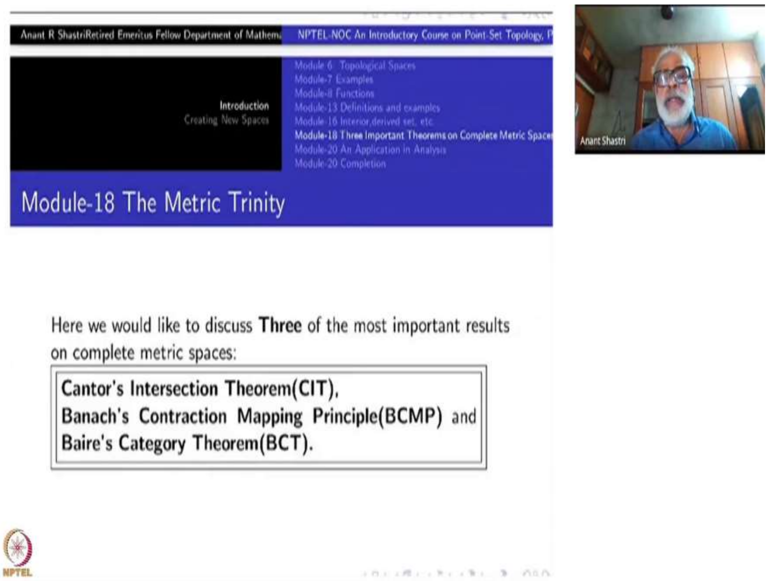
**Introduction to Point Set Topology, (Part I)**
**Prof. Anant R. Shastri**
**Department of Mathematics**
**Indian Institute of Technology, Bombay**

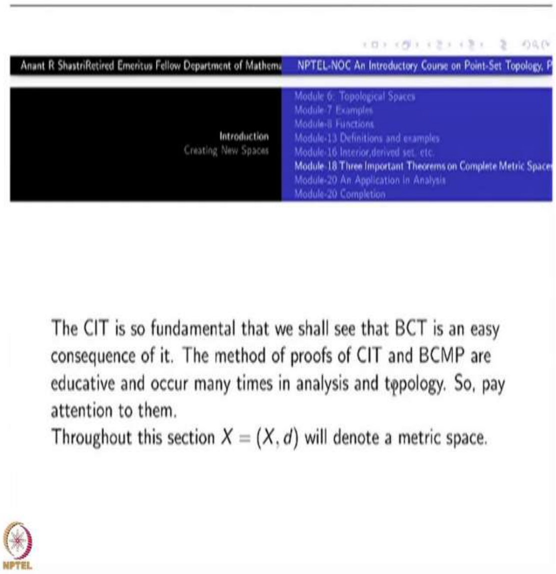
**Module - 18**
**Lecture - 18**
**The Metric Trinity**

(Refer Slide Time: 00:15)




Welcome to module 18, I have boldly named it as The Metric Trinity. What we are going to do today is to discuss three most important theorems, according to me, in metric space theory. So, one of them is Cantor's Intersection Theorem, second one is Banach's Contraction Mapping Principle, the third one is Baire's Category Theorem. You can call all of them principles,  from theorems to they have become principles anyway.

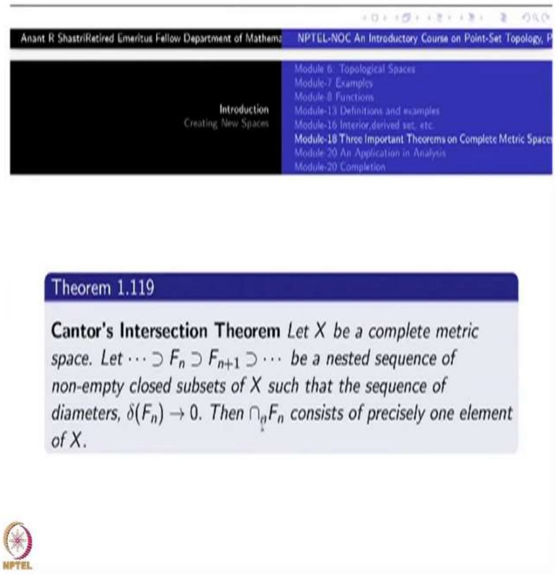
(Refer Slide Time: 01:14)




The high status of a theorem is to become principle. The CIT; CIT I mean Cantor's Intersection Theorem. This is so fundamental that even in the proof of Baire's category theorem we will be using it. Banach's contraction mapping theorem is slightly of a different a flavour. It does not use CIT directly, ok?

The method of the proofs of all these three are themselves quite educative. So, often you may have to employ that in your own research work if you want to do, you know, deep analysis or topology. So, I would like you to pay attention to not only the statement here but how things have been arrived at. The proofs also should be learned properly.

(Refer Slide Time: 02:38)

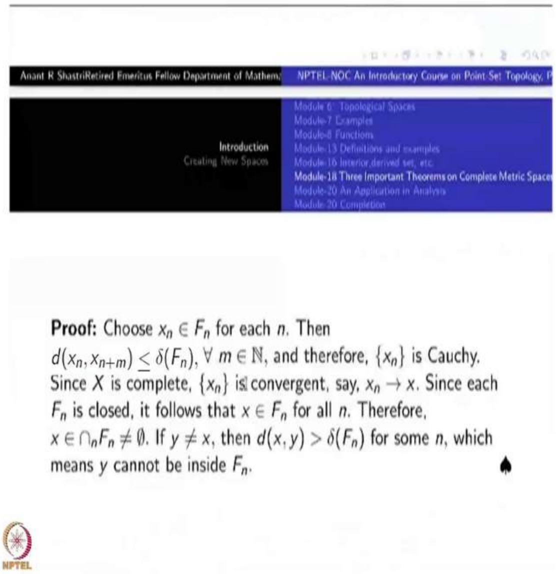



So, let us begin with freezing this symbol $(X, d)$ or just $X$ for a metric space once for all in this section. The first theorem Cantor's Intersection Theorem. So, start with a complete metric space $X$. Take a nested sequence of non empty closed subsets of $X$. Nested means what? one contained in the other but the other way around. $F_n$ contains $F_{n+1}$ and so on, they are decreasing here, ok? Nested could be the other way around also.

So, here they are decreasing, but condition comes automatically after the second condition here. So, $\delta(F_n)$ denotes the diameter of each of them. So, this sequence of real number converges to 0. So, this is also a condition. So, each $F_n$ is closed which contains $F_{n+1}$ and the diameter tends to 0. Then the statement is that intersection of all these $F_n$'s consists precisely of one element, ok?

This is stronger than saying that it is non empty. This has exactly one element. The proof is surprisingly very simple, ok? You will see that.

(Refer Slide Time: 04:12)



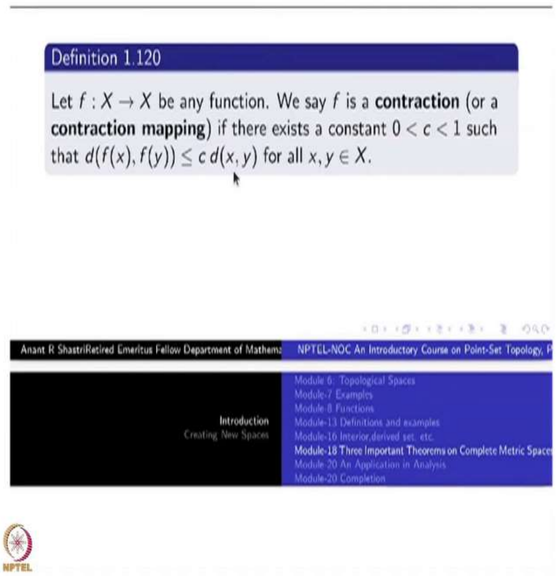

All that I do is $F_n$'s are non empty, so, pick up $x_n$ belong to $F_n$ for each $n$. Then look at this sequence $d(x_n, x_{n+m})$ is less than $\delta(F_n)$ because both of the elements are inside $F_n$. This $\delta$ is what? $\delta$ is the diameter, it is the supremum of all such numbers $d(x, y)$, where $x$ and $y$ range over $F_n$. Therefore, the distance between $x_n$ and $x_{n+m}$ becomes less than $\delta(F_n)$, but $\delta(F_n)$ itself tends to 0. Therefore, it can be made less than epsilon etc.

So, this implies that $(x_n)$ is a Cauchy sequence, but now I use that $X$ is complete therefore, $(x_n)$ is convergent. So, let us take $x$ to be the limit, ok? Now, we also know that if you have a sequence $(x_n)$, you see the entire sequence $(x_n)$ is not in any one of the sets, but suppose you choose some $F_k$, here then $k$ onwards that sequence is in $F_k$. The limit of that portion of the sequence is also $x$, right?

The first $k-1$ terms does not matter here. Therefore, each of these sequence can be thought of as a sequence inside $F_n$, ok and the limit therefore, must be inside the closed sets $F_n$. So $x$ is in the intersection. It follows that this limit point is inside all the $F_n$'s. So, here we are used that $F_n$'s are closed and they are monotonically you know decreasing one contained in the other that is also you have to use. That just means that intersection is non empty.

The second part: that it must be a singleton; this comes very easily because of this $\delta(F_n)$ goes to $0$, you can see that this the diameter of the intersection will also become 0. So, diameter is 0 the set can be what? Only singleton. Anyway if $y \neq x$, you know then $d(x, y)$ will be positive therefore, it will be bigger than $\delta(F_n)$ for large $n$ because $\delta(F_n)$ is converging to 0. Once it is bigger than $\delta(F_n)$ both $x$ and $y$ cannot be inside $F_n$, right? If it is not in one $F_n$, it cannot be in the intersection. So, any point which is not equal to $x$ is not inside the intersection. So, there is only one point. So, that is a proof ok?

(Refer Slide Time: 07:24)

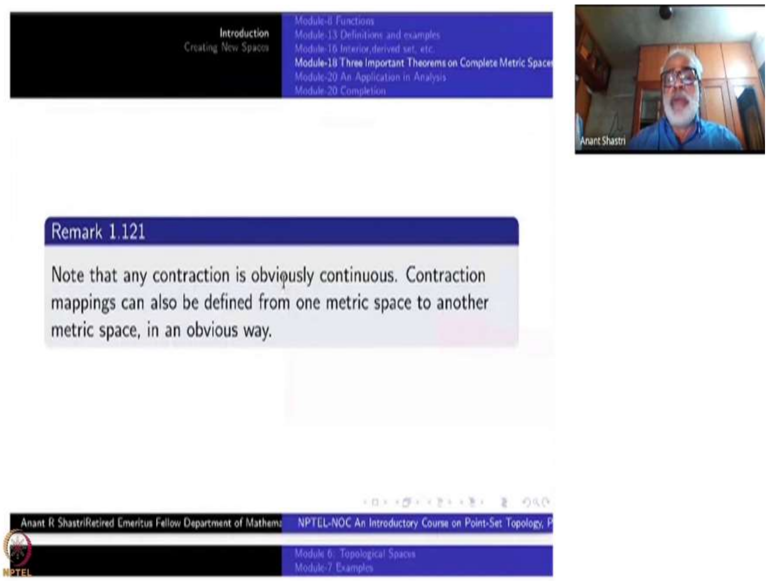



Now, I have to make a definition because there is a new term in the Banach contraction mapping, in the statement itself. So, take a function from one metric space to itself OR to some other metric space also, this will work. But right now, I need only this definition.

We say $f$ from $X$ to $X$ is a contraction mapping or just a contraction if there exists a positive constant $c < 1$, this strictly less than 1 is important, such that the distance between $f(x)$ and $f(y)$ is less than or equal $c$ times the distance between $x$ and $y$. For example, if $c$ is half each time you apply $f$, ok? What happens? So, two points at distance one will go to points less than half the distance apart next time one fourth and so on ok, that is the meaning of contraction mapping ok?

If you talk in the Layman's language and take a map of your campus or a country or just a state, say. If the map is up to scale definitely, it will be a contraction mapping, ok? From the actual object of which it is a map to the map. You can think of holding the map in your hand, you know, you are standing in the campus or the country. So, the map is inside. So, the function $f$ is from the country into the country, but it is a contraction mapping.

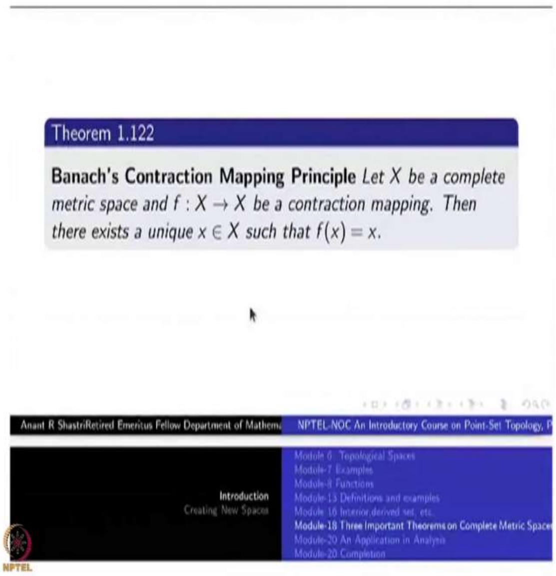
(Refer Slide Time: 09:32)




Now, what does this theorem say let us see, ok? So, for example, contraction mapping is always continuous because I want to control distance between $f(x)$ and $f(y)$, I can choose $\delta$ equal to $\epsilon/c$, that is all. So, this will be less than $\epsilon$. So, continuity is obvious. By the way this condition constant $c_1, c_2$ on both sides we are familiar with that. That gives equivalence of metrics right? Similarity of metrics. So, this is not a strange condition at all it is quite a nice condition it implies continuity.

Contraction mapping can also be defined from one metric space to another metric space which I already told you.

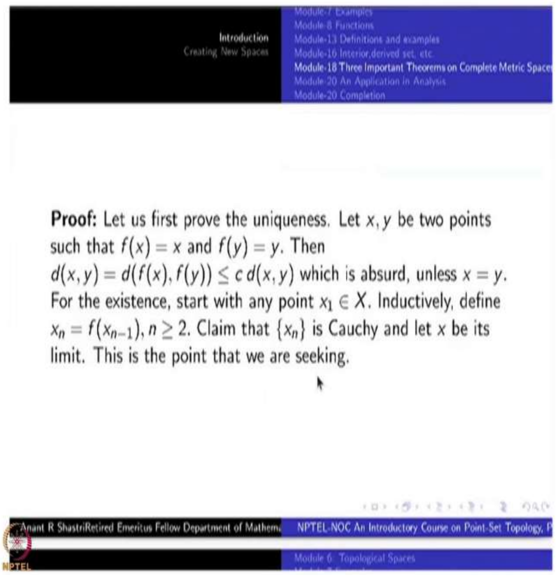
(Refer Slide Time: 10:19)





So, the theorem says that if $X$ is a complete metric space and you have a contraction mapping $f$ from $X$ to $X$, then $f$ fixes a point. Fixes a point means that there is some $x$ in $X$ such that $f(x)$ equal to $x$. Moreover such an $x$ is unique. So, then there exists a unique $x$ belonging to $X$ such that $f(x)$ equal to $x$, that is the statement.

(Refer Slide Time: 10:52)



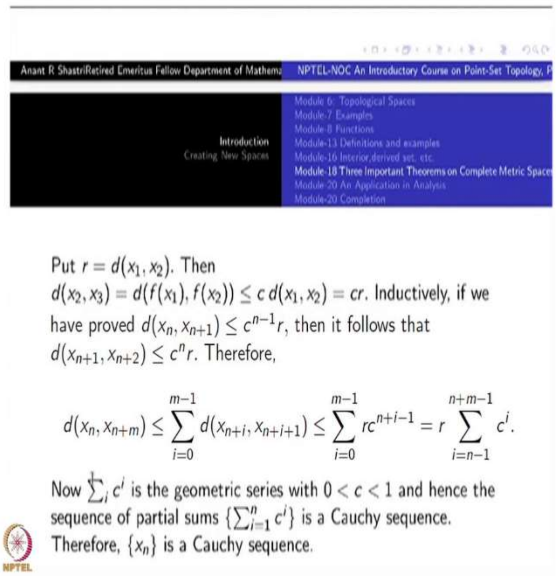
The proof this time is slightly little longer than that of Cantor's Intersection Theorem, alright. So, let us first take care of uniqueness. Suppose there are two points $x$ and $y$ such that $f(x)$ is equal to $x$ and $f(y)$ equal to $y$. Then you apply you know, $d(f(x), f(y))$ ok? $d(f(x), f(y))$; $f(x)$ is $x$ ok $f(y)$ is equal to $y$. So, it is $d(x, y)$; $d(x, y)$ equal to $d(f(x), f(y))$, but because it is a contraction mapping it is less than equal to $cd(x, y)$, but $c < 1$. So, this is cannot happen unless $d(x, y)$ is 0, which is same thing as saying $x$ is equal to $y$. So, the proof of that uniqueness theorem uniqueness part is as easy as in the first part.

Now for the existence. So, one may say that you know, this Banach's contraction mapping where he got the idea? maybe he got the idea from Newton. People who are familiar with some elementary numerical analysis, they will know this kind of iteration method, you know, which was also used by Picard and others. So, start with any point ok? Apply $f$. If it is a different point apply $f$ again, if it is different apply $f$ again. Your $f(x)$ will be equal to $x$ eventually. That is the whole idea.

But it may not happen at all. So, that is where the ingenuity comes. So, do not give up the method yet. So, so look deeper into it that is the point. So, you start with some $x_1$ belonging to $X$. Inductively define $x_n$ to be the image, $f(x_{n-1})$. So, apply $f$ to that previous one and take that as $x_n$.

So, $x_1$ is any point. I do not mind. $x_2$ will be $f(x_1), x_3$ will be $f(x_2)$ and so on. So, claim is that this $x_n$ is a Cauchy sequence. So, this time it is not all that obvious you have to do a little more work that is all. You know in the case of CIT, it was easy to see that $(x_n)$ is a Cauchy sequence ok. Once it is a Cauchy sequence completeness comes into picture. There will be a limit point; the limit point of this sequence, ok? The beauty is that that limit point cannot go anywhere else, $f(x)$ has to be $x$ itself ok? So, that is the point we are seeking. So, let us see how the proofs of all, these claims are coming.

(Refer Slide Time: 14:24)





So, put $r$ equal to distance between $x_1$ and $x_2$. We are assuming that $x_2$ is not equal to $x_1$, ok. If $x_1$ is equal to $x_2$, then there is nothing to bother about going further at all because then what is $x_2$? $x_2$ is $f(x_1)$, if it is equal to $x_1$, that is fixed point, we are home already. So, anyway we are not going to use that. Put $r$ equal $d(x_1, x_2)$. Then look at distance between $x_2$ and $x_3$. That is, by definition distance mean $f(x_1)$ and $f(x_2)$ because $x_2$ is $f(x_1)$, $x_3$ is $f(x_2)$, but then this is, by contraction mapping part, less than $c$ times distance between $x_1$ and $x_2$. This we have denoted by $r$. So, it is less than $cr$. So, distance between $x_2$ and $x_3$ will be less than or equal to $cr$, alright? What happens to distance between $x_3$ and $x_4$? One more $c$ will come, one more $c$ will come and so on right?

So, inductively distance between $x_n$ and $x_{n+1}$ will be less than or equal to $c^{n-1}r$, ok? So, $d(x_2, x_3)$ is less than $cr$, $d(x_n, x_{n+1})$ will be $c^{n-1}r$; the index will be one less here. Then it follows that distance between $x_{n+1}$ and $x_{n+2}$ ok, is less than or equal to $c^n r$, ok, once you have proved this one.

Therefore, if you look at distance between $x_n$ and $x_{n+m}$, see we are trying to prove that this is a Cauchy sequence right. So, distance between $x_n$ and $x_{n+1}$ is less than or equal to $r \sum_{i=n-1}^{n+m-1} c^i$.

I start with $x_n$ go to $x_{n+1}, x_{n+1}$ go to $x_{n+2}, x_{n+2}$ go to $x_{n+3}$, I use triangle inequality and then I get the summation. So, distance will $x_{n+i}$ to $x_{n+i+1}$, ok. So, this is a summation.

But each of them is $rc^{n+i-1}$. We can rewrite it; $r$ will come out from the summation $i$ ranges from $n-1$ to $n+m-1$ of $c^i$. What you know is $c < 1$, therefore this is a geometric series ok? Mother of all series. So, this is a convergent series we know how to compute the limit also anyway.

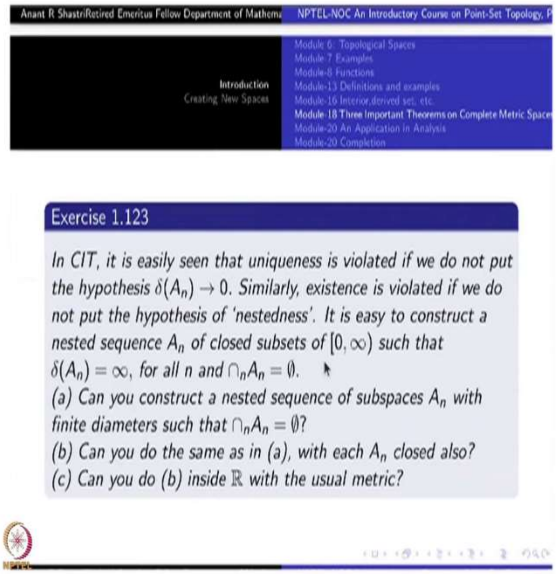
So, geometric series because $c < 1$; and hence the partial sums $i$ ranging from $0$ to $n$ of $c^i$, they are Cauchy sequences ok. So, this is this is just the difference between $(n-1)^{th}$ partial sum and $(n+m)^{th}$ partial sum, I have taken right? So, if this sequence of partial sums is a Cauchy sequencem this number can be made less than $\epsilon/r$. So, $r$ times that will be less than $\epsilon$ for sufficiently large $n$. That means, this is a Cauchy sequence, alright.

(Refer Slide Time: 18:48)


So, starting with a point $x_1$ and repeatedly applying $f, f, f$ and so on, we have got a Cauchy sequence, that sequence converges. Take the limit say $x$, ok? Now, we apply $f$, which is a continuous function right? We apply $f$ on $x$. What is $x$?$x$ is limit of all these $x_n$'s, but $x_n$, I can write it as $f(x_{n-1})$. So, I can take it as $x_{n+1}$ and write it as $f(x_n)$, ok? because $f$ is continuous, applying $f(x_n)$ is same thing as limit of $x_{n+1}$, but limit of $x_{n+1}$ or $x_{n+2}$ or whatever, as $n$ tends to infinity is same limit which is $x$.

So, $x$ is a fixed point of $f$, uniqueness you have already proved ok.

(Refer Slide Time: 19:41)




So, the third theorem takes a little more time. Therefore, let us first consolidate these two theorems today and we will do the third theorem next timem ok? So, let us look at some easy comments here on this. Look at the Cantor's Intersection Theorem ok? What are conditions? $\delta(F_n)$ goes to 0, Oh-ho! $F_n$ or $A_n$ whatever ok, $A_n$'s are closed, $X$ is complete ok? and $A_n$'s are one contained in the other. The point is if you violate any of those conditions something will go wrong, ok?

So, conclusion will not be as strong as you like, or conclusion may not be there at all ok? For example, suppose $\delta(A_n)$ does not tend to 0 ok? but because they are smaller and smaller it may tend to something finite, but a positive number, then you will not get the uniqueness. You may be able to prove the existence, that is, intersection is non empty. There is some point, but you will not be able to prove that it is a unique point ok?

Similarly, existence itself will be violated if we do not put the hypotheses nested. If you take arbitrary subsets ok? $F_1, F_2, \ldots, F_n, \ldots$ You know without any relation between them maybe half of it is contained in there or they are disjoint and so on, there may not be any intersection, intersection can be easily empty right? now that is not very surprising ok? The third comment I make is that here it is easy to construct a nested sequence of closed subsets of 0 to infinity, by the way 0 to infinity, is a complete metric space such that all the diameters are infinity yet the intersection is empty ok?

So, here the diameters are not decreasing to zero or any finite number, they are all infinity. Yet the intersection is empty ok? Can you think of such a sequence? what should be $A_n$ for each $n$? Remember they must be closed subsets and $A_n$ should be containing $A_{n+1}$ containing $A_{n+2}$ and so on smaller and smaller. Can you think of such a sequence?

Student: Yes sir, if I take $[n, \infty)$.

Right put $A_n$ equal to $[n, \infty)$ very good. So, intersection will be empty it satisfies all other things, but now $\delta(A_n)$ it is not converging to 0 all the time, it is infinity ok? Yet the intersection can be empty ok. So, here is another question. Can you construct a nested sequence of subspaces $A_n$ with finite diameter such that the intersection of $A_n$ is empty? You can do it in $\mathbb{R}$ itself? Note that this time I am requiring nested sequences of diameters finite also but I am not saying diameters converge to 0.

But what I am not saying is that they should be also closed right? there is no word `closed' here. So, is it possible? No? Think of some subsets of the open interval $(0, 1)$.

Student: Open interval $(0, 1/n)$.

Yes. Instead of going up, n to infinity, here you take $(0, 1/n)$, right? Intersection will be empty. Now, can you do the same thing as you did, but this time all $A_n$'s closed; closed and nested, that is all. I am not asking for diameters converging to 0. Is it possible? This is a little more harder right? So, think about this.

The next one is little more harder also. Can you do the same thing as (b), but in $\mathbb{R}$ with the usual metric?

So, this is a hint for the other one. maybe you can change the metric ok? And then try to do that not in the usual metric.

So, (c) is much harder. So, think about them we will answer them in due time, maybe as an assignment. You can work it out. Then we will explain it after checking your answers. We will explain you the result ok? So, that is all for today. So, let us meet next time.

Thank you.