Introduction to Point Set Topology, (Part I)
Prof. Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay

Module - 02
Lecture - 02
Normed Linear Spaces

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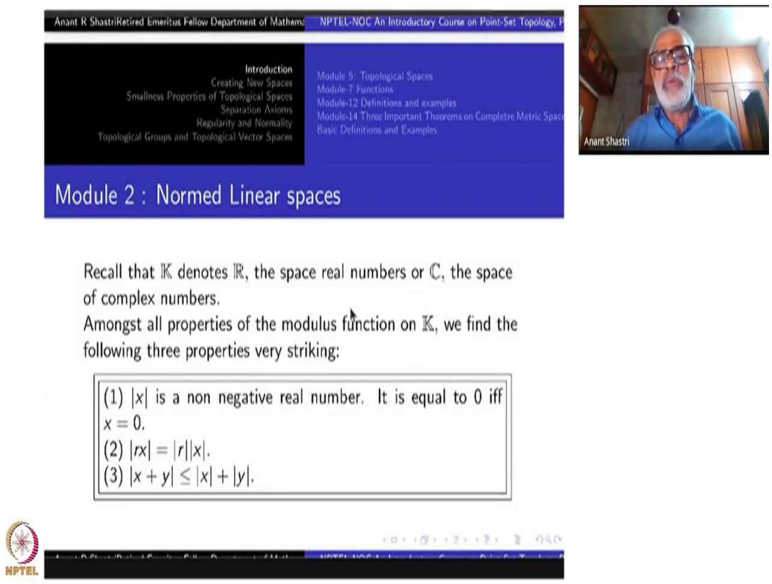

Welcome to the 2nd module of Point Set Topology course part I. This is an NOC course of NPTEL. So today's topic is Normed Linear Spaces. Recall that we use this notation $\mathbb{K}$ for either the set of real numbers or the set of complex numbers. Amongst all properties of the modulus function on $\mathbb{K}$- (you know we talk about modulus of z which is equal to what? Take $x^2 + y^2$ and then taking the square root, ok?)

So, there are modulus functions on both $\mathbb{R}$ and $\mathbb{C}$. So, I am trying to handle both of them together. So, what is the striking properties of this modulus function? I would like to list three of them. The first thing is that modulus is always a non negative real number. Whether x is a real number or the complex number, $|x|$ is a non negative real number.

It is 0 if and only if x is 0. So that is the first property. Second property is that if you multiply by some number, r times x, then the modulus of that is modulus of r into modulus of x. I deliberately write r and x instead of x and y because in the multiplication, this r is supposed to be scalar multiplication. It is not a real number, it is real or complex, no problem.

But just to indicate that they have different roles here, ok. So, that is why I am writing r here, that is all. Even if you write $|xy| = |x||y|$ is correct, there is no problem. But do not take r as a real number, it represents real and complex number as the case may be. And the third property is that $|x+y| \leq |x| + |y|$, ok?

So, these things we have been using all the time, that is why they are the most important ones. So, these properties will now taken and made into an axiom, ok? So that is what we are going to do now, ok?

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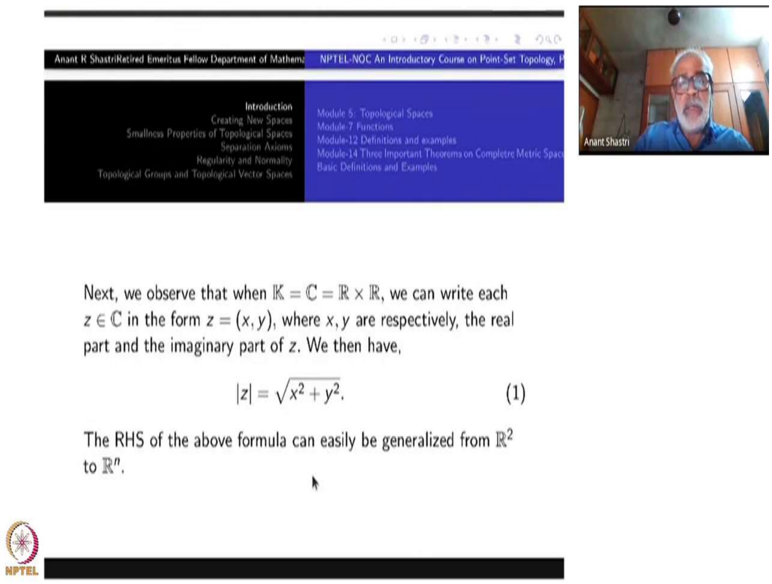

So, I have already told you that what I am going to do. If $\mathbb{K}$ is not real but $\mathbb{K}$ is $\mathbb{C}$ and $\mathbb{C}$ can be taken as $\mathbb{R} \times \mathbb{R}$. Namely every complex number can be written as real part comma imaginary part (x, y).

Then $|z|$ is nothing but square root of $x^2 + y^2$, ok? So, this looks like I have the same modulus function the same properties, wherein $\mathbb{K}$ could be $\mathbb{R}$ or $\mathbb{R}^2$. But now you can do it for all $\mathbb{R}^n$. So, that is our idea, it is easily generalized to all $\mathbb{R}^n$, right?

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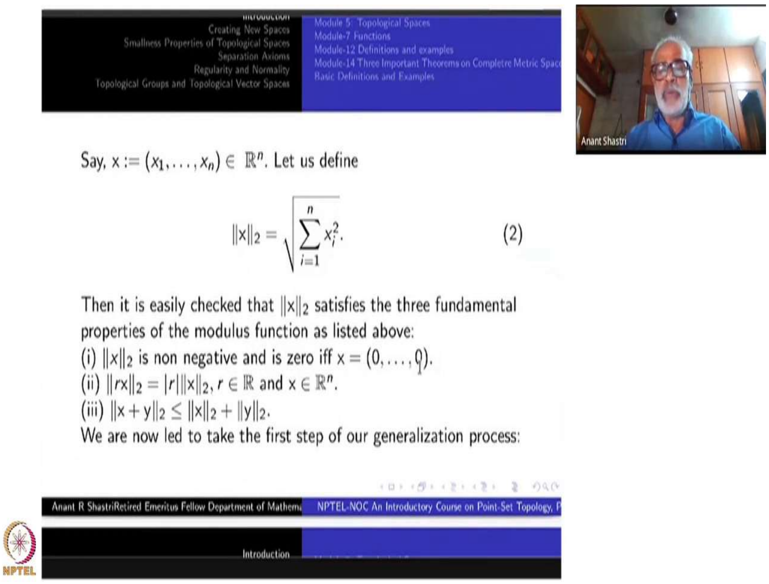

How? If you just take a vector, now an n vector $x = (x_1, x_2, \dots, x_n)$, all $x_1 x_2, \dots, x_n$ are real numbers. Just like when n was 2, this was real part and imaginary part of a complex number.

So, in more general case you have n coordinate real numbers here. The coordinates of x, they are all real. Then I denote it, for the sake of future reference also, this

$$\|x\|_2 = \sqrt{\sum_{n=1}^{\infty} x_i^2},$$

the 2 indicates that I am going to take squares here and then square root here, ok? Take the squares of each x_i, sum it up and then take the square root. This is what is called as Euclidean norm, ok?

So, we have just generalized the complex number modulus for all $\mathbb{R}^n$, you know you can call it as a modulus. So, just not to get confused, instead of calling it a modulus, I am having a different notation here that is all. So, the same properties can be checked for this `norm ' now. The $\|x\|_2$ is non a negative real number. It is 0 if and only if $x = 0$. Now, $x = 0$ means what? Each $x_1, x_2, \dots, x_n$ are all 0.

Similarly, now comes the difference, I cannot take r as $(r_1, r_2, \dots, r_n)$ and x as $(x_1, x_2, \dots, x_n)$. No, x will be taken as $(x_1, x_2, \dots, x_n)$ and r will be now taken only from $\mathbb{K}$

either a complex number or it is just a real number. Now, I want to take it a real number because all x_i, we have taken are real number.

So, this is the second property. Third property is again $x+y$ is the same thing $(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n)$. Here it will be $(x_1+y_1, x_2+y_2, \dots, x_n+y_n)$. The norm of that, instead of modulus, is less than or equal to $\|x\|_2 + \|y\|_2$. So, these are properties which will be, these three properties which will be, carried over to this norm function also. This is number three, ok.

So, many of you have seen this one, I am just listing it: $\|x\|_2$ is non negative and is 0 if and only if all the coordinates are zero. Norm of rx is modulus of r times norm x. Here r is a real number and x is in $\mathbb{R}^n$. Norm of $x+y$, namely this two-norm, is less than equal to $\|x\|_2 + \|y\|_2$, ok?

So, this was alright and an easy step. But what we want to take as the the first step is something more you know, and substantial, that is our first step of generalization. What is this, it is called normed linear space, that is what we are going to define now.

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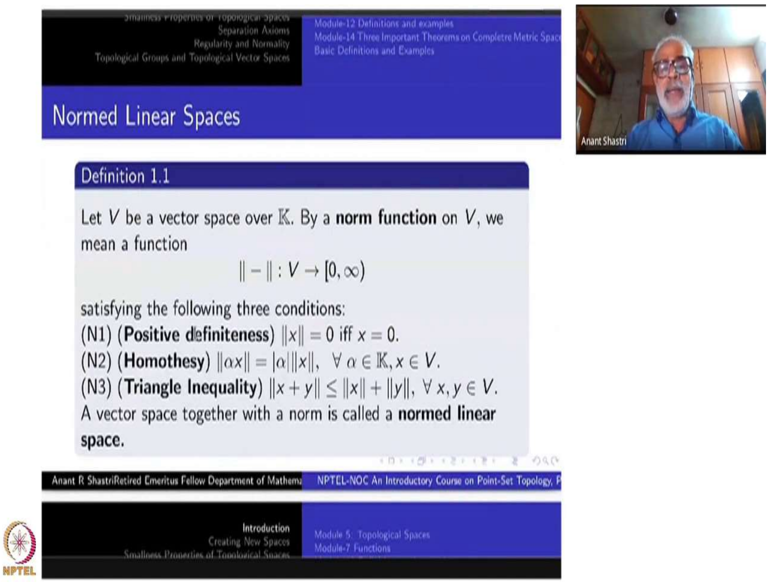

So, instead of taking $\mathbb{R}^n$, what you want to take is any vector space over $\mathbb{K}$. So, I can take $\mathbb{C} \times \mathbb{C} \times \mathbb{C}$ also, ok. So, in that case this r will become a complex number. Then also this make sense, that is what we want to know.

So, take any vector space over $\mathbb{K}$, ok? $\mathbb{K}$ is a field ok, whether $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, it is a field. We have to take a vector space over $\mathbb{K}$, need not even be a finite dimensional one, it could be infinite dimensional also, ok? Take any vector space V. So, by a norm function on V we mean a function, $||\cdot||$ this notation is not functional notation, but this is the way we would like to write it namely, this is the slot for the function ok?

So, norm is a function from this vector space V into what?-- non negative real numbers the closed interval $[0, \infty)$, ok? The entire ray here, so that is the codomain. So, a function which satisfies these three hypothesis $N1, N2, N3$. N for norm, ok? What are they? They are exactly the same thing as what we have seen here, same thing as what we have seen for the modulus function here, ok?

The very easy thing that we have done the same property we write here only, we have to take care here namely that x is a vector. x is always inside V, but wherever r appears, I have written deliberately you know α, where α is a scalar, i.e., α is in the field $\mathbb{K}$ whether $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$, ok?

The first property is, positive definiteness is the name : $||x|| = 0$ if and only if $x = 0$. The positive part corresponds to that it is taking non negative numbers here and it is 0 if and only if $x = 0$ is definiteness, ok. This is our standard terminology here. The `homothesy' says that take scalar multiplication and then take the norm that is the same thing as multiplying by the modulus of the scalar into the norm of x, ok? This happens for every scalar α and every vector x.

The third one is again a copy of the corresponding third property there. This has a name triangle inequality, which is derived from the two dimensional school geometry you know. If you take a triangle, then sum of the two sides, you know, is always slightly bigger than or equal to the third side. So, that is what this triangle inequality means here.

Norm of $x + y$ is less than equal to $||x|| + ||y||$. These x and y are now arbitrary vectors inside this vector space. When you have such a function, it is called a norm. Together with such a function the vector space V will be called a normed linear space, ok? This is the standard terminology now, nobody else speaks about anything else. Earlier in the development of these things there were different names ok?

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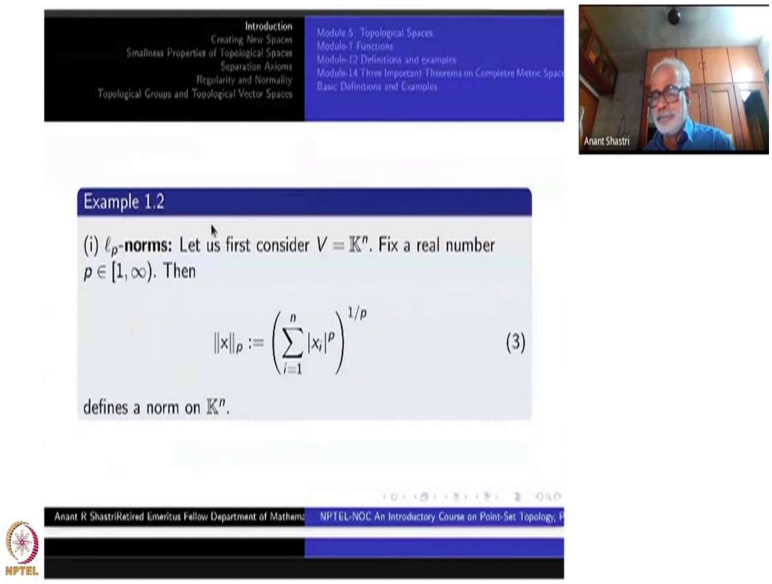

Let us have some examples here, of what are called as ℓ_p norms. See now why I denoted this x with a $||x||_2$ before with a suffix 2, that will be clear here. Instead of this 2, I would like to write any positive real number, ok. I would like to, but I cannot. I will have to restrict myself namely for $p \geq 1$.

So, of course, it is 1, then what is the meaning of this? This is summation of $|x_i|$, we have to take. Do not take x_i as they are, with squares, it was was ok. Real numbers square are already non negative, so you do not have to take modulus. But the correct thing is to take always modulus, then it will work for complex numbers also ok.

So, this thing which is very simple as to be correctly generalized and correctly modified. So instead of 2, I can write p here then $|x_i|^p$. Then instead of square root what should I do? I should take the p^{th} root. Raised to the p first and then take the p^{th} root of that. That is what I am doing here. Come here, $|x_i|^p$, take the summation, and take the p^{th} root of the total summation.

So, that will define again a norm is the claim here. So, what you have to verify? You have to verify that if I put this whole thing to 0 then all x_i must be 0. That is an easy thing, some power raised to $1/p$ is 0, means the summation itself is 0. If the summation is all non negative number, so each $|x_i|^p$ must be 0. Therefore, each x_i must be 0.

So, the first part is totally obvious right? Second thing is if I multiply this one by a scalar α, all the x_i will be multiply by scalar α. First of all, modulus of αx_i raised to p is $|\alpha|^p$ into $|x_i|^p$. Then $|\alpha|^p$ will come out here, when you take p^{th} root of that it will be just $|\alpha|$, right.

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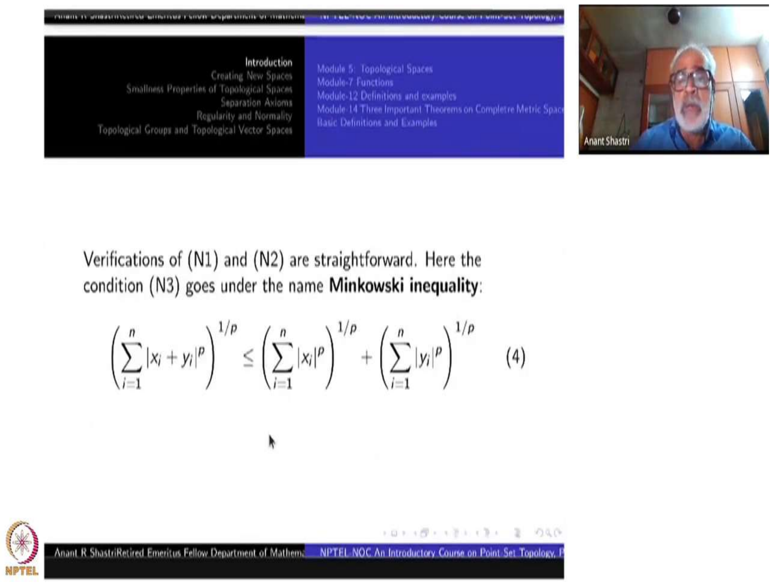

So, that is the second one. So, the first two $N1$ and $N2$ are straight forward, right. Some computation, little more analysis, is needed to prove the third one namely triangle inequality. Triangle inequality now for p, what it becomes? Take $|x_i + y_i|^p$, summation and then take the p^{th} root. It should be less than or equal to individually, you do the same thing for both x and y, namely $|x_i|^p$ summation then raised to $1/p$ plus $|y_i|^p$ summation and then $1/p$, right?

So, this is what you have to verify. This has a name. This is called Minkowski's inequality, ok? It is not very straightforward, but it is not very difficult also, there are theories here. So, these things are very nicely done in elementary real analysis books. So, what I want you to do is, if you have not seen it, you please read it from some book. I am assuming that you know already a bit of real analysis. Real analysis courses will discuss these kind of things ok?

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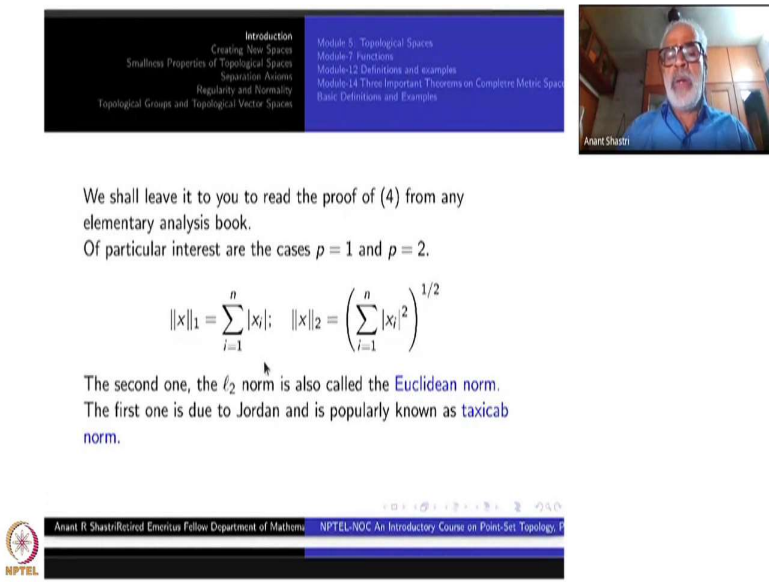

There are many books. If you do not know you can approach me, then I will give you exact reference to some books. No problem. So, we shall leave it to you to read the proof of this Minkowski inequality from any elementary analysis book, ok? Of particular interest for us is the cases where $p = 1$ and $p = 2$. When you put $p = 1$, what is it? You have to take $|x_i|$ raised to 1.

So, there is 1 you do not have to write it, take the summation and then take the 1^{th} root and 1^{th} root is the same thing. So, it is just summation$p$ of $|x_i|$. The second one, which we already started with, what is called as Euclidean norm. Started with the observation that how it happens, what it happens in $\mathbb{R} \times \mathbb{R}$ namely in the plane, ok.

Summation $|x_i|^2$ ok? (If x_i real numbers, we do not need to put modulus, but for complex numbers also it is valid; you can take $z = (z_1, z_2)$ then you have to take $|z_1|^2 + |z_2|^2$ and then take the square root.)

So, that is Euclidean norm ok. The first one is called the ℓ_1 norm. If $1, 2, 3$ etc. or p, then ℓ_p. That is what we have: the ℓ_p spaces and ℓ_p norms. This ℓ_1 norm has a different name, it is called taxicab norm also. This was actually introduced by Jordan ok?

Before Jordan, people were usually talking of this one single norm namely l2 norm. The l1 is a contribution from Jordan, ok? Very simple minded thing it is. Now of course, we have many other norms which we will have to study.

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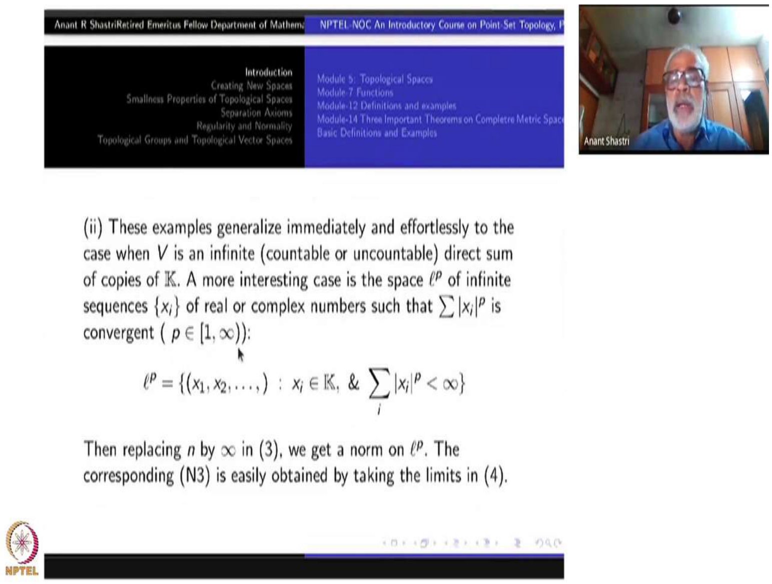

So, these examples generalize immediately and effortlessly to the case when V is an infinite direct sum of copies of $\mathbb{K}$. You know direct sum of copies of $\mathbb{K}$ means what? It is say $z_1, z_2, \ldots, z_n$ finitely many and then 000 to where you want to stop nobody tells you.

If it is only $\mathbb{K}^n$ then you have to stop it $z_1, z_2, \ldots, z_n$ and n tuples, right. If that is the infinite direct sum, it could be countable or uncountable also. Then you cannot write it is say $(z_1, z_2, \ldots, z_n)$. We can write (z_α) where α ranges over an indexing set so on.

So, but for each vector only finitely many entries will be nonzero. So, you can add, you can take the modulus of them, you can take the raised to p, then you can add the indices, all these things make sense, immediately and everything works because each time you have to restrict to only finitely many coordinates. So, if you have proved Minkowski inequality, ok other things are just easy or right now you think of that you have proved it then you can apply it ok.

Now, there is yet another interesting case namely it is not direct sum of anything. This ℓ^p is infinite sequences of real or complex number such that you can take this sum. You see if it is finite sum there is no problem, if it is infinite sum what you have to do?

You have to say its convergent ok, now it becomes very crucial that I assume p is between one and infinity. Otherwise, Minkowski inequality will not be true ok. Other things are ok, but Minkowski inequality will not be true.

So, p belong to $[1,\infty)$ we have taken, then look at all this. ℓ^p that is a notation now ℓ upper p. Remember ℓ_p was for the norm, ℓ^p was the space here. All sequences are inside it ok, infinite sequence with all x_n inside $\mathbb{K}$ ok, summation modulus is absolute convergent, sum of modulus raised to p^{th} power, that must be finite.

Take two of them the sum will be also finite, by Minkowski inequality. it means that you have proved that this ℓ^p is a vector space, α times a vector in this is finite is obvious, ok.

So, it is a vector space. In proving that this is vector space itself you had to use Minkowski inequality ok. So, all that you have to do is you replace this n here by ∞, to justify this what you have to do? You have to show that these are convergent and so on ok. Once you have made convergent, you have made it as a hypothesis here you have proved this hypothesis.

So finally, what you have to is why this one is true for when you have infinite sum. You take just the limit. It is true for each n ok, it is true for partial sums so, you can take the limit. Then for the limit also it will be true ok. You do not have to prove Minkowski inequality separately for the convergent sums, that will automatically follow, alright.

So, we have all these ℓ^p spaces what are they? They are convergent sequence, convergence is with respect to this $\|\cdot\|_p$, $|x_i|^p$ summation. So, those things must be convergence, ok.

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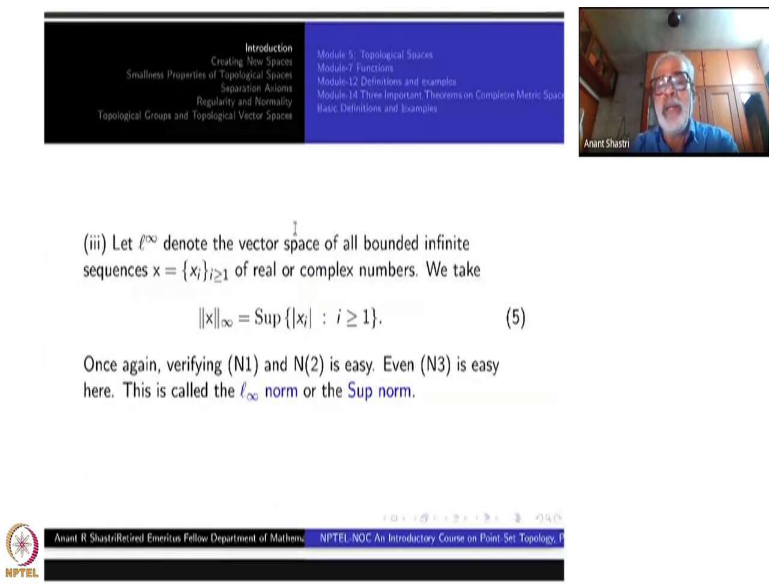

There is one more interesting thing. Do not worry about convergence, but put another condition, just weaker condition namely all those sequence which are bounded by some number. See when you have infinite sequence you do not know it may be $1, 2, 3, 4, 5, \ldots$ you can keep going you do not know whether they are bounded. You want them bounded ok.

Take real or complex number, no problem, take only bounded sequences. Then you can talk about the supremum. Remember supremum is not the same thing as maximum because we are taking infinite sets here ok. Supremum will be also some finite number because it is a bounded sequence. You put that supremum as this $\|x\|_\infty$. It is just a symbol. Here I am not taking p first and then taking the limit ok.

In some sense that is also true if you understand the geometry, but this is just a symbol here for supremum ok. Now, verifying $N1$ and $N2$ is easy as before. Verifying $N3$ is also easier here, you think about it. If we have not done it, these are not difficult.

For the supermum you do not need a lot of analysis, some elementary inequalities will do. So, that will give you Minkowski inequality, something corresponding Minkowski inequality, which I have called a triangle inequality in this case ok. For $\|x\|_\infty$, so $N3$ will be also easy here. So, this is called ℓ^∞ or sometimes $\sup$ norm, ok.

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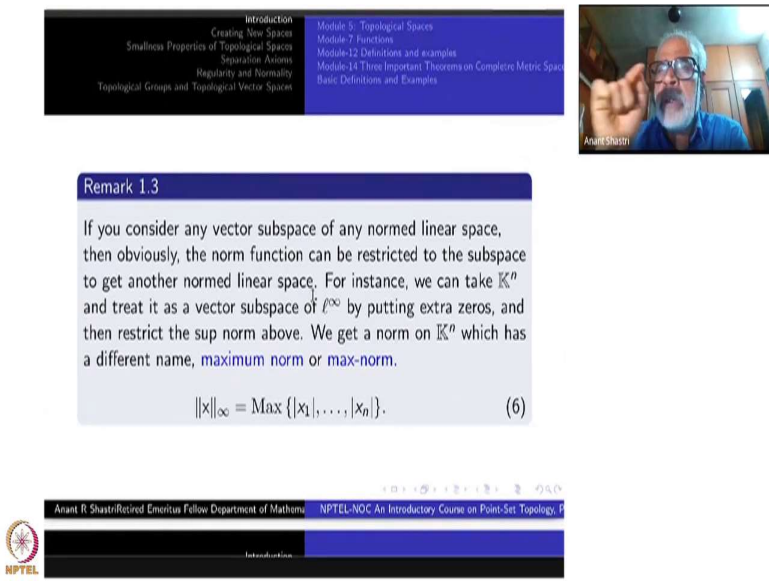

So, if you take a vector subspace of vector space which has a norm, then everything works for this vector subspace also. If you just restrict the function - norm function to the subspace. That way there will be another normed linear space ok.

So, for example, you can take this $\mathbb{K}^n$ itself, sitting inside all this ℓ^∞ spaces. How? $\mathbb{K}^n$ is what? $\mathbb{K}^n$ is just $(z_1, z_2, \ldots, z_n)$ then put $0s$. So, a finite sequence can be made into infinite sequence, you know, by extending, by putting extra $0s$ ok. So, then take the maximum norms, supremum norm, various norms. All old things are there already, restricted ℓ_p norms will be ℓ_p norms on $\mathbb{K}^n$.

But now I get one extra thing which I had not done, namely, the supremum norm becomes maximum - the maximum of $|x_1|, |x_2|, \ldots, |x_n|$ ok. So, that is called $||x||_\infty$, the same symbol because it obtainable by restriction. But now it has a different name on the right hand side because this is maximum, supremum becomes maximum ok, only when these things are finite dimensional vector spaces.

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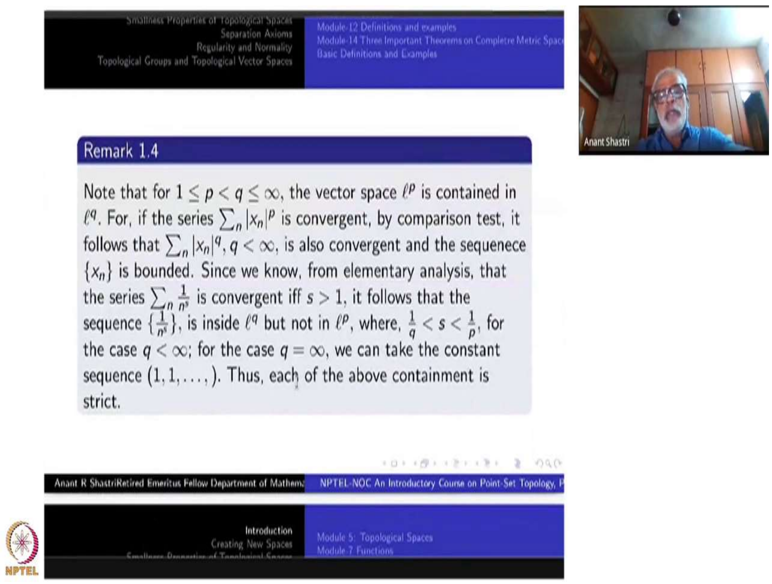

So, here is a remark about these ℓ^p spaces. Which just for getting because these are examples that I am giving you I am telling, but you should be learning these things in analysis. But let me just explain this point because I may sometimes use this one. Namely if $p \leq q$, then ℓ^p is contained inside ℓ^q.

What is the meaning of that? If a sequence, if a series $\sum x_n$ is convergent after raising power p in absolute convergence raised to power p, then it will be convergent if you take q also.

Why this is? This is just a comparison test. See a sequence which is convergent like this in which I am taking modulus here, you can always assume they are real number positive real numbers ok. So, it is absolutely convergent to a positive real number. So, for any positive real numbers summation convergence means after certain stage, ok, these things must be small. In fact, limit of x_n as n tends to ∞ must be 0.

So, they are smaller than 1 therefore, when you take a higher power it will be smaller. Therefore, each x_n raised to q after certain stage will be smaller than x_n raised to p. Therefore if this is convergent this will be also convergent. So, that is the comparison test here ok.

So, this is all elementary analysis. So, I have given you full explanation here, ok. Why if you take a sequence like this, this one will also be convergent. Convergence of such a thing $1/n^s$ for example, ok. You have to use why this convergent if and only if $s \geq 1$. You take the sequence $1/n^s$, where s is between $1/q$ and $1/p$. This sequence will be inside ℓ^q, but not inside ℓ^p.

So, that will give you that ℓ^p is contained inside ℓ^q, but there are points in ℓ^q which are not in ℓ^p. So, containment is strict here ok. One single example wherein s you have to choose correctly, ok, so $1/n^s$. This is a beautiful example, this series itself is very important one ok?

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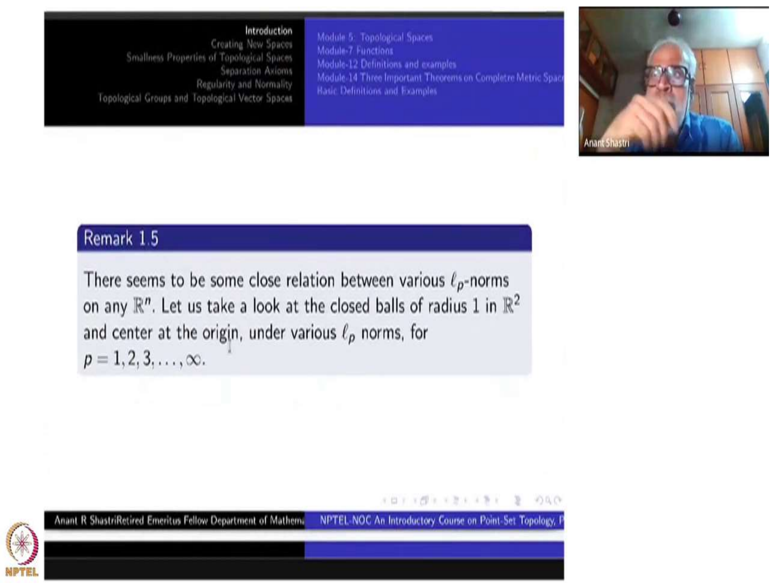

So, containment is fine. But there are some other kind of relations also close relations between ℓ^p norms. So now, I am coming to the geometric aspect of this one. Look at just the numbers $1, 2, 3, p$ and then ∞. Do not worry about the in between real numbers there, only take the integers ok.

Just to concentrate on what is going on. Not for the sake of logical statement, just for getting some ideas ok. You can take one and half and so on that may be more difficult to imagine what is happening ok.

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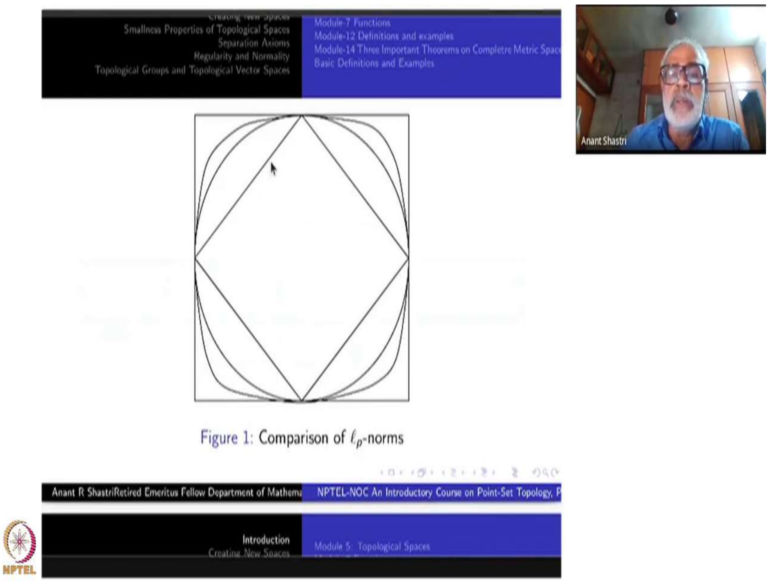

Figure 1: Comparison of ℓ_p-norms

So, just take these numbers and look at this picture. This is in $\mathbb{R}^2$, ok. because I am drawing a picture means it has to be in the plane right? So, plane is $\mathbb{R}^2$. $\mathbb{R}^3$, $\mathbb{R}^4$ etc difficult to imagine ok. So, what is this picture? This square here, this is set of all points (x, y) such that $|x| + |y| \leq 1$.

This is x plus y less than or equal to 1. You can say this is origin 0, this is x axis this is y axis right? What is this one? This is the circle. And, what is this last thing? This is the maximum of $|x|$ and $|y|$ is less than or equal to 1. All these things are unit discs inside corresponding ℓ^p spaces. This is ℓ^1, this is ℓ^2(the standard Euclidean). This is ℓ^3 or ℓ^4 or ℓ^5, I do not know I have only drawn one of them then this last one is ℓ^∞.

So, you see now if you keep on taking bigger and bigger p, this line becomes flatter and flatter like these keeps going. So, in the limiting case it will become the outer square. So, in that sense ℓ_∞ is actually the limiting case of these things. That is what I meant earlier ok.

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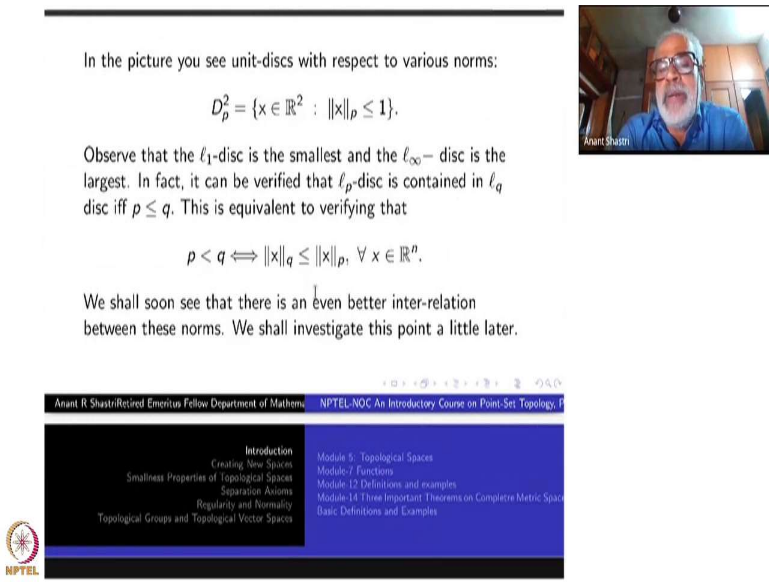

So, these are D_p^2, for you know $n = 2$, p for the p norm or $x \in \mathbb{R}^2$ such that norm of x is less than or equal to 1. So, various discs I have shown ok. So, when p is smaller the corresponding disc is also smaller. This is ℓ^1, this is smallest one is containing all of them. As p increases the size of the disc increases you know bigger and bigger and finally, it will become the square like this. So, that is the picture in $\mathbb{R}^2$, similar pictures you can see in all $\mathbb{R}^n$ ok?

So, that I cannot draw. But I can make this conclusion - If $p \leq q$, the $\|x\|_q$ is always less than or equal to (same x, x is same ok) the $\|x\|_p$. The q^{th} norm, you know ℓ_q norm is less than ℓ_p norm. So, this easy to verify ok. Argument is similar to what we have done, this kind of argument we have to use.

So, this is the roughly for today. There is little more deeper relations between this ℓ^p spaces. So, that we will investigate a little later ok.

Thank you.