

Introduction to Point Set Topology, (Part I)
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Module - 18
Lecture - 18
The Metric Trinity

(Refer Slide Time: 00:15)

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Introduction
Creating New Spaces

Module 6 Topological Spaces
Module 7 Examples
Module 8 Functions
Module 13 Definitions and examples
Module 16 Interior, derived set, etc.
Module-18 Three Important Theorems on Complete Metric Spaces
Module-20 An Application in Analysis
Module-20 Completion

Module-18 The Metric Trinity

Anant Shastri

Here we would like to discuss **Three** of the most important results on complete metric spaces:

**Cantor's Intersection Theorem(CIT),
Banach's Contraction Mapping Principle(BCMP) and
Baire's Category Theorem(BCT).**

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Welcome to module 18, I have boldly named it as The Metric Trinity. What we are going to do today is to discuss three most important theorems, according to me, in metric space theory. So, one of them is Cantor's Intersection Theorem, second one is Banach's Contraction Mapping Principle, the third one is Baire's Category Theorem. You can call all of them principles, from theorems to they have become principles anyway.

(Refer Slide Time: 01:14)

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Introduction
Creating New Spaces

Module 6: Topological Spaces
Module 7: Examples
Module 8: Functions
Module 11: Definitions and examples
Module 16: Interrelated results
Module 18: Three Important Theorems on Complete Metric Spaces
Module 20: An Application in Analysis
Module 20: Completion

The CIT is so fundamental that we shall see that BCT is an easy consequence of it. The method of proofs of CIT and BCMP are educative and occur many times in analysis and topology. So, pay attention to them.
Throughout this section $X = (X, d)$ will denote a metric space.

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The high status of a theorem is to become principle. The CIT; CIT I mean Cantor's Intersection Theorem. This is so fundamental that even in the proof of Baire's category theorem we will be using it. Banach's contraction mapping theorem is slightly of a different flavour. It does not use CIT directly, ok?

The method of the proofs of all these three are themselves quite educative. So, often you may have to employ that in your own research work if you want to do, you know, deep analysis or topology. So, I would like you to pay attention to not only the statement here but how things have been arrived at. The proofs also should be learned properly.

(Refer Slide Time: 02:38)

The image shows a presentation slide with a table of contents at the top and a theorem statement below. The table of contents lists modules 6 through 20, with Module 10 highlighted. The theorem statement is Theorem 1.119, Cantor's Intersection Theorem.

Introduction Creating New Spaces	Module 6: Topological Spaces Module 7: Examples Module 8: Functions Module 13: Definitions and examples Module 15: Interior, derived set, etc. Module 10: Three Important Theorems on Complete Metric Spaces Module 20: An Application in Analysis Module 20: Completion
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Theorem 1.119
Cantor's Intersection Theorem *Let X be a complete metric space. Let $\cdots \supset F_n \supset F_{n+1} \supset \cdots$ be a nested sequence of non-empty closed subsets of X such that the sequence of diameters, $\delta(F_n) \rightarrow 0$. Then $\bigcap F_n$ consists of precisely one element of X .*

So, let us begin with freezing this symbol (X, d) or just X for a metric space once for all in this section. The first theorem Cantor's Intersection Theorem. So, start with a complete metric space X . Take a nested sequence of non empty closed subsets of X . Nested means what? one contained in the other but the other way around. F_n contains F_{n+1} and so on, they are decreasing here, ok? Nested could be the other way around also.

So, here they are decreasing, but condition comes automatically after the second condition here. So, $\delta(F_n)$ denotes the diameter of each of them. So, this sequence of real number converges to 0. So, this is also a condition. So, each F_n is closed which contains F_{n+1} and the diameter tends to 0. Then the statement is that intersection of all these F_n 's consists precisely of one element, ok?

This is stronger than saying that it is non empty. This has exactly one element. The proof is surprisingly very simple, ok? You will see that.

(Refer Slide Time: 04:12)

NPTEL NOC An Introductory Course on Point Set Topology, P

Module 6: Topological Spaces
 Module 7: Compact
 Module 8: Functions
 Module 13: Definitions and examples
 Module 16: Interior, derived set, etc.
 Module 18: Three Important Theorems on Complete Metric Spaces
 Module 20: An Application in Analysis
 Module 20: Completion

Introduction
 Creating New Spaces

Proof: Choose $x_n \in F_n$ for each n . Then
 $d(x_n, x_{n+m}) \leq \delta(F_n), \forall m \in \mathbb{N}$, and therefore, $\{x_n\}$ is Cauchy.
 Since X is complete, $\{x_n\}$ is convergent, say, $x_n \rightarrow x$. Since each
 F_n is closed, it follows that $x \in F_n$ for all n . Therefore,
 $x \in \bigcap_n F_n \neq \emptyset$. If $y \neq x$, then $d(x, y) > \delta(F_n)$ for some n , which
 means y cannot be inside F_n .

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All that I do is F_n 's are non empty, so, pick up x_n belong to F_n for each n . Then look at this sequence $d(x_n, x_{n+m})$ is less than $\delta(F_n)$ because both of the elements are inside F_n . This δ is what? δ is the diameter, it is the supremum of all such numbers $d(x, y)$, where x and y range over F_n . Therefore, the distance between x_n and x_{n+m} becomes less than $\delta(F_n)$, but $\delta(F_n)$ itself tends to 0. Therefore, it can be made less than epsilon etc.

So, this implies that (x_n) is a Cauchy sequence, but now I use that X is complete therefore, (x_n) is convergent. So, let us take x to be the limit, ok? Now, we also know that if you have a sequence (x_n) , you see the entire sequence (x_n) is not in any one of the sets, but suppose you choose some F_k , here then k onwards that sequence is in F_k . The limit of that portion of the sequence is also x , right?

The first $k - 1$ terms does not matter here. Therefore, each of these sequence can be thought of as a sequence inside F_n , ok and the limit therefore, must be inside the closed sets F_n . So x is in the intersection. It follows that this limit point is inside all the F_n 's. So, here we are used that F_n 's are closed and they are monotonically you know decreasing one contained in the other that is also you have to use. That just means that intersection is non empty.

The second part: that it must be a singleton; this comes very easily because of this $\delta(F_n)$ goes to 0, you can see that this the diameter of the intersection will also become 0. So, diameter is 0 the set can be what? Only singleton. Anyway if $y \neq x$, you know then $d(x, y)$ will be

positive therefore, it will be bigger than $\delta(F_n)$ for large n because $\delta(F_n)$ is converging to 0. Once it is bigger than $\delta(F_n)$ both x and y cannot be inside F_n , right? If it is not in one F_n , it cannot be in the intersection. So, any point which is not equal to x is not inside the intersection. So, there is only one point. So, that is a proof ok?

(Refer Slide Time: 07:24)

Definition 1.120

Let $f : X \rightarrow X$ be any function. We say f is a **contraction** (or a **contraction mapping**) if there exists a constant $0 < c < 1$ such that $d(f(x), f(y)) \leq c d(x, y)$ for all $x, y \in X$.

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- Module 6: Topological Spaces
- Module 7: Examples
- Module 8: Functions
- Module 11: Definitions and examples
- Module 16: Interior, derived set, etc.
- Module 18: Three Important Theorems on Complete Metric Spaces
- Module 20: An Application in Analysis
- Module 20: Completion

Introduction
Creating New Spaces

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Now, I have to make a definition because there is a new term in the Banach contraction mapping, in the statement itself. So, take a function from one metric space to itself OR to some other metric space also, this will work. But right now, I need only this definition.

We say f from X to X is a contraction mapping or just a contraction if there exists a positive constant $c < 1$, this strictly less than 1 is important, such that the distance between $f(x)$ and $f(y)$ is less than or equal c times the distance between x and y . For example, if c is half each time you apply f , ok? What happens? So, two points at distance one will go to points less than half the distance apart next time one fourth and so on ok, that is the meaning of contraction mapping ok?

If you talk in the Layman's language and take a map of your campus or a country or just a state, say. If the map is up to scale definitely, it will be a contraction mapping, ok? From the actual object of which it is a map to the map. You can think of holding the map in your hand,

you know, you are standing in the campus or the country. So, the map is inside. So, the function f is from the country into the country, but it is a contraction mapping.

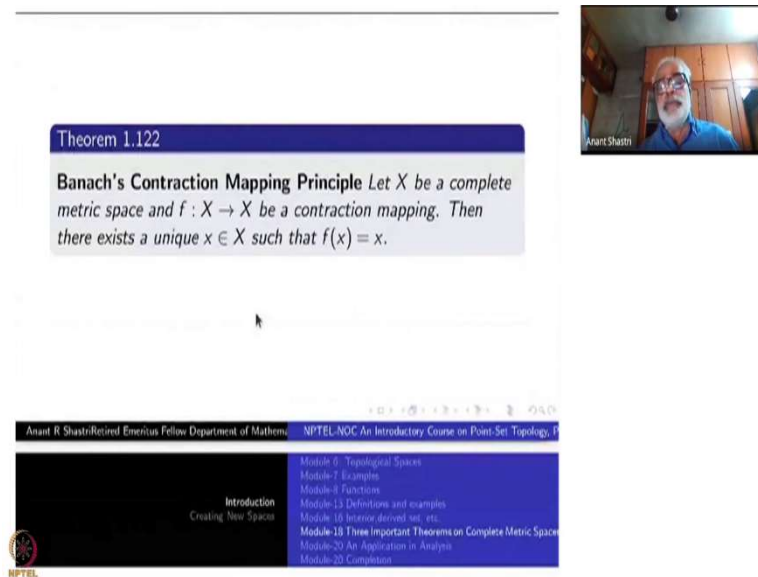
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The image shows a screenshot of a video lecture. At the top, there is a navigation menu with the following items: Introduction (Creating New Spaces), Module-4 Functions, Module-13 Definitions and examples, Module-18 Introductory Topology, etc., Module-10 Three Important Theorems on Complete Metric Spaces, Module-20 An Application in Analysis, and Module-20 Completion. A small video window in the top right corner shows the speaker, Anant Shastri. The main content of the slide is a blue box with the text: 'Remark 1.121 Note that any contraction is obviously continuous. Contraction mappings can also be defined from one metric space to another metric space, in an obvious way.' At the bottom, there is a footer with the text: 'Anant R. Shastri Retired Emeritus Fellow Department of Mathematics NPTEL-NOC An Introductory Course on Point-Set Topology, P' and 'Module-6: Topological Spaces Module-7: Examples'.

Now, what does this theorem say let us see, ok? So, for example, contraction mapping is always continuous because I want to control distance between $f(x)$ and $f(y)$, I can choose δ equal to ϵ/c , that is all. So, this will be less than ϵ . So, continuity is obvious. By the way this condition constant c_1, c_2 on both sides we are familiar with that. That gives equivalence of metrics right? Similarity of metrics. So, this is not a strange condition at all it is quite a nice condition it implies continuity.

Contraction mapping can also be defined from one metric space to another metric space which I already told you.

(Refer Slide Time: 10:19)



Theorem 1.122

Banach's Contraction Mapping Principle Let X be a complete metric space and $f : X \rightarrow X$ be a contraction mapping. Then there exists a unique $x \in X$ such that $f(x) = x$.

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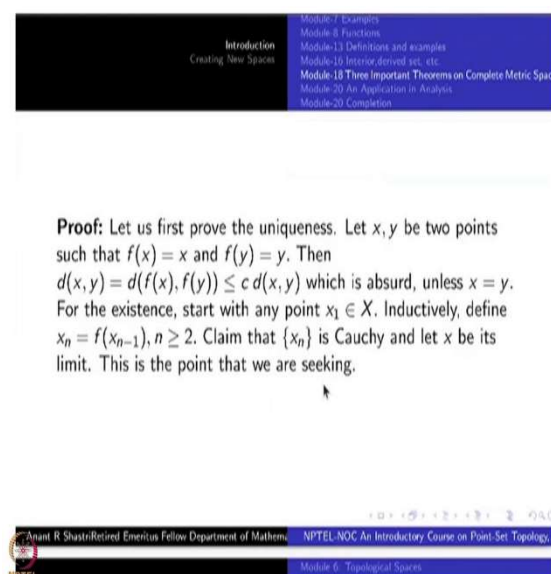
Introduction
Creating New Spaces

Module 6: Topological Spaces
Module 7: Examples
Module 8: Functions
Module 13: Definitions and examples
Module 16: Interior, derived set, etc.
Module 18: Three Important Theorems on Complete Metric Spaces
Module 20: An Application in Analysis
Module 20: Completion

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So, the theorem says that if X is a complete metric space and you have a contraction mapping f from X to X , then f fixes a point. Fixes a point means that there is some x in X such that $f(x)$ equal to x . Moreover such an x is unique. So, then there exists a unique x belonging to X such that $f(x)$ equal to x , that is the statement.

(Refer Slide Time: 10:52)



Introduction
Creating New Spaces

Module 7: Examples
Module 8: Functions
Module 13: Definitions and examples
Module 16: Interior, derived set, etc.
Module 18: Three Important Theorems on Complete Metric Spaces
Module 20: An Application in Analysis
Module 20: Completion

Proof: Let us first prove the uniqueness. Let x, y be two points such that $f(x) = x$ and $f(y) = y$. Then $d(x, y) = d(f(x), f(y)) \leq c d(x, y)$ which is absurd, unless $x = y$. For the existence, start with any point $x_1 \in X$. Inductively, define $x_n = f(x_{n-1}), n \geq 2$. Claim that $\{x_n\}$ is Cauchy and let x be its limit. This is the point that we are seeking.

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Module 6: Topological Spaces

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The proof this time is slightly little longer than that of Cantor's Intersection Theorem, alright. So, let us first take care of uniqueness. Suppose there are two points x and y such that $f(x)$ is equal to x and $f(y)$ equal to y . Then you apply you know, $d(f(x), f(y))$ ok? $d(f(x), f(y))$; $f(x)$ is x ok $f(y)$ is equal to y . So, it is $d(x, y)$; $d(x, y)$ equal to $d(f(x), f(y))$, but because it is a contraction mapping it is less than equal to $cd(x, y)$, but $c < 1$. So, this is cannot happen unless $d(x, y)$ is 0, which is same thing as saying x is equal to y . So, the proof of that uniqueness theorem uniqueness part is as easy as in the first part.

Now for the existence. So, one may say that you know, this Banach's contraction mapping where he got the idea? maybe he got the idea from Newton. People who are familiar with some elementary numerical analysis, they will know this kind of iteration method, you know, which was also used by Picard and others. So, start with any point ok? Apply f . If it is a different point apply f again, if it is different apply f again. Your $f(x)$ will be equal to x eventually. That is the whole idea.

But it may not happen at all. So, that is where the ingenuity comes. So, do not give up the method yet. So, so look deeper into it that is the point. So, you start with some x_1 belonging to X . Inductively define x_n to be the image, $f(x_{n-1})$. So, apply f to that previous one and take that as x_n .

So, x_1 is any point. I do not mind. x_2 will be $f(x_1)$, x_3 will be $f(x_2)$ and so on. So, claim is that this x_n is a Cauchy sequence. So, this time it is not all that obvious you have to do a little more work that is all. You know in the case of CIT, it was easy to see that (x_n) is a Cauchy sequence ok. Once it is a Cauchy sequence completeness comes into picture. There will be a limit point; the limit point of this sequence, ok? The beauty is that that limit point cannot go anywhere else, $f(x)$ has to be x itself ok? So, that is the point we are seeking. So, let us see how the proofs of all, these claims are coming.

(Refer Slide Time: 14:24)

Put $r = d(x_1, x_2)$. Then $d(x_2, x_3) = d(f(x_1), f(x_2)) \leq c d(x_1, x_2) = cr$. Inductively, if we have proved $d(x_n, x_{n+1}) \leq c^{n-1}r$, then it follows that $d(x_{n+1}, x_{n+2}) \leq c^n r$. Therefore,

$$d(x_n, x_{n+m}) \leq \sum_{i=0}^{m-1} d(x_{n+i}, x_{n+i+1}) \leq \sum_{i=0}^{m-1} r c^{n+i-1} = r \sum_{i=n-1}^{n+m-1} c^i.$$

Now $\sum_{i=0}^{\infty} c^i$ is the geometric series with $0 < c < 1$ and hence the sequence of partial sums $\{\sum_{i=1}^n c^i\}$ is a Cauchy sequence. Therefore, $\{x_n\}$ is a Cauchy sequence.

So, put r equal to distance between x_1 and x_2 . We are assuming that x_2 is not equal to x_1 , ok. If x_1 is equal to x_2 , then there is nothing to bother about going further at all because then what is x_2 ? x_2 is $f(x_1)$, if it is equal to x_1 , that is fixed point, we are home already. So, anyway we are not going to use that. Put r equal $d(x_1, x_2)$. Then look at distance between x_2 and x_3 . That is, by definition distance mean $f(x_1)$ and $f(x_2)$ because x_2 is $f(x_1)$, x_3 is $f(x_2)$, but then this is, by contraction mapping part, less than c times distance between x_1 and x_2 . This we have denoted by r . So, it is less than cr . So, distance between x_2 and x_3 will be less than or equal to cr , alright? What happens to distance between x_3 and x_4 ? One more c will come, one more c will come and so on right?

So, inductively distance between x_n and x_{n+1} will be less than or equal to $c^{n-1}r$, ok? So, $d(x_2, x_3)$ is less than cr , $d(x_n, x_{n+1})$ will be $c^{n-1}r$; the index will be one less here. Then it follows that distance between x_{n+1} and x_{n+2} ok, is less than or equal to $c^n r$, ok, once you have proved this one.

Therefore, if you look at distance between x_n and x_{n+m} , see we are trying to prove that this is a Cauchy sequence right. So, distance between x_n and x_{n+1} is less than or equal to

$$r \sum_{i=n-1}^{n+m-1} c^i.$$

I start with x_n go to x_{n+1} , x_{n+1} go to x_{n+2} , x_{n+2} go to x_{n+3} , I use triangle inequality and then I get the summation. So, distance will x_{n+i} to x_{n+i+1} , ok. So, this is a summation.

But each of them is rc^{n+i-1} . We can rewrite it; r will come out from the summation i ranges from $n - 1$ to $n + m - 1$ of c^i . What you know is $c < 1$, therefore this is a geometric series ok? Mother of all series. So, this is a convergent series we know how to compute the limit also anyway.

So, geometric series because $c < 1$; and hence the partial sums i ranging from 0 to n of c^i , they are Cauchy sequences ok. So, this is this is just the difference between $(n - 1)^{th}$ partial sum and $(n + m)^{th}$ partial sum, I have taken right? So, if this sequence of partial sums is a Cauchy sequence this number can be made less than ϵ/r . So, r times that will be less than ϵ for sufficiently large n . That means, this is a Cauchy sequence, alright.

(Refer Slide Time: 18:48)

Therefore, $\{x_n\}$ is a Cauchy sequence.

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Module 6: Topological Spaces
 Module 7: Examples
 Module 8: Functions
 Module 13: Definitions and examples
 Module 16: Inverse derived set, etc.
Module-18 Three Important Theorems on Complete Metric Spaces
 Module 20: An Application in Analysis
 Module 20: Completion

Let $\lim_n x_n = x$. (This is where we use the completeness of X .)
 Since f is continuous

$$f(x) = \lim_n f(x_n) = \lim_n x_{n+1} = x.$$

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So, starting with a point x_1 and repeatedly applying f, f, f and so on, we have got a Cauchy sequence, that sequence converges. Take the limit say x , ok? Now, we apply f , which is a continuous function right? We apply f on x . What is x ? x is limit of all these x_n 's, but x_n , I can write it as $f(x_{n-1})$. So, I can take it as x_{n+1} and write it as $f(x_n)$, ok? because f is

continuous, applying $f(x_n)$ is same thing as limit of x_{n+1} , but limit of x_{n+1} or x_{n+2} or whatever, as n tends to infinity is same limit which is x .

So, x is a fixed point of f , uniqueness you have already proved ok.

(Refer Slide Time: 19:41)

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Introduction Creating New Spaces	Module 6: Topological Spaces Module 7: Examples Module 8: Functions Module 11: Definitions and examples Module 16: Interior, derived set, etc. Module 18: Three Important Theorems on Complete Metric Spaces Module 20: An Application in Analysis Module 20: Completion
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Exercise 1.123

In CIT, it is easily seen that uniqueness is violated if we do not put the hypothesis $\delta(A_n) \rightarrow 0$. Similarly, existence is violated if we do not put the hypothesis of 'nestedness'. It is easy to construct a nested sequence A_n of closed subsets of $[0, \infty)$ such that $\delta(A_n) = \infty$, for all n and $\bigcap_n A_n = \emptyset$.

- Can you construct a nested sequence of subspaces A_n with finite diameters such that $\bigcap_n A_n = \emptyset$?
- Can you do the same as in (a), with each A_n closed also?
- Can you do (b) inside \mathbb{R} with the usual metric?

So, the third theorem takes a little more time. Therefore, let us first consolidate these two theorems today and we will do the third theorem next time ok? So, let us look at some easy comments here on this. Look at the Cantor's Intersection Theorem ok? What are conditions? $\delta(F_n)$ goes to 0, Oh-ho! F_n or A_n whatever ok, A_n 's are closed, X is complete ok? and A_n 's are one contained in the other. The point is if you violate any of those conditions something will go wrong, ok?

So, conclusion will not be as strong as you like, or conclusion may not be there at all ok? For example, suppose $\delta(A_n)$ does not tend to 0 ok? but because they are smaller and smaller it may tend to something finite, but a positive number, then you will not get the uniqueness. You may be able to prove the existence, that is, intersection is non empty. There is some point, but you will not be able to prove that it is a unique point ok?

Similarly, existence itself will be violated if we do not put the hypotheses nested. If you take arbitrary subsets ok? $F_1, F_2, \dots, F_n, \dots$. You know without any relation between them maybe half of it is contained in there or they are disjoint and so on, there may not be any intersection, intersection can be easily empty right? now that is not very surprising ok? The third comment I make is that here it is easy to construct a nested sequence of closed subsets of \mathbb{R} from 0 to infinity, by the way \mathbb{R} is a complete metric space such that all the diameters are infinity yet the intersection is empty ok?

So, here the diameters are not decreasing to zero or any finite number, they are all infinity. Yet the intersection is empty ok? Can you think of such a sequence? what should be A_n for each n ? Remember they must be closed subsets and A_n should be containing A_{n+1} containing A_{n+2} and so on smaller and smaller. Can you think of such a sequence?

Student: Yes sir, if I take $[n, \infty)$.

Right put A_n equal to $[n, \infty)$ very good. So, intersection will be empty it satisfies all other things, but now $\delta(A_n)$ it is not converging to 0 all the time, it is infinity ok? Yet the intersection can be empty ok. So, here is another question. Can you construct a nested sequence of subspaces A_n with finite diameter such that the intersection of A_n is empty? You can do it in \mathbb{R} itself? Note that this time I am requiring nested sequences of diameters finite also but I am not saying diameters converge to 0.

But what I am not saying is that they should be also closed right? there is no word 'closed' here. So, is it possible? No? Think of some subsets of the open interval $(0, 1)$.

Student: Open interval $(0, 1/n)$.

Yes. Instead of going up, n to infinity, here you take $(0, 1/n)$, right? Intersection will be empty. Now, can you do the same thing as you did, but this time all A_n 's closed; closed and nested, that is all. I am not asking for diameters converging to 0. Is it possible? This is a little more harder right? So, think about this.

The next one is little more harder also. Can you do the same thing as (b), but in \mathbb{R} with the usual metric?

So, this is a hint for the other one. maybe you can change the metric ok? And then try to do that not in the usual metric.

So, (c) is much harder. So, think about them we will answer them in due time, maybe as an assignment. You can work it out. Then we will explain it after checking your answers. We will explain you the result ok? So, that is all for today. So, let us meet next time.

Thank you.