Introduction to Point Set Topology, (Part I)
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Module - 21
Lecture - 21
Completion of a metric space

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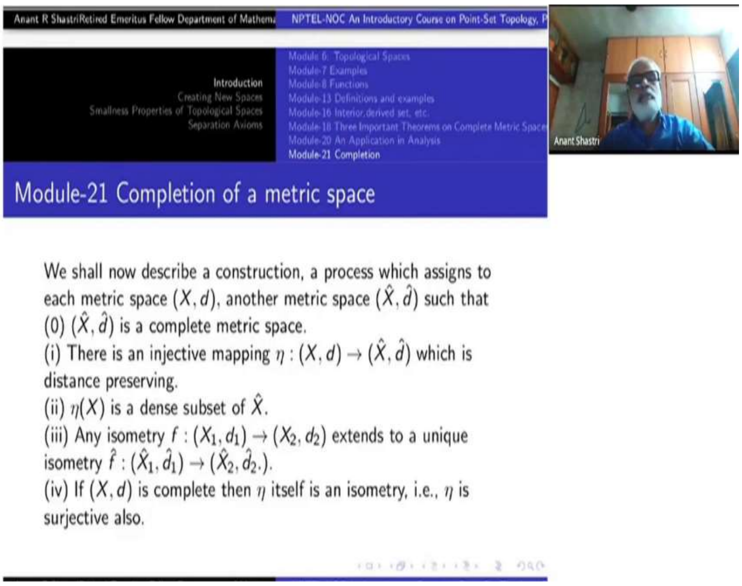

Welcome to module 21 Completion of a metric space. We have defined the concept of a complete metric space. So, we want to now define, what is the meaning of completion of a given metric space. So, we shall describe this construction, a process, which assigns to each metric space $(X d)$, another metric space $(\hat{X}, \hat{d})$, that is the notation we have set up, such that the first condition is this $(\hat{X}, \hat{d})$ is a complete metric space. Then it has to do something with the given metric space after all. So, all those conditions are put down, in this list 0, 1, 2, 3. 4.

So, there is an injective mapping eta from (X, d) to $(\hat{X}, \hat{d})$ which is distance preserving. In fact, any distance preserving map is automatically injective that we know ok. So, injectivity is emphasized here but that is a consequence of distance preserveness.

The image $\eta(X)$ is dense in $\hat{X}$. Any isometry f from (X_1, d_1) to (X_2, d_2) extends to a unique isometry $\hat{f}$ from $(\hat{X_1}, \hat{d_1})$ to $(\hat{X_2}, \hat{d_2})$. So, this is what I meant by process, instead of just taking $(\hat{X}, \hat{d})$ for one single space, it is a process ok. So, I cannot explain it more than that right now, but I will just throw a few words, what one calls a canonical or categorical, you know it is like a functor and so on ok.

So, that is this third part here. Any isometry f from X_1 to X_2 extends uniquely from $\hat{X}_1$ to $\hat{X}_2$. The fourth one is that if (X, d) is already complete then η itself is an isometry. In other words up to isometry, we are not doing any new thing here at all, it is the old (X, d) itself. Something is already complete then there is no more completing, you know (X, d) itself is its completion. But then why the word `isometrically'? Because the construction has to be there for more general metric spaces, then in the special case it will not be just (X, d), but some copy of (X, d) copy means what again isometrically.

So, this η itself is there the canonical embedding, that itself will be surjective that is all ok? It is already injective, it is an isometry and surjective, the inverse will be also an isometry. So, these 1, 2, 3, 4 explain ok the relation between $(\hat{X}, \hat{d})$ and the space (X, d) the one which we have started with ok? So, let us start with the construction.

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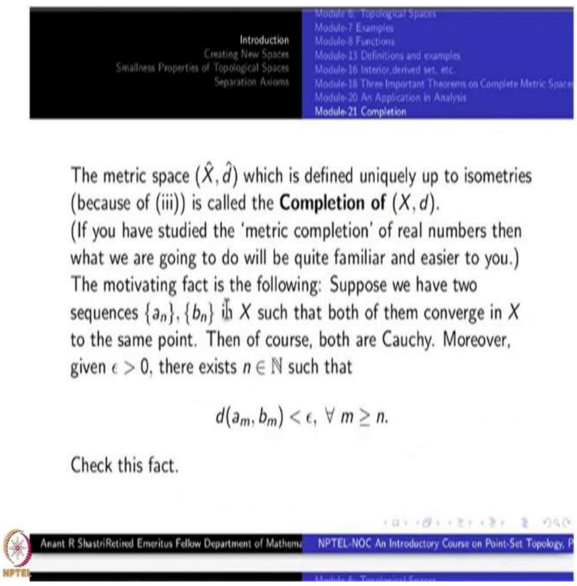

So, this $(\hat{X}, \hat{d})$ will be called completion of (X, d) that is what. One point I want to remind you if you have done the metric completion of $\mathbb{R}$ in your analysis, then the process that I am going to do is just half of that work, only 50 percent of that work. The metric completion of $\mathbb{R}$ starting with $\mathbb{Q}$ is exactly like this in the first half and then there is more work to do there, here we have less work to do.

On the other hand, if you have not done it, at least half of that part you will be learning today I will not do the other half, but later on if time permits, I will just explain how that is also done in the case of real numbers.

So, what I mean to say is here, you start with the rational numbers with the standard metric namely distance metric; that we know that is not complete right? When you complete it in this process what you get is the real numbers this part we will not be proved here ok?

We will only remark that the process of constructing the real numbers is similar to this one. That is what I want to emphasize that is all. So, the motivating fact is the following. Suppose we have two sequences; $(a_n), (b_n)$ in X such that both of them converge to the same point in X. Then of course, the first thing to note is that both are Cauchy sequences. Moreover both the sequences are coming nearer to that point after certain stage that is the meaning of that right? So, in particular they will be nearer to each other.

So, how do we even express that? For each $\epsilon > 0$, there exist n such that $m \geq n$ implies the distance between a_m and b_m (the same m ok) will be less than ϵ, ok? All that you have to do is you have to use triangle inequality here. So, I will leave it to you to verify this one. Ok? Two Cauchy sequences converging to the same point then they will be near to each other by this definition ok?

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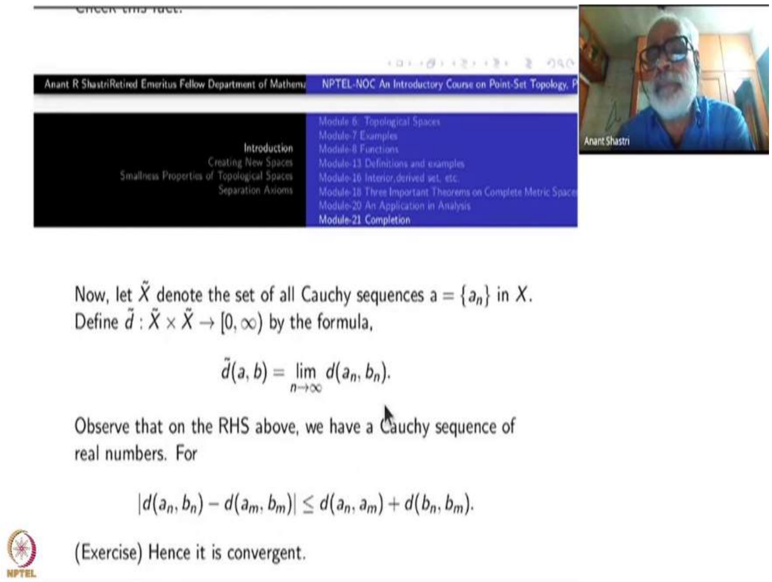

So, that is the starting point; which just means that you could take instead of elements of X you could take neatly chosen Cauchy sequences in X. Since there is no way to choose a particular Cauchy sequence what you would like to do is, take all sequences Cauchy sequences converging to a given point to represent that point. Now when that point is missing that what? See if that point is there for every Cauchy sequence then the space X is complete already.

When that point is missing, the Cauchy sequence will be there still. Look at all the Cauchy sequences, how to pick among them? They must be nearer to each other. So, that is how we arrive at this definition. Let $\tilde{X}$ denote the set of all Cauchy sequences in X. Now I define a function $\tilde{d}$ from $\tilde{X} \times \tilde{X}$ to 0 infinity by the formula $\tilde{d}(a, b)$, here $a = (a_n)$ and $b = (b_n)$ are now Cauchy sequences in X, take the limit of $d(a_n, b_n)$ as $n \to \infty$.

So, this is motivated by the observation: if (a_n) and (b_n) were converging to the same point then this limit of this distance would have been 0; so, that is our motive. In general what we do? We take this function to measure the distance between two Cauchy sequences. So, this is the measure that is what the $\tilde{d}(a, b)$. So, $\tilde{d}(a, b) = \lim_{n\to\infty} d(a_n, b_n)$.

Now, here I want to caution you why this limit exists? Both (a_n) and (b_n) are Cauchy sequences. therefore, whether they are convergent or not they are bounded ok; when both of them are bounded you can expect that this limit will exist alright? So, that is whole idea ok. Actually, you can directly verify this. Consider $|d(a_n, b_n) - d(a_m, b_m)|$ and use triangle inequality correctly, modulus of this difference is always less than equal to $d(a_n, a_m) + d(b_n, b_m)$.

Now, if n and m are very large we know that this is less than $\epsilon/2$ and this is also less than $\epsilon/2$, or both of them are less than $\epsilon/2$, ok? So, the sequence $d(a_n, b_n)$ is Cauchy. Here we are using the fact that $\mathbb{R}$ is complete. So, this will be a convergent sequence that is all.

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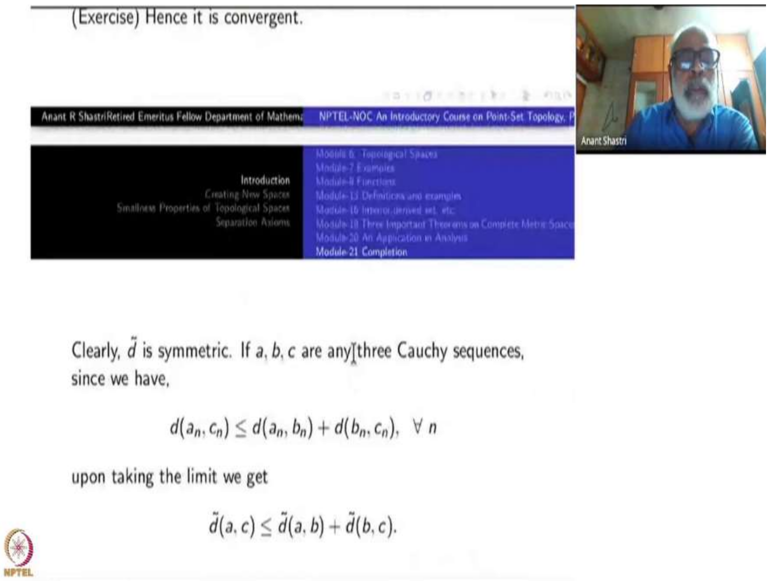

Now, by the very definition if you interchange b and a here, (b_n) and (a_n) will interchange, the $d((a_n), (b_n))$ is the same. So, limit will just be the same. That shows $\tilde{d}$ is symmetric. For each a, b and c; three sequences distance between a and c is limit of this one, but (a_n) to (c_n) you can write it as (a_n) to (b_n) plus (b_n) to (c_n). So, each term can be replaced by sum of two terms when we take the limit it is the limit of the sums right which is sum of the limits. So, therefore, what you get this triangle inequality.

$d((a_n),(c_n))$ is less than equal to $d((a_n),(b_n)) + d((b_n),(c_n))$. So, when you pass the limit you have to triangle inequality for $\tilde{d}$. Triangle inequality is true, symmetry is true, it is a non negative function. The only thing is why $d(a,b) = 0$ will imply $a = b$? That is not true obviously, not true right?

You can take any two seqeunces and you can change just one element in one of them, it will have the same limit. Say, the first element a_1 you change the limit will be the same thing. So, this is not true at all ok? However, the only thing that is not true is that the positive definiteness. Non negativity is still true; the definiteness part is missing here; the condition (d0) is missing herem ok? (d2), (d3) axioms for $\tilde{d}$ are true ok?

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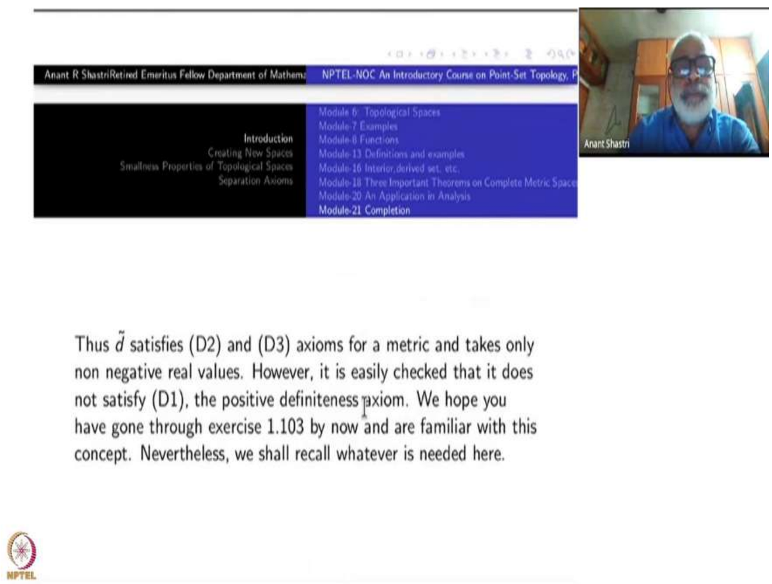

So, the only thing which needs to be doe now, this is one of the exercise given to you earlier. If we have spent sometime on that, even if you have not solved completely, these things should be easy for you to understand. Now, in any case whatever is needed here in the proof of this theorem I am going to explain it now, Ok?

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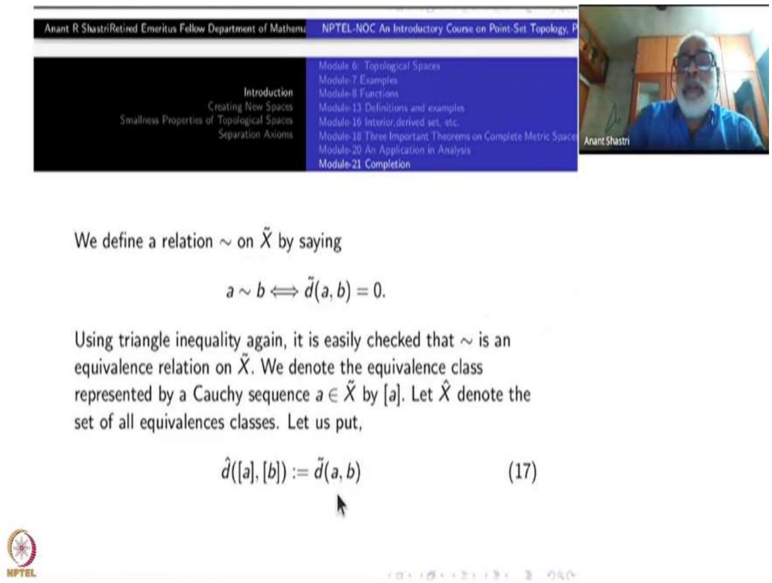

So, what we are going to do? We introduce a relation on $\tilde{X}$ by saying that a is related to b if and only if $\tilde{d}(a,b)$ is 0; just because $\tilde{d}(a,b)$ is 0 does not imply a and b are equal. So, we are making this relation, otherwise there was no need. Using triangle inequality again, you can check that this is an equivalence relation.

Symmetry is obvious by the definitionbecause $\tilde{d}$ is symmetric function ok? Reflexivity is obvious $\tilde{d}(a,a)$ is always 0 alright? If $\tilde{d}(a,b)$ is 0 and $\tilde{d}(b,c)$ is 0 then you should see that $\tilde{d}(a,c)$ is also 0 by triangle inequality, that is all.

So, this relation is an equivalence relation. Let us denote the equivalences classes by square bracket a i.e. $[a]$ the class of a the equivalence class of a ok? So, all Cauchy sequences represented by this $[a]$, they are all equivalent to each other in the sense that $\tilde{d}(a,b)$ is 0 for all b in this class ok? Let us denote the set of all these equivalence classes by $\hat{X}$. $\hat{X}$ denotes the set of all equivalence classes.

Now, I want to take this definition $\hat{d}$ on two equivalence classes is nothing but $\tilde{d}(a,b)$; where a and b are representatives. To say that this is well defined, what I have to do? I have to show that this right hand side is independent of what representatives I choose. So, if they are all same then this left hand side can be defined to be equal to right hand side. That will makes sense. Otherwise, if $\hat{d}(a,b)$ keeps changing as you change the representative then it is will not be well defined right? Well definiteness means that, whenever you have a choice in the definition you have to verify that the whatever you have taken on the right hand side is independent on the choice.

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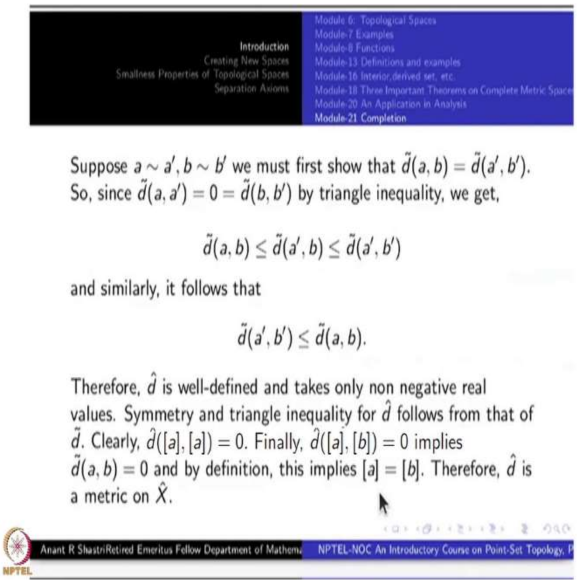

So, suppose a is equivalent to a', b is equivalent to b', we must show that $\hat{d}(a,b)$ is the same as $\hat{d}(a',b')$. $\tilde{d}(a,a')$ is 0 that is the meaning of a is equivalent a'. Similarly, $\tilde{d}(b,b')$ is also 0 right, because b is equal to b'.

Again by triangle inequality $\tilde{d}(a,b)$ is less than or equal to $\tilde{d}(a',b) + \tilde{d}(a,a')$ which is equal to $\tilde{d}(a',b)$. So, we have changed from a to a'. Now I change b to b', $\tilde{d}(a',b)$ is less than or equal to $\tilde{d}(a',b') + \tilde{d}(b',b)$ which is equal to $\tilde{d}(a',b')$.

So, this is less than or equal this one. So, one is less than or equal to the other. But this relation is symmetric. I can start from here with $\tilde{d}(a',b')$ and show that it is less than or equal to $\tilde{d}(a,b)$ right? So, they are equal. Now that is what we wanted to show. We see all once, the idea is there all verification is very canonical very easy; therefore, $\hat{d}$ is well defined and takes only non negative real values why? Because $\tilde{d}$ has that property; now $\tilde{d}$ is symmetric also it satisfied triangle inequality also.

Therefore $\hat{d}$ will also satisfy triangle inequality; The new thing is definiteness. $\hat{d}([a],[a])$ is also 0, that is also clear, but $\hat{d}([a],[b])$ is 0 implies that a and b are equal that is just the definition; that is just the definition. Equality means what here? They are equivalent, this equivalence class is same. So, this is a trick we are using to get equality ok. So, here equality is just equivalence classes alright.

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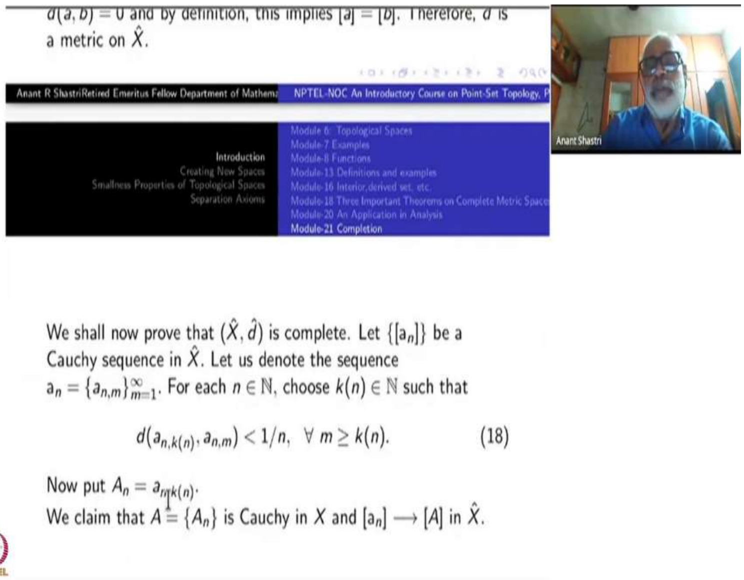

So, $\hat{d}$ becomes a metric on X hat alright. So, the first part is over. We have constructed this metric space $(\hat{X},\hat{d})$. Now take $[(a_n)]$ to be a Cauchy sequence in $\hat{X}$. Each (a_n) itself is is a Cauchy sequence in X, right? Equivalence class is one single element. Now you take a Cauchy sequence of them, ok? That means, a Cauchy sequence in $\hat{X}$.

Let us denote the sequence (a_n), (a_n) is a sequence now ok, by this $a_{n,m}$, so, multi index you have to use for each fixed n take $a_{n,m}$ for m from 0 to infinity. Some people write it with m on the top, but I do not want to do that because that may cause confusion. If they are numbers then the power has some other meaning, ok?

So, for each n, choose $k(n)$ such that $d(a_{n,k(n)}, a_{n,m})$ is less than $1/n$. So, this is where I am using the fact that each (a_n) is a Cauchy sequence ok. So, it will be less than $1/n$. When for some choice of $k(n)$, for all m bigger than this one this will be true. So, this is where I have used that each n, $(a_{n,m})$ is a Cauchy sequence ok? It is the first step.

Now, look at the sequence A_n equal to $(a_{n,k(n)})$; $a_{n,k(n)}$ is one element of (a_n), of the sequence Cauchy sequence (a_n). What have I done? What is the choice of $k(n)$, it should satisfy this condition ok? For $n = 1, k(1)$ is such that $d(a_{1,k(1)}, a_{1,m})$, where m is bigger than $k(1)$ will be less than 1. For n equal to 2, it will be less than $1/2$; successively you have to make it nearer and nearer to 0, alright?

Claim is that this sequence in X ok? (A_n) is a sequence in X, is Cauchy in X and this (a_n) the equivalence class of this sequence inside $\hat{X}$ ok? $\hat{X}$ is what? Equivalence classes of Cauchy sequences, the square brackets here ok, converges to the class of $[A_n]$.

I started with a Cauchy sequence in $\hat{X}$, I am producing a Cauchy sequence in X, its class is an element of $\hat{X}$ anyway, the class of the Cauchy sequence. So, that will the limit of this one, that is the meaning of $(\hat{X}, \hat{d})$ is complete; every Cauchy sequence is convergent. So, this is the claim ok? I have to assume that, I have to prove this one.

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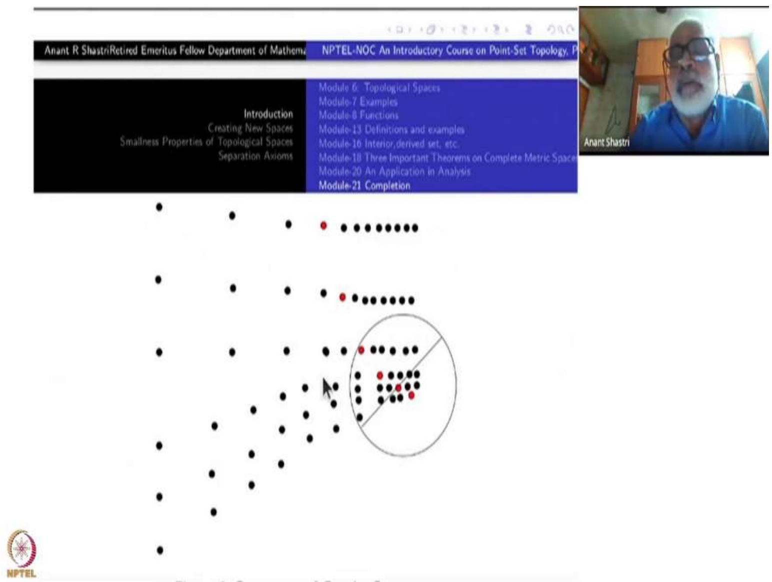

So, most of the work goes here in showing that it is complete here alright. So, let us carefully see that how this is coming out. So, here is a picture: this is your (a_1), I have chosen this red point here somewhere this is $a_{1,k(1)}$, this is (a_2) this is $a_{2,k(2)}$; $a_{3,k(3)}$ and so on $a_{n,k(n)}$ is the sequence (A_n). So, these are all original sequences $(a_1), (a_2), \ldots, (a_n), \ldots$What is being claimed is that this sequence (A_n) is the limit. All these sequences, after a certain stage are here in this, some ϵ-ball here, actually this is an $\epsilon/2$-ball.

So, this radius is $\epsilon/2$, all of them are within this, the tail end of each one of them will be here. So, some initial elements may be omitted; many things here omitted, but the tail ends of the tail ends are here that is the meaning of this given ϵ—ball. Now, this is what we have to prove. This is a picture you can keep it in mind that is all, alright.

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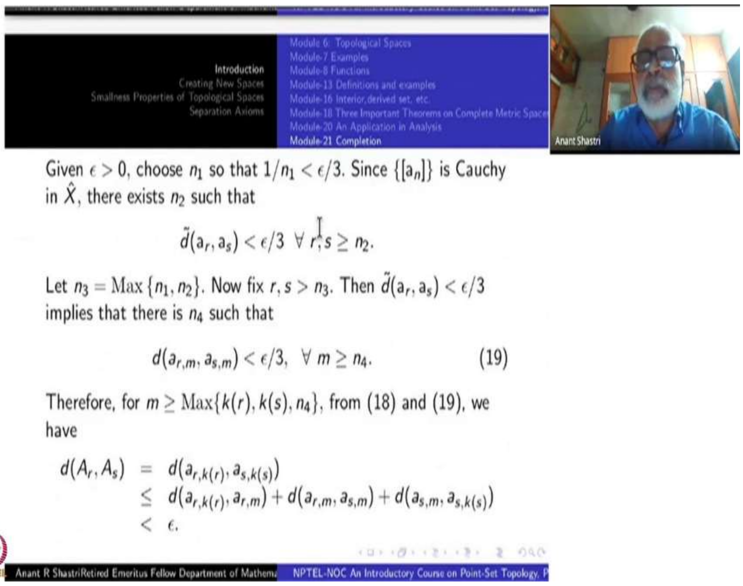

So, given epsilon positive, you have to find a number which fits all the sequences after that stage in both ways that is how I have to choose. The first step is given ϵ, choose the n_1 itself such that $1/n_1 < \epsilon/3$; you can make it less than $\epsilon/4$ also if you want to bit careful ok. So, here there is nothing new. This is just the property of real numbers.

Since (a_n) is Cauchy in $\hat{X}$, ok, there exists an n_2 such that $\tilde{d}((a_r),(a_s)) < \epsilon/3$ for every $r, s \geq n_2$; remember this $\hat{d}$ was nothing but $\tilde{d}$ of the representatives ok? These are classes; if we put a bracket here; $\hat{d}([(a_r)],[(a_s)]) < \epsilon/3$ is the statement for $r, s \geq n_2$.

But I do not have to put hat here I can directly put $\tilde{d}((a_r),(a_s))$; remember (a_r) and (a_s) are themselves Cauchy sequences. And what is the definition of $\tilde{d}$? $\tilde{d}$ was the limit as n tends to infinity of $d(a_{r,n}, a_{s,n})$ of the original distance between $a_{r,n}$ and $a_{s,n}$ ok? So, finally, everything is happening in the metric space X, in terms of d.

Now, you take the maximum of this n_1 and n_2 ok? that is n_3; I want to show that (A_r) is a Cauchy sequence first of all; I have not yet shown that. Without that it will not be an element of $\tilde{X}$. So, it will not be inside $\hat{X}$. The first thing is to show that (A_n) itself is a Cauchy sequence; that means what? For any $\epsilon > 0$, I must find some integer n_3 such that the distance between A_r and A_s is less than ϵ right? Or $\epsilon/3$ whatever you want to say for $r, s \geq n_3$. So, that is what I have to show.

The fix r and s to be bigger than n_3. For all of them I must show that $d(A_r, A_s)$ where d is the given metric on X, these are points of X now right, is less than epsilon ok? So, how do I show that? So, starting with these choices see r and s are bigger than n_3.

So, they will be bigger than n_1 and n_2. So, both these hypotheses will be applicable $\tilde{d}((a_r),(a_s))$ is already less than $\epsilon/3$; because r, s are bigger than n_2 ok? This implies that there exists an n_4 now I am using the definition of $\tilde{d}$; because it was obtained by taking limits right? The limit is less than $\epsilon/3$ that is what I have.

So, $d(a_{r,m}, a_{s,m})$ for large enough m is less than $\epsilon/3$. r and s are fixed here. The variable here is the index m, ok? But m is sufficiently large namely bigger than n_4, ok? So, this is one equation ok? Now you choose m to be bigger than what? m is to be bigger than some numbers, I can choose it even bigger than all of them viz., m to be bigger than $k(r), k(s)$ and n_4 ok?

Then you can apply both this (18) and (19) here. See for this one, m should be bigger than $k(n)$, right? So, I am applying it for $n = r$. So, it must be bigger than $k(r)$. Similarly, applying it for $n = s$, viz, for m bigger than $k(s)$. So, I am taking m to be bigger $k(r), k(s)$ as well as bigger than n_4 also ok? So, both (18) and (19) are available to me. So, I just use triangle inequality. Now $d(A_r, A_s)$. What are A_r and A_s? By definition, $a_{r,k(r)}$ and $a_{s,k(s)}$.

So, we are in the metric space X right? So, triangle inequality right? Three quantities on the RHS, from $a_{r,k(r)}$ to $a_{r,m}$ where m is chosen to be larger than all these, plus $a_{r,m}$ to $a_{s,m}$ plus from $a_{s,m}$ come back to $a_{s,k(s)}$. So, this is triangle inequality ok? From (18) these first and the last one are less than $\epsilon/3$. From (19) this middle one will be less than $\epsilon/3$.

So totally, it is less than ϵ. So, (A_n) is a Cauchy sequence. That is what we have proved.

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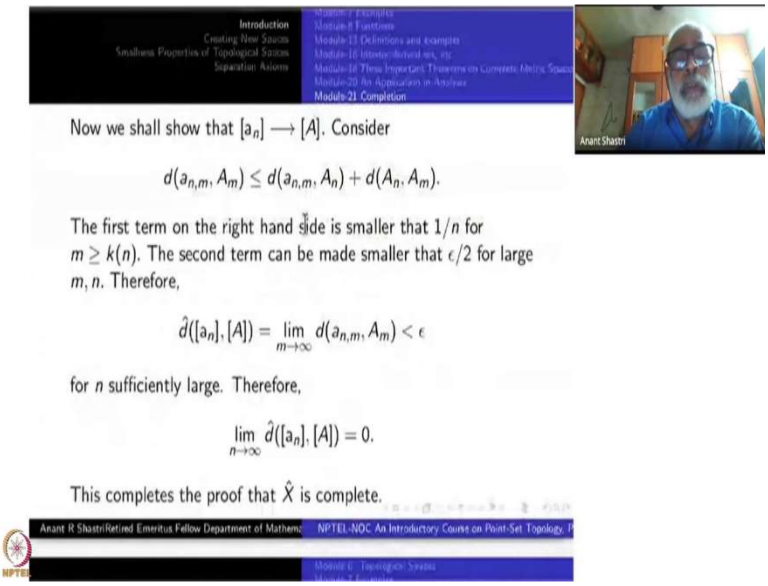

The next task is that the original sequence of classes $[(a_n)]$ converges to the class of A that is what I have to show ok? Look at $d(a_{n,m}, A_m)$, what is A_m? Remember this one it is little $a_{m,k(m)}$, right. So, that is a whatever it is this is a number this this is just an element of X, they are all elements of X.

So, $d(a_{n,m}, A_m)$ make sense which is less than equal to $d(a_{n,m}, A_n) + d(A_n, A_m)$. Now this is a just triangle inequality ok. Now the first term on the right hand side this one is less than $1/n$ for $m \geq k(n)$ right; this $k(n)$ was chosen such that look at this very first choice $d(a_{n,k}, a_{n,m})$ is less than $1/n$. So, that I am applying here, ok?

So, this term is less than, as soon as m is bigger than $k(n)$ this term is less than $1/m$ alright. This second term can be made smaller than any anything once you choose n and m large enough; because just now we have shown that this is a Cauchy sequence.

Therefore $\hat{d}([(a_n)], A)$, the class A which is by definition $\tilde{d}([(a_n)], A)$ which is again by definition the limit as m tends to infinity of d of this sequence $(a_{n,m})$ and the other sequence (A_m). Now here m tends to infinity ok? Each term is less than ϵ after a certain stage, so, limit will be also less than ϵ, ok. So, for this will apply when n is large enough because I have an n here ok. So, this is $\epsilon/2$, I have to make this n less than $\epsilon/2$ to make this to less than ϵ. So, n must be large enough how large: $1/n$ should be less than $\epsilon/2$, that is all.

Therefore if this is true for every ϵ, look at here this is one sink one one single number here this less than ϵ for every ϵ. Therefore, the limit as n tends to infinity, this part $\hat{d}([(a_n)], [A])$ ok is 0 for large n, this is less than ϵ. So, you should take the limit of this one this will be 0; that is precisely the statement that this this sequence converges to this one ok. When does a sequence (x_n) converge to x in a metric space? Limit of the distance between x_n and x is 0 ok?

So, what we have done? we have just completed the construction and property 0; that $\hat{X}$ is a complete metric space. Now we have to verify (i), (ii), (iii), (iv) they are all easy alright. Hard work is already done.

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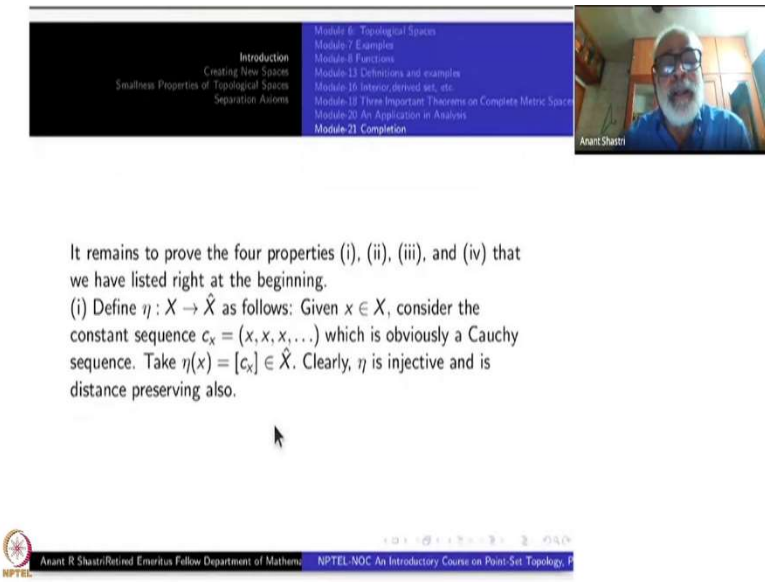

So, it remains to prove these four properties; (i), (ii), (iii), (iv) ok. So, first thing is I have to define η, η from X to $\hat{X}$ that is what I have to define. Given any x belonging to X, consider the constant sequence $c_x = (x, x, x, \dots)$ all the nth term is x for all n. Look at that sequence. Obviously, it is a Cauchy sequence, take its class that is an element of $\hat{X}$, ok?

So, this way I get the function; $\eta(x)$ is the class of c_x; obviously, η is injective ok? Why? Suppose $\eta(x)$ is equal to $\eta(y)$, then c_x is in the same class of c_y. What is the meaning of that? That distance the limit of that you see you have to take the limit is equal to 0, but limit is of the sequence $d(x, y), d(x, y), d(x, y), \dots$ the limit itself is $d(x, y)$. That is 0 means x equal to y. Remember X is a metric space ok?

So, this η is clearly an injective mapping ok. Actually we shall prove that it is distance preserving. What is the meaning of this? Take two points x and y, then c_x and c_y distance between them also $d(x, y)$. So, that is the meaning of this distance preserving $\hat{d}([c_x], [c_y])$ is equal to $d(x, y)$ because both of them are constant sequences ok. So, it is automatically distance preserving and hence injective mapping.

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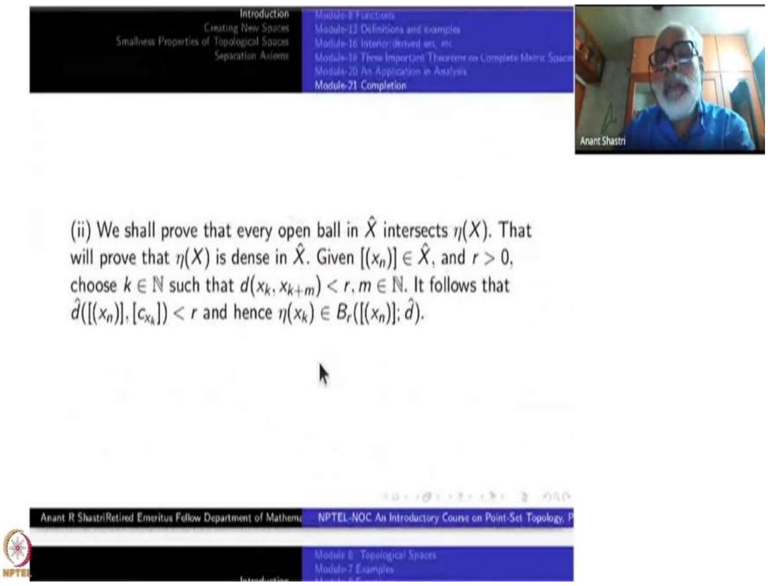

What is the second thing here? I forgot. What is it? η is dense inside $\hat{X}$, the image of η is dense in $\hat{X}$. Take any open ball in X hat you must find a point $\eta(x)$ inside that and intersect with $\eta(X)$. So, that is the meaning of density of $\eta(X)$, ok?

So, here it is. We shall prove that every open ball in $\hat{X}$ intersect $\eta(X)$, ok? Given (x_n) a sequence Cauchy sequence and its class ok, take r positive and take an open ball around that of radius r. I want to get an element of the metric space X, so that the constant sequence, the class of the constant sequence is in that ball; that means, its distance from (x_n) that must be less than r.

So, for that choose $k \in \mathbb{N}$ such that, this is a Cauchy sequence right, $d(x_k, x_{k+m})$ is less than r for all m, ok? Now it follows that once you have chosen k like this, look at the constant sequence c_{x_k}. That sequence has distance less than r from (x_n), ok. So, that is why this $\eta(c_{x_k})$ belongs to this open ball ok?

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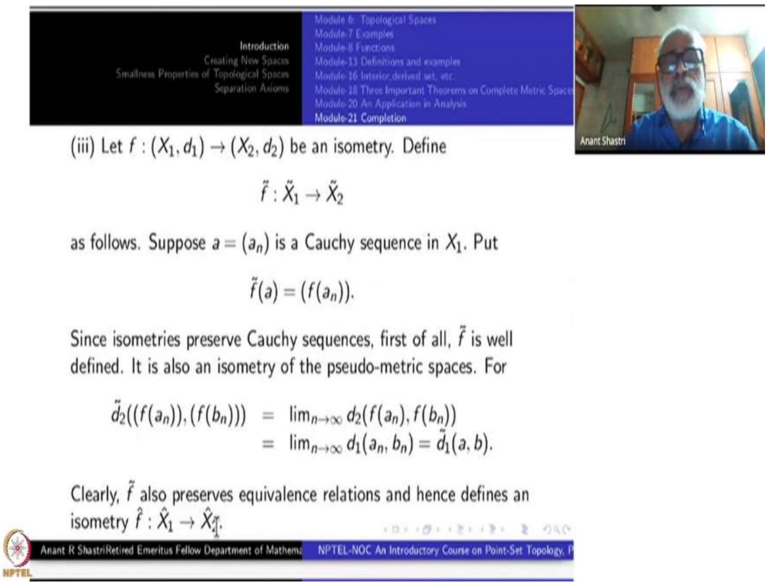

Finally take any two metric spaces and an isometry f, that isometry itself extends to the whole of $\hat{X}_1$ to $\hat{X}_2$. This is stronger than saying that $\hat{X}_1$ is unique up to isometry. That is the meaning of this. That is what we want to prove, but we are proving instead that the given isometry extends to an isometry, ok.

So, how do we prove that? Isometry means distance preserving. Look at the construction of $\hat{X}_1$ and $\hat{X}_2$. They just depend upon the metrics there. That is all. Under an isometry, (even under an equivalence of metrics) Cauchy sequences are preserved. And then equivalence classes are preserved equivalence and finally, distance between these equivalence classes is also preserved. So, that is what you have to take a sequence Cauchy sequence in $X_1, \hat{f}$ of a which is $f(a_n)$'s that sequence that will be Cauchy inside X_2.

Even just a similarity would have given you this one right? Here we have an isometry ok. Since isometries preserve Cauchy sequences first of all, $\hat{f}$ is well defined from $\tilde{X}_1$ to $\tilde{X}_2$, ok. Here take a Cauchy sequence, f of that will be a Cauchy sequence here ok? Now you have to see that this $\hat{f}$ is an isometry of the corresponding $\tilde{d}_1, \tilde{d}_2$ the pseudo metric spaces the $\tilde{d}_2$ distance is preserved by this $\tilde{f}$. $\tilde{d}_2(f(a_n), f(b_n))$ by definition is the limit as n tends to infinity.

$\tilde{d}_2(f(a_n), f(b_n))$ is what? Each $f(a_n), f(b_n)$ is the same thing as d_1 distance between these one $d_1(a_n, b_n)$ right because f itself is a isometry its this limit $\tilde{d}(a,b)$ the given sequence a_n, b_n. So, that is the meaning of that f is distance preserving from d_1 to d_2. From $\tilde{d}_1$ to $\tilde{d}_2$, ok clearly $\hat{f}$ also preserves equivalence relations.

And hence defines an isometry $\hat{f}$ from $\tilde{X}_1$ to $\tilde{X}_2$, why this preserves equivalence? What is the equivalence? (a_n) is equivalent to (b_n) if the limit of distance between a_n and b_n is 0 right, distance between $f(a_n)$ and $f(b_n)$ also 0 in the limit. So everything is preserved that is what you have verify. So, you get a map from $\hat{f}$ from $\hat{X}_1$ to $\hat{X}_2$ here the class goes to the class there, ok.

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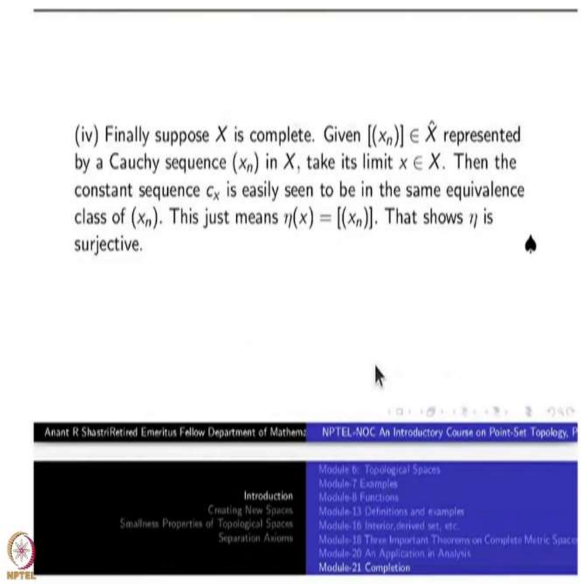

And the distance $\hat{d}$ in same thing as distance we have defined for representative. So, they they are preserved. So, it is an isometry all already. So, the net conclusion here is the construction depends only on the isometry class of X. If X_1 and X_2 are isometric, $\hat{X}_1$ and $\hat{X}_2$ are isometry, ok.

The last one is: suppose X itself is complete, then I have to prove that this η is surjective. Then eta itself will be an isometry ok. So, why it is surjective? Take any (x_n) in $\hat{X}$, ok? The class represented by a Cauchy sequence x in X that is the meaning of this $\hat{X}$, right. Since X is complete it has a limit this n has a limit this limit is independent of the representative Cauchy sequence of the equivalence class. All the sequences whatever sequence you take here ok? Representing this one they will all be convergent to the same point x in X.

Because the distance limit of distance between (x_n) and (y_n) that itself will be 0. So, if limit of (x_n) is x and that one is y, then x will be equal to y. So, there is a unique limit here inside X, ok? Now it is very easy to verify that the constant sequence c_x itself is in the equivalence class of (x_n).

So, you have to show that these two are equivalent what is the meaning of that? Distance between (x_n) and this c_x that is tends to 0, but that is the definition of that (x_n) converges to X, ok. So, that is the meaning of this one. So, this just shows that $\eta(x)$ is equal to the class (x_n). So, η is surjective ok?

So, that more or less completes several things that we have in mind about metric spaces; that does not mean that the study of metric space is completed there are many more things to come. But now onwards we will concentrate more and more on general topological spaces, bringing in metric spaces only to strengthen the results that the topological results will give you about metric spaces ok.

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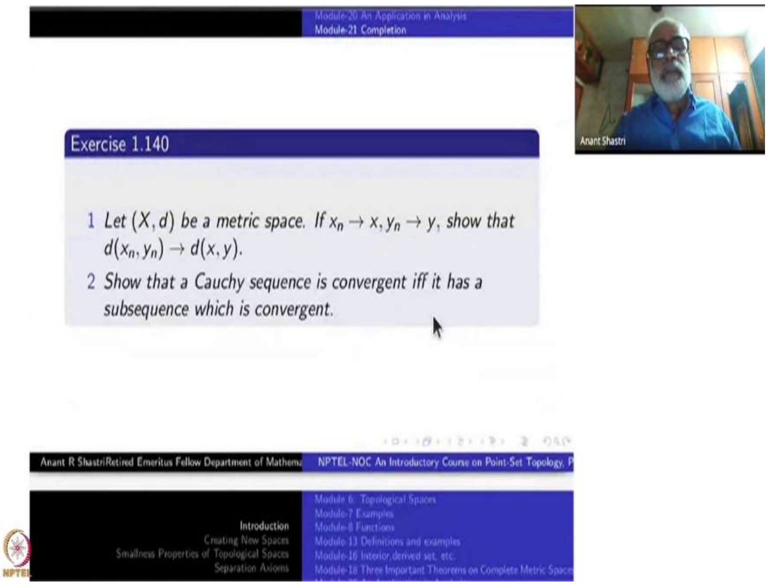

Any questions? Here are some exercises all straightforward exercises nothing to do with completion of course, ok. Just ordinary exercises this is about the product finite product of metric spaces alright.

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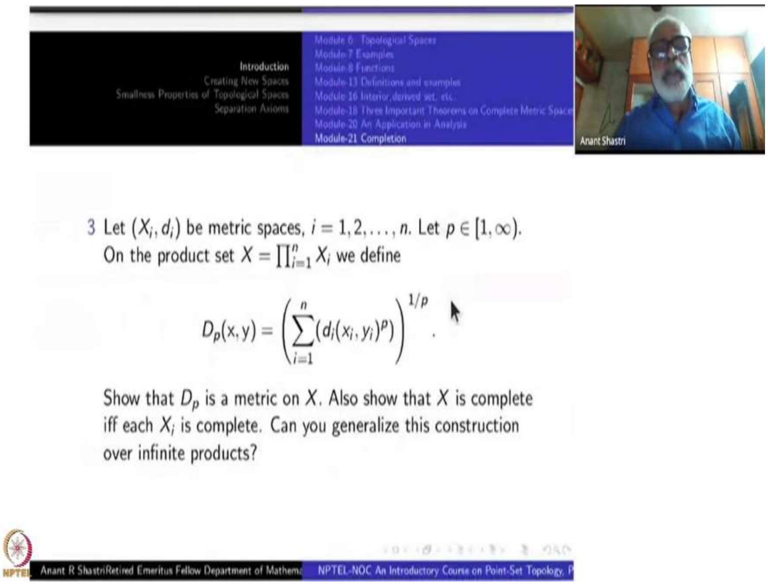

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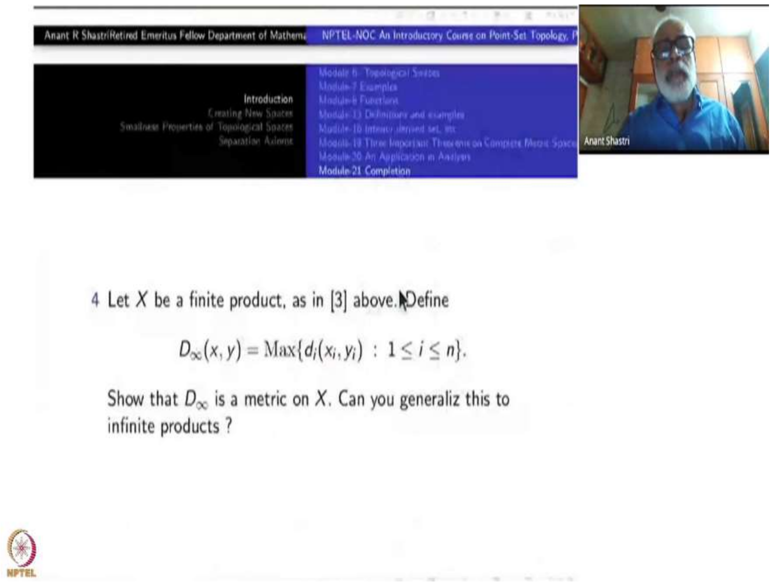

This is about finitely many metric spaces. Then you what to want to take the maximum metric for the product set. Just like ℓ_∞ norm you can do ℓ_∞ kind of construction here. You have D_∞ metric here so, D_∞ corresponding to maximum that infinity symbol ok. So, that is the next chapter now.

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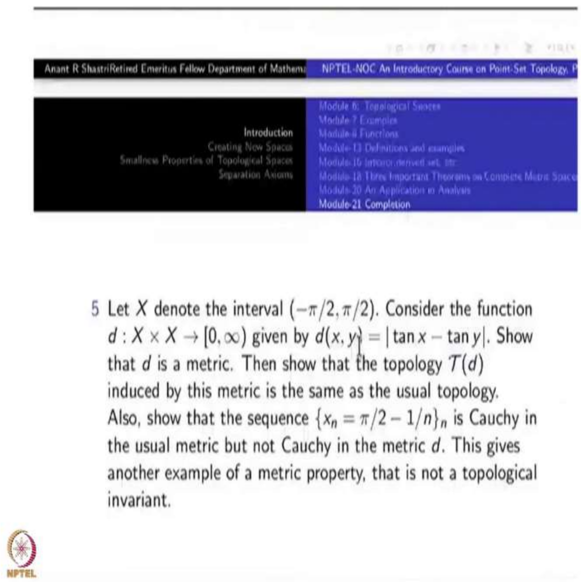

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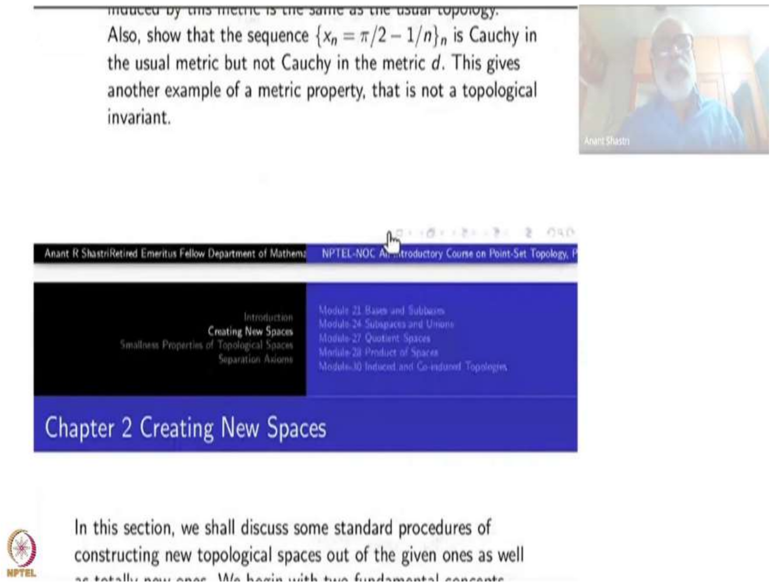

If there are no questions we will stop here.

Thank you.