of $\mathcal{U}$, if it is equal to the whole of X, then we say $\mathcal{U}$ is a cover for X, this we will use later on ok?

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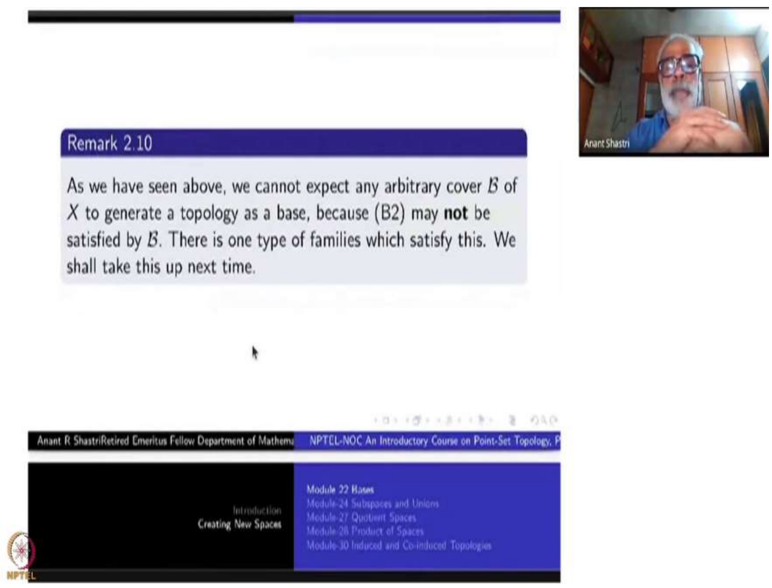

So, what we have seen above ok, we cannot expect any arbitrary cover $\mathcal{B}$ of X to generate a topology, we need condition (B2) also right? This (B2) may not be satisfied by a given family $\mathcal{B}$. However, there is one type of families which satisfy this property ok? It is stronger than (B2) that is why I, I want to call your attention to this alright. We will take up that one next time and that will lead to the concept of Subbases.

Thank you. So, we will meet next time.