Introduction to Point Set Topology, (Part I)
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Module - 20
Lecture - 20
An Application in Analysis

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Welcome to module 20 of Point Set Topology Part 1. So, today we will discuss one small application of the Baire's category theorem. As I have told you all these three big theorems in matric spaces have many many applications in analysis. For example, the Banach contraction mapping principle is used in the existence and uniqueness of initial value problems, boundary value problems and so on in differential equations.

So, instead of calling them theorems, people who like to call them as principles because they are often the principle rather than just final result stated in the theorem ok? So, today we will just give you a small application in elementary analysis. Since we cannot discuss the big applications of this in function analysis, in this course.

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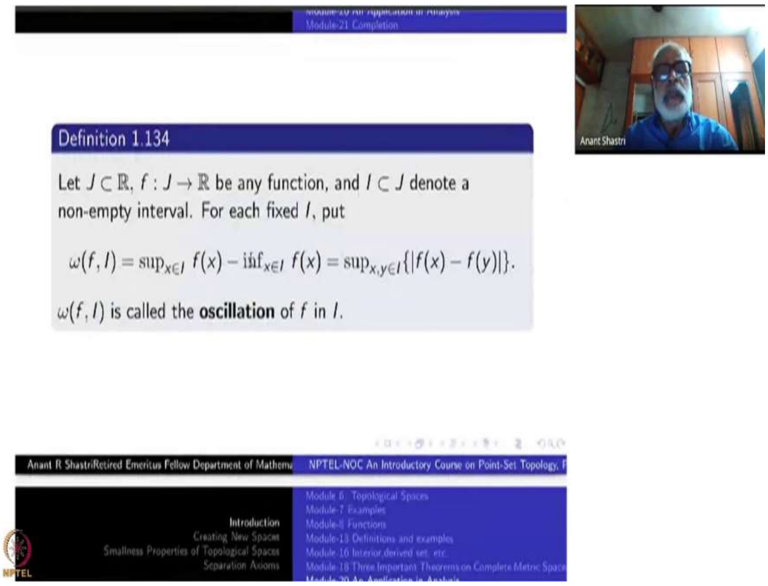

You start with a domain J inside $\mathbb{R}$, take a function any function real valued function. Take subsets of J which are open intervals, non-empty open intervals. Those things I will denote by I, J need not be any open subset of $\mathbb{R}$ will do, or even a close subsets of $\mathbb{R}$ like a closed interval or whatever. The function has to have a domain after all.

For each fixed non empty open interval I contained in J, let us put this notation $\omega(f, I)$ which is the difference of the supremum and the infimum of $f(x)$ taken over the entire of I. Do you take the supremum and infimum first and then take the difference, or you can just look at $|f(x) - f(y)|$ and take the supremum, these two quantities are the same, as x and y freely range over I, I is fixed here.

But each I, I have this number ok, may be this could be infinity also, I do not mind ok? ∞ is also allowed alright. So, only thing is I am assumed that I is non empty so that the RHS is not something like $-\infty + \infty$ etc. which may not make sense. So, this always makes sense. So, this quantity is called the oscillation of f in I. So, you can say that this measures the difference between $f(x)$ and $f(y)$ how large it could be. So, that is oscillation.

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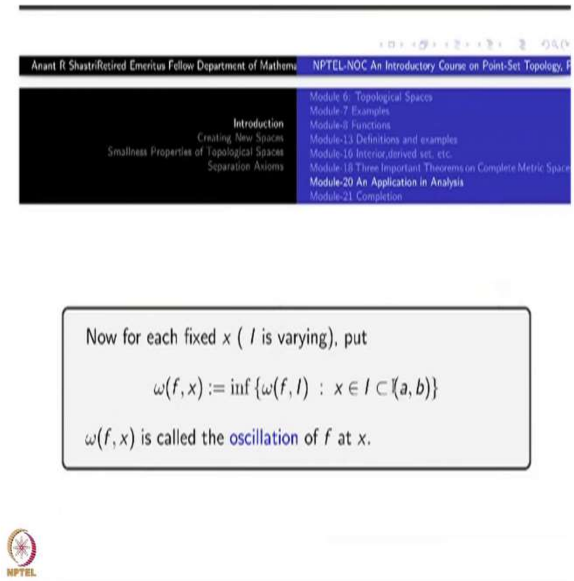

Now, for each fixed x, you take all neighborhoods I of x which are interval. So, open intervals around I, ok? Put $\omega(f,x)$ equal to infimum of all this $\omega(f,I)$; where I ranges and now x is fixed, ok? So, these I are all subsets of the domain J.

Student: This (a,b) should be J, here?

(a,b) could be J yes. If so you know, depends upon what you have taken for the domain of f. Normally, it does not matter because you will have to take I smaller and smaller. So, it is around x that is what is important. So, $\omega(f,x)$ is called the oscillation of f at x.

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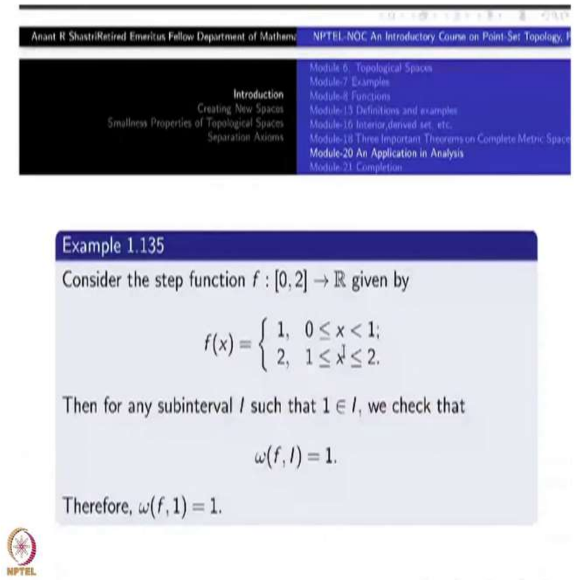

For example, I will just give you a very simple minded example, look at a step function. So, here I am taking the domain as $[0, 2]$ ok, it does not matter what J you take, does not matter, but I want to include a step function. So, it should include some points around the point where the breaking occurs, I believe that you all know step function. So, here is 1; $f(x)$ is equal to 1 in the interval $[0, 1)$ open, it is equal to 2 from closed interval $[1, 2]$.

So, at 1, when x equal to 1, there is a discontinuity here you can see, ok? Then take any sub interval I around 1, ok? Look at all such sub intervals open subset you have to take which contains I, then $\omega(f, I)$ will be exactly 1 ok? Now, if you take the infimum it will be also 1, ok. So, $\omega(f, 1)$ will be 1; remember $\omega(f, x)$ is the infimum of our $\omega(f, I)$ ok.

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Here is a picture what happens, this is a step function at $(0, 1)$, it is this 1 ok, I have deliberately changed this thing here. You could send this point here or above, does not matter. The modulus of the difference ok? For any point here, the value the difference is always 1. So, the infimum is also 1 ok.

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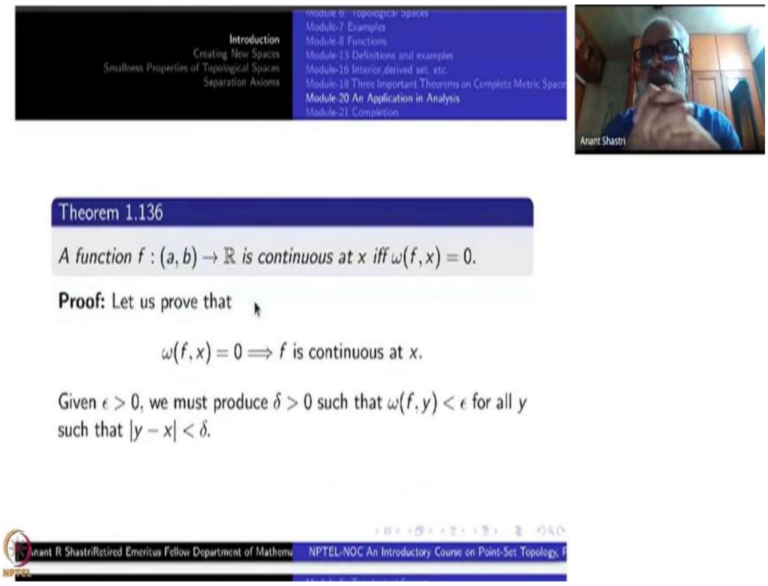

Now, you have definitions of continuous functions, usually you take the domain to be an open set right. So, I have taken an open interval here. You can also discuss the continuity in a closed intervals also finally, but to begin with you take open intervals anyway. So, take an open interval J ok, take a point x inside that, take a function it is continuous at that point if and only if the oscillation at that point is 0 ok? So, this could have been right in the beginning your definition of continuity because this statement is if and only if.

But you have defined continuity in a different way, so let us just check the validity of this statement. It is a very straightforward ok? Suppose $\omega(f, x)$ is 0 then we want to prove that f is continuous at x in the $\epsilon - \delta$ definition. So, given $\epsilon > 0$, we must produce δ such that $|y - x| < \delta$ must imply modulus of $|f(x) - f(y)| < \epsilon$.

If that is true for all x, y inside that interval, that the supremum will be also less than ϵ which is same thing as $\omega(f, I) < \epsilon$. So, I am using this part of the definition for the oscillation here these two are equal is obvious to verify, only thing you have to know what is the meaning of supremum and infimum, alright.

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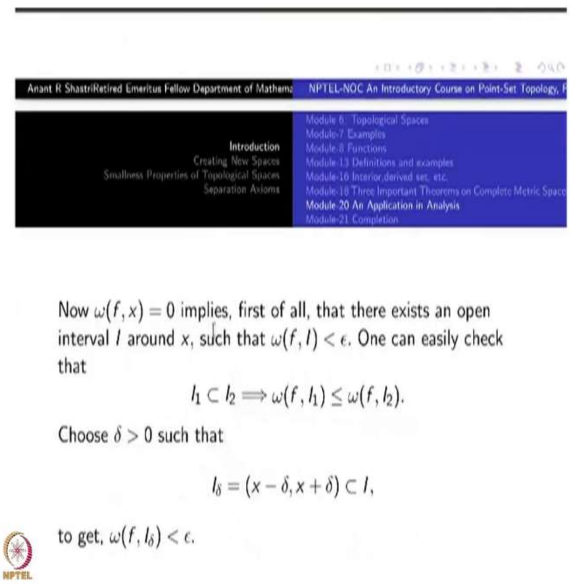

So, now $\omega(f, x)$ is 0 first of all implies there exists an open interval I around x, such that $\omega(f, I)$ must be less than ϵ. If the infimum is 0, this has to be less than ϵ some I, right. Also, you can see that when you are taking the supremum on a smaller set of of values is smaller than the supremum on larger set of values.

So, omega of I 1 will be less than omega at I 2 f is common here, so there is a f here. Choose delta positive, such that I delta is x minus delta x plus delta contained inside I. So, this I delta is just in a short notation. Then omega of f, I delta either will be less than epsilon because I have already made omega of f I is less than epsilon.

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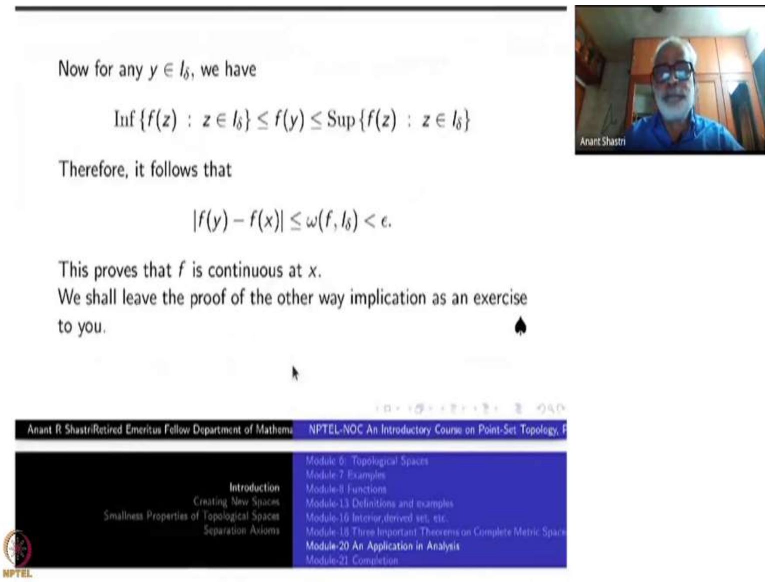

So, if I_1 is contained in I_2, then $\omega(f, I_1)$ will be less than or equal to $\omega(f, I_2)$. Choose $\delta > 0$, such that I_δ, the interval $(x - \delta, x + \delta)$ is contained inside I. So, this I_δ is just a short notation. Then $\omega(f, I_\delta)$ will be less than ϵ because I have already made $\omega(f, I)$ is less than ϵ.

So, now for any y in I_δ, what is the meaning of I_δ? That $|y - x|$ is less than δ ok? Look at all the infimum of $f(z)$, such that z belongs to I_δ, that that will be less than $f(y)$ because I am taking infimum here, and y is one of the points here. Similarly, that $f(y)$ will be less than the supremum of all the $f(z)$, z varying over I_δ. Because $f(y)$ occurs here as well as here ok?

Therefore, when you take the difference $f(y) - f(x)$ that will be less than $\omega(f, I_\delta)$ ok, which is supremum of this minus infimum of that, alright. So, therefore, that will be less than ϵ ok. $\omega(f, I_\delta)$ is already chosen to be less than ϵ because it is less than $\omega(f, I)$. So, this proves the continuity. The other way around is even simpler. So, I will leave it to you as an exercise.

This is just to warm up so that you may refresh your memory of continuity, $\epsilon - \delta$ continuity definition and so on, that is all.

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So, what I am interested in is, take f from $\mathbb{R}$ to $\mathbb{R}$, any function and $\mathfrak{D}(f)$ be the set of all points at which f is discontinuous. So, $\mathfrak{D}$ for discontinuous, the set $\mathfrak{D}(f)$ of points of discontinuity of f is an F_σ set ok? So, this is the proposition ok? So, here there is no continuity remember, f may be continuous at some points, may not be continuous at all the points, whatever. Earlier, you have applied this G_δ and F_σ for continuous functions.

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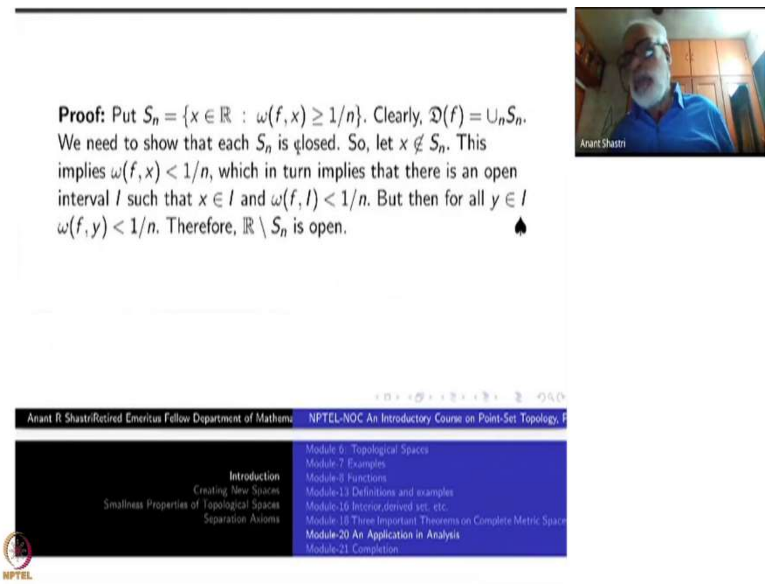

Now, I am taking set of discontinuities, that is F_σ. So, how to prove this? These are standard methods in real analysis. It is the method that should be paid attention to. You know, they are educative that is why you have to pay them attention. Look at S_n, set of all points $x \in \mathbb{R}$ such that $\omega(f,x) \geq 1/n$. Why I am looking at this? I am looking at the points where in $\omega(f,x)$ is not 0; $\omega(f,x)$ is always non negative. If it is 0, those points correspond to continuities of f.

So, I am looking at points where it is positive. If it is positive it must be bigger than or equal to $1/n$ for some n. So, that is why look at this set, ok? Then if you take $\mathfrak{D}(f)$ which is all points where in it is positive will be the union of S_n's right? Because every point must be bigger than $1/n$ for some n.

So, $\mathfrak{D}(f)$ will be union of S_n's. We need to show that each S_n is closed. Then this will be F_σ. Over, ok? So, here is something. I am not claiming that $\omega(f,x)$ is a continuous or anything like that, ok? But points wherein it is greater than equal to $1/n$ is a closed set is what I am claiming. If it were a continuous function this would have been obvious.

So, let x not in S_n ok? I should show that there is an open subset around x some I_δ or whatever, such that the whole open set is not in S_n; that means, I am trying to prove that complement of S_n is open ok.

Suppose x is not in S_n, then $\omega(f, x)$ by definition is less than $1/n$ ok. What is $\omega(f, x)$? It is the infimum of all $\omega(f, I)$, where I is some interval around x. So, that this is less than $1/n$ means that there is an open interval I_1 say, such that x is inside I_1 and that oscillation of f on I_1 is less than $1/n$, ok? On other intervals it may be bigger, but at least one of them must be there, otherwise infimum will be bigger than equal to $1/n$.

So, but then for all y in I_1 what happens? I_1 is an open interval remember. $\omega(f, y)$ will also less than $1/n$, because now I have to take infimum over all these open intervals containing y. So, for every fixed y in I_1, $\omega(f, y)$ is less than $1/n$. So, $\omega(f, y)$ will be less than $1/n$, ok.

So the whole interval I_1 is contained in the complement of S_n. Thus for each point you have got an interval. So, the complement of S_n is open alright?

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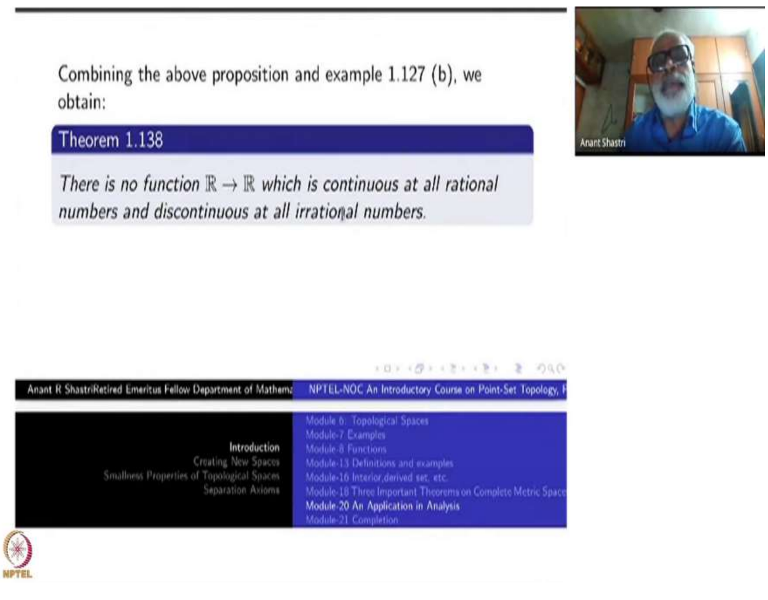

So, coming to the proof of one of the finest theorems. What we are going to do? Combining this proposition about this set being F_σ, ok? Along with what? Along with our example 1.127. Let us have a look at it. This example said that the set of irrational numbers is not F_σ.

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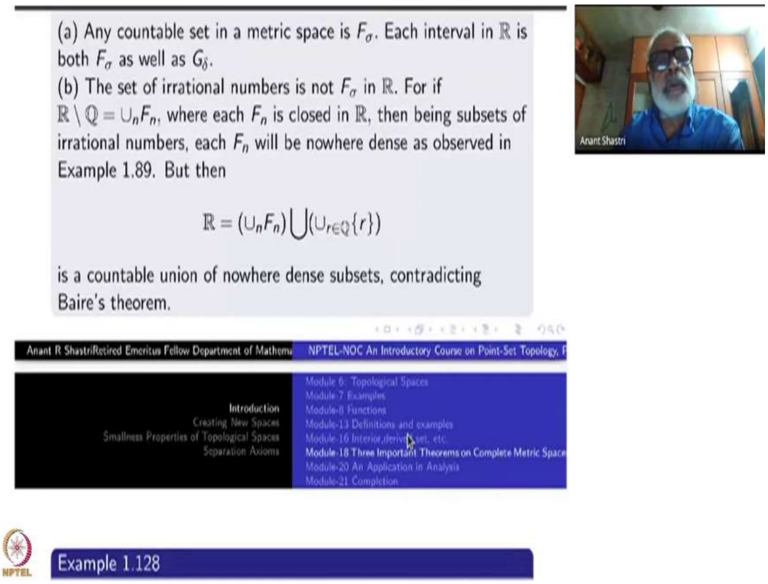

So, let us go here that is the example here ok. The set of irrational numbers is not F_σ right? Indeed, if a closed set is contained in the set of irrational numbers then it is nowhere dense. So, that is that is the discussion here. In fact, what we have done is even if it is contained in the set of rational numbers, it is F_σ. Because either rational numbers or irrational numbers they do not contain any interval. But closed subset you have to take alright.

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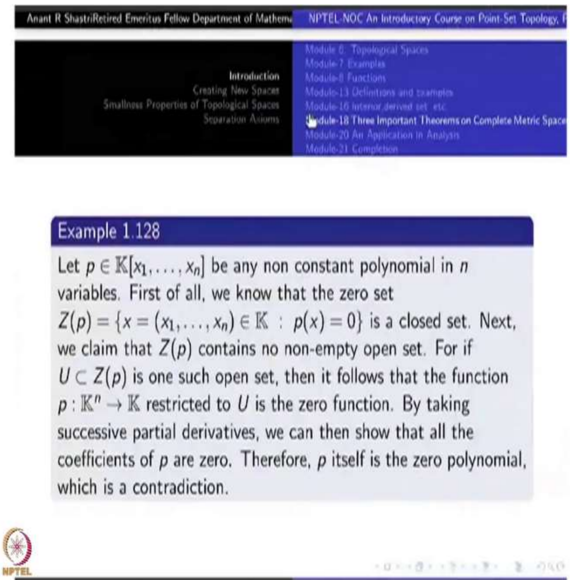

So, the conclusion here is there is no function from $\mathbb{R}$ to $\mathbb{R}$ which is continuous at all rational numbers and discontinuous at all irrational numbers. Why? $\mathfrak{D}(f)$ for such a thing would be exactly equal to $\mathbb{R} \setminus \mathbb{Q}$, then this proposition says that; $\mathbb{R} \setminus \mathbb{Q}$ is F_σ there is no problem about that.

But F_σ means what? All these S_n's are closed subsets of $\mathbb{R} \setminus \mathbb{Q}$ therefore, it is nowhere dense. Therefore, I have written the irrational numbers as a countable union of nowhere dense sets; that is not possible ok? So, there is no such function.

So, this is not at all that important result, but why I have included it here is just for an illustration of the power of Baire's Category Theorem. from very many things to very strong things it can control.

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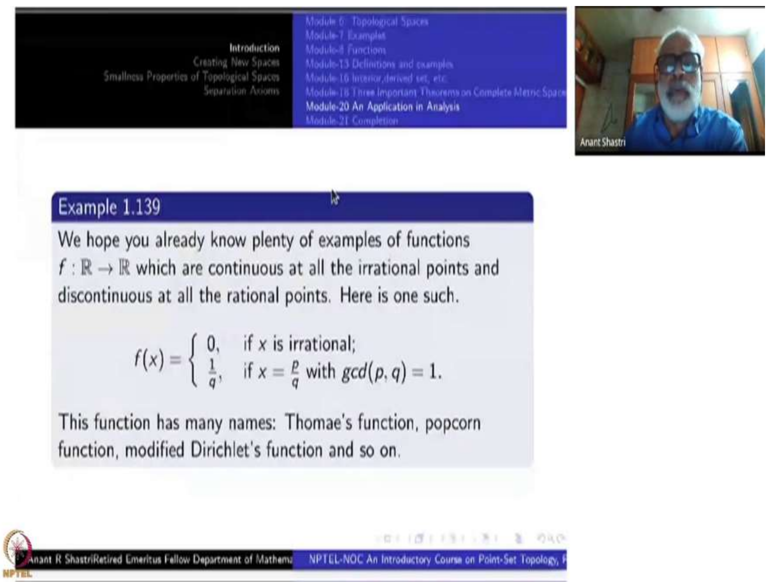

Just to complete the picture, I want to remind you that if you interchange the role of rational and irrational here there are such functions, there are continuous functions such that at all rational points they are discontinuous and at all irrational points they are continuous.

So, so the $\mathfrak{D}(f)$ can be the set of rational numbers or any subset of it also, no problem. At say few points which are rational and few others irrational numbers also that is also allowed. It is just the irrational numbers as set of discontinuities that is not allowed. That is the conclusion of this theorem ok. So, here is that function which has many many names.

So, the idea goes back to Dirichlet. Dirichlet's function is slightly different, put 1 here instead of 0 here. Thomae has slightly modified it and it becomes quite an interesting example. So, $f(x) = 0$ if x is irrational and $1/q$ if $x = p/q$ with $gcd(p, q) = 1$ ok. So, just to remind you what will be $f(0)$ according to this definition?

See 0 is a rational number and how do you write it as p/q where p and q are coprime. The only way $gcd(p, q)$ can be equal to 1 when $p = 0$, is to take $q = 1$. Therefore, the value of this function at $x = 0$ is $1/1$. I am just explaining this definition that is all. Suppose we take $5/10$, then you should write it as $1/2$ and then then $f(5/10) = 1/2$.

Suppose you have $15/25$, then you will have to write it as $3/5$ then $f(15/25) = 1/5$ and so on. So, continuity of this function uses an interesting property of all rational numbers or more so of irrational numbers, that I will not discuss. This is very standard ok? So, let us stop here next time we will do one more serious result and that will be the end of this chapter ok. So, tomorrow we will do one more serious result about metric spaces, namely, completion of metric spaces. Any questions?

Student: Let us.

Ok, Tell me.

Student: So, this theorem about the set of discontinuitues is being F_σ set, is there any analogous result for Metric spaces? Can it be extended?

Complete metric spaces yes, see complete metric spaces you can formally say you know what we need to show first of all, you can try to see that this $\omega(f, x) \geq 1$, you should be able to show that it is closed that is it. So, where how far you have used here, what is the meaning of the oscillation etc, you will have to be very careful you have to define properly ok?

So, what is it that you have to take neighborhoods instead of I you can you can just restrict yourself to just open balls around that point, no problem. Look at all (x, y) ranging in the open ball ok, $|f(x) - f(y)|$. But the function has to be real valued ok, and not arbitrary then there is no infimum and supremum. So, domain can be arbitrary metric space, but you have to put completeness ok? Then at least it makes sense you have to check it it is a good exercise try it. Good question.

So, try it and see how far all these things go ok, countable union automatically comes just like this one. So, if you can show that these are closed subsets which is not difficult anyway. Complement is open, similar proof is ok. Now, the problem here is why these S_n's ok are nowhere dense. So, you do not have irrational numbers and so on there in an arbitrary metric space ok. So, what is the conclusion? The set $\mathfrak{D}(f)$ is F_σ. Up till there you are fine ok

So, in the absence of any fixed nohwere dense subsets, final conclusion will be missing.

So, I what I mean to say there is no concept of the set of irrational numbers or complementary irrational numbers and so on, in an arbitrary metric space. So, you have to replace the corresponding by correct hypothesis that is all ok. Alright, let us stop here.