Introduction to Point Set Topology, (Part I)
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Module - 17
Lecture - 17
More Examples

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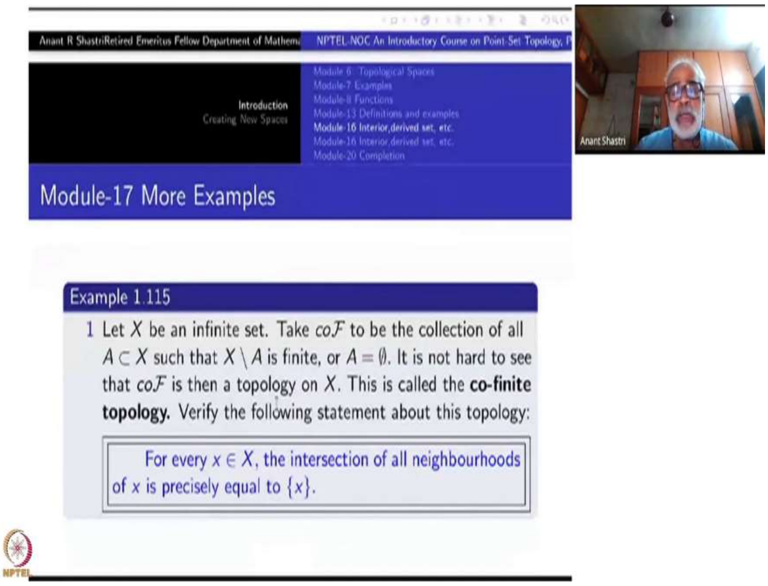

Welcome to module 17 of Point Set Topology course. This time we shall consider a few more interesting examples of topological spaces. The most interesting one that we are going to do today will be a subspace of $\mathbb{R}$ itself. Before that let me consider a few things which are out of $\mathbb{R}$, not from the usual topology.

Start with any infinite set then I am denoting this topology $co\mathcal{F}$; $co\mathcal{F}$ cofinite. So, you can read it as co finite. So, what is this? It is the collection of all subsets A of X such that A^c is finite. Of course, I will have to allow A to be empty also because complement of an empty set is the whole space, and in this definition I do not have that one right? $X \setminus A$ is finite. If A is X of course, it is allowed, A is empty is not allowed in this rule. So, I have to allow it specifically.

So, either A is empty or $X \setminus A$ is finite, then I put it inside this collection co-finite topology. So, I want to say this is a topology ok? So, this is called the co-finite topology on the given set X. Once you consider such a topology, it is no point in considering X to be a finite set. For then what happens? This becomes just the discrete, because all subsets will be there because their complements are also finite.

So, it is interesting only when you take X to be an infinite set, ok? How to verify that this is a topology? That is not very difficult because what you have to do? Once you take two of them $X \setminus A_1$ and $X \setminus A_2$, right? What is the complement of intersection? It is the union of the complements right? So, each of them is finite so, union is also finite.

So, intersection of two open sets is open or intersection of finitely many open set is open The union of arbitrary families is open because as soon as one of them complement is finite if you enlarge it larger and larger say it would be automatically finite. So, that part is easier here. So, that will verify that this co-finite becomes a you know family of co-finite subsets can become a topology. We will have many many instances and usefulness of this one throughout the course, ok?

So, this is going to be kind of model for us. One of the interesting property you can directly verify it right now namely take any point in x look at all the neighbourhoods of this point intersection is precisely the $\{x\}$, that is the only point. x is common to all the neighbourhoods ok? So, verify this statement just to begin with getting familiarity with this co-finite topology. We will have many instances of studying this one. So, this is just an introduction right now.

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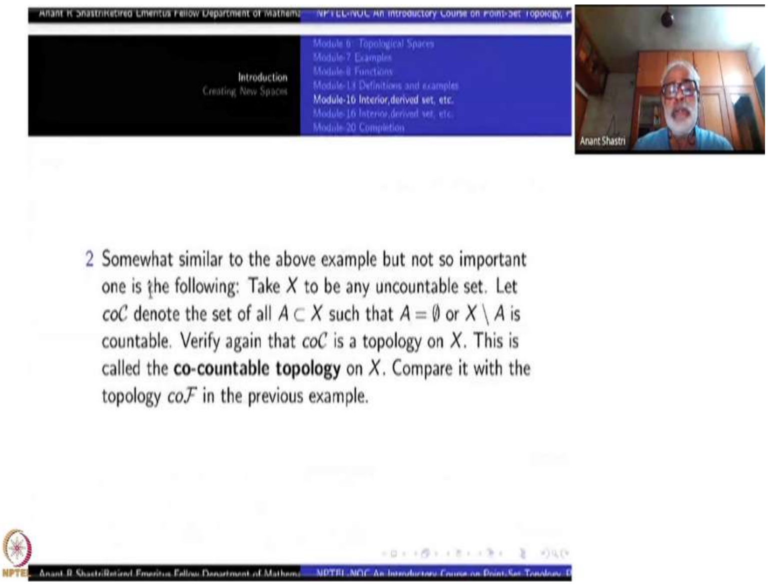

Similar to this one, but not so important is the following, ok. Take X to be an uncountable set. Let $co\mathcal{C}$ is in a co-countable denote the set of all subsets such that either its empty that empty set is allowed or the complement is countable. Now countable includes finiteness as well as infinite countable also ok. So, countable means that alright.

So, once again verification that this is a topology is identical to the previous one there is no problem ok, because countable finite union of countable sets is countable. That is all I am using here ok. So, this is called the co-countable topologies just similar to co-finite topology ok. So, it has similar properties. Once again there is no point in taking this topology when X is a countable set then again it will be a discrete topology. Therefore, I take X to be starting with with an uncountable set ok, right.

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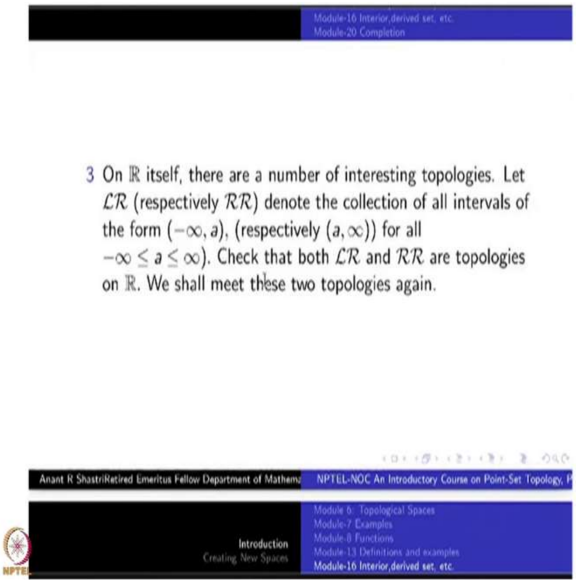

Let us go to the 3rd example here. Actually there are two of them. This time I consider the underlying set $\mathbb{R}$ itself. But topology is different. So, I better write it with different symbols. So, here I have used $\mathcal{LR}$ (respectively $\mathcal{RR}$) to denote collection of all intervals of the form $(-\infty, a)$ open, to denote that these are left handed rays $\mathcal{LR}$ (respectively $\mathcal{RR}$), $\mathcal{R}$ for ray. Instead of arbitrary notation.

To indicate that they are unbounded, we are using the term rays. They are unbounded intervals right? Unbounded below in $\mathcal{LR}$. The other ones, in $\mathcal{RR}$, intervals of the form (a, ∞) , a varies from $-\infty$ to ∞. a to infinity, right rays ok. Where, this a ranging? a can be taken as all the way from $-\infty$ included to ∞ including. What does it mean? Open interval infinity to infinity means empty set; minus infinity to infinity means the whole of $\mathbb{R}$. Open interval I am taking remember that. So, that will be just the whole of $\mathbb{R}$, ok?

So, this forms a topology because if you take intersection of $(-\infty, a)$ with $(-\infty, b)$, look at whether a is bigger than b or b is bigger than a, intersection would be just smaller one. If you take some a_α's here and take the union then the union will be what? Take the supremum of these a_α's and take that open interval from minus infinity to the supremum.

Exactly similar things hold for $\mathcal{RR}$ also. Here you may have to take infimum to show that arbitrary union of open rays is again an open ray. So, this is even simpler than the topology in which all open intervals are allowed right? the usual euclidean topology. So; obviously, in both of them there are fewer open sets than in the usual topology on $\mathbb{R}$.

Also, they are different topologies in any case. For example, the open ray here to the right will not be an open subset in the $\mathcal{LR}$, see and vice versa alright. So, these things are again important in ah ah analysis. So, when an opportunity arises I will again refer to this one.

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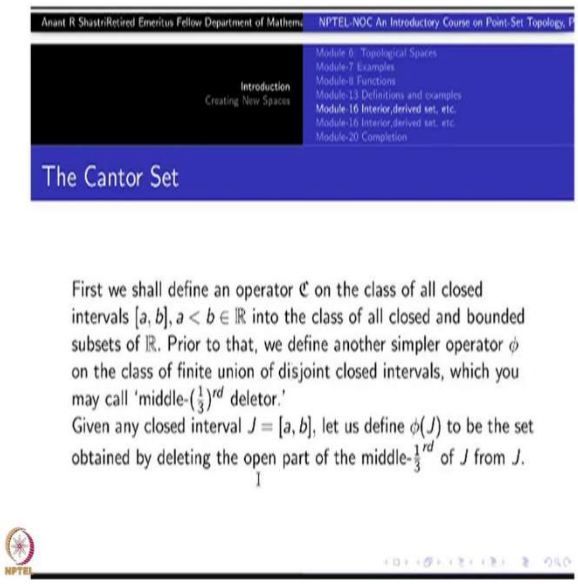

Now I come to the what I told you the most important example. This time I have a subset of $\mathbb{R}$ itself with the usual topology, ok? So, this is called the Cantor set which is a landmark result in topology ok, a landmark result by Cantor. First we shall define an operator. You may read it as Cantor I do not know how to read this symbol here I read it as Cantor. On the class of all closed intervals $[a, b]$, a less than b, ok? I am not interested in singleton intervals, a less than b.

So, closed intervals like that into the class of all closed and bounded subsets of $\mathbb{R}$, ok? So, what is this $\mathfrak{C}$?$\mathfrak{C}$ is going to be an operator. It takes a closed interval and then produces a closed and bounded subset of the same interval $[a, b]$, ok? So, prior to that we define another simpler operator ϕ on the class of finite union of disjoint closed intervals which you may call the middle one-third deletors, ok? So, this funny name we will explain it right now. Start with any closed interval, you may denote it by J, ok? or say $[a, b]$.

Let us define $\phi(J)$ to be the set obtained by deleting the open part of the middle one-third of J, open part. So, in notations how to state this?

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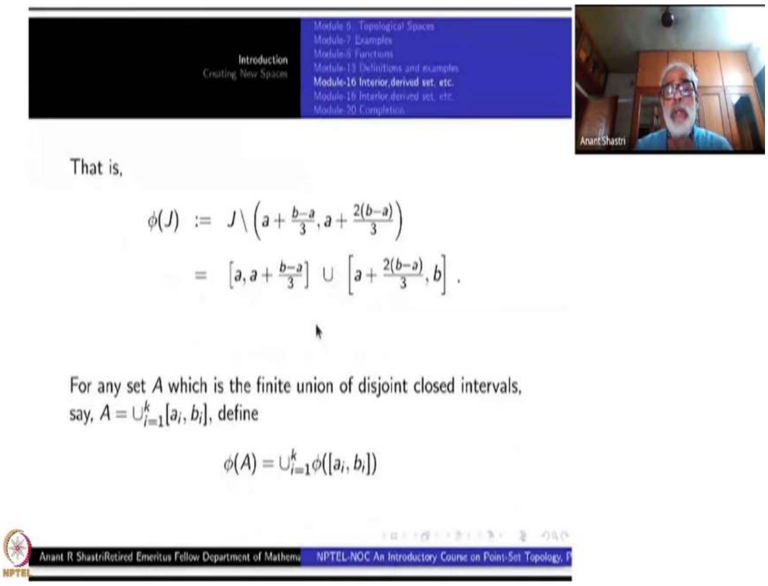

The open one-third is nothing but $(a + (b - a)/3, a + 2(b - a)/3)$. So, delete that from J. So, what you get is a disjoint union of two closed intervals. So, that is the definition of $\phi(J)$. Take any closed interval $[0, 1]$ or $[a, b]$ or $[5, 15]$ whatever. So, it will produce two disjoint closed interval each of length one-third of the original one.

Right from the starting point to that one-third distance and then two-third distance the end, ok, very simple operator. The middle one-third is deleted that is why it is called middle one-third deletor, ok? Now you extend ϕ to the union of finitely many disjoint closed intervals ok.

So, A is union of disjoint closed intervals $[a_i, b_i]$, define $\phi(A)$ to be union of $\phi([a_i, b_i])$. So, here I have defined $\phi([a, b])$. So, that is how you will define $\phi([a_i, b_i])$, then take the union of these ones they will be again disjoint because each $\phi([a_i, b_i])$ is contained inside $[a_i, b_i]$ and they are disjoint to begin with. ok? So, this is just extending the definition from one interval to finitely many intervals. So, the definition of $\phi(A)$ is over.

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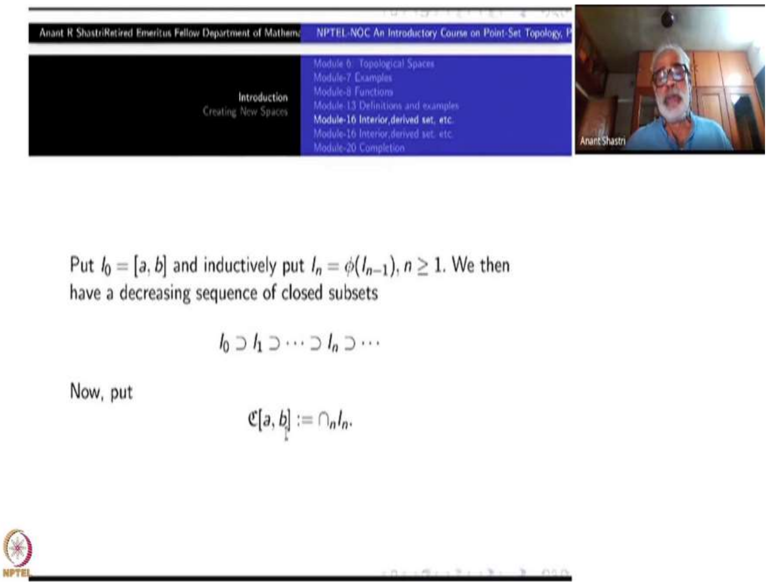

Now, what we do? Now fix one interval $[a, b]$, ok? Let us call it the I_0 inductively define say $I_1, I_2, \ldots, I_n$ will be what? $\phi(I_0)$; that means, first I delete middle one-third of $[a, b]$. Now I have got two intervals there. Operate ϕ on that, ok? So, that is what I have to do, ok? $\phi(I_1)$ will have four intervals right? Take that as I_2 and so on.

So, I_n will be $\phi(I_{n-1})$. So, when you come to n^{th} level there will be how many intervals? 2^n intervals will be there. They are all disjoint with each other. All of them will be contained in the previous ones. So, this is a strictly monotonically decreasing sequence of closed and bounded sets right? The number of intervals strictly monotonically increasing. What I am going to do now?

Finally the Cantor of $[a, b]$ it is in operator, it is just an intersection of all these. It is like taking the limit. Limit of sets, but limits of decreasing sequence of sets is defined very easily-- just take intersection of all of them, ok? This is the definition.

One thing is very clear that because all these are closed subsets this will be closed. So, this is a closed and bounded subset starting with an interval I have produced a closed and bounded subset of $[a, b]$ itself. This I_0 is $[a, b]$, then all these are subsets of that ok?

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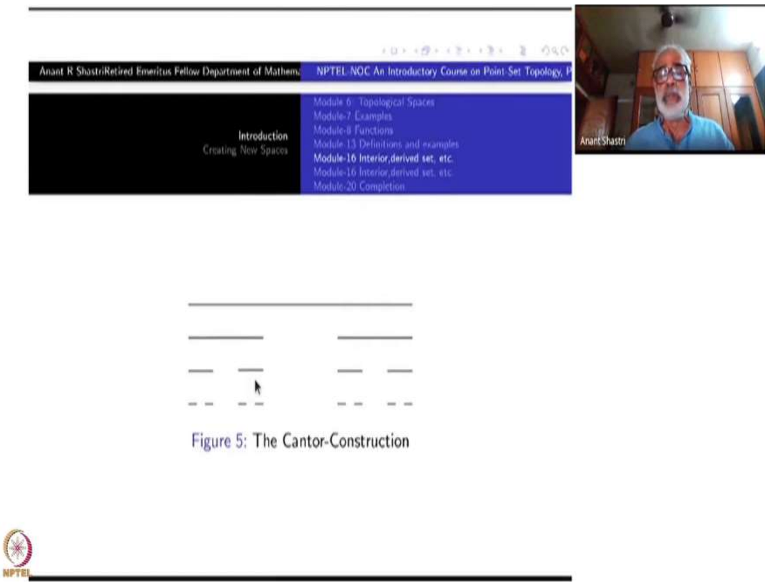

So, here is a picture what I have done. Start with one interval delete the middle one-third, we have two intervals here. Now look at this one delete the middle one-third. So, this will give you 2 here, 2 here. So, at the second level already you have 4 of them. Now delete one-third here delete one-third here and here also. So, the third level you will have 8 of them and so on.

So, keep going on so on so, you will get lot of things looking just like dot dot dot dot dots ok. So, that the limiting thing is the Cantor set ok. The function $\mathfrak{C}$ is called the Cantors construction ok.

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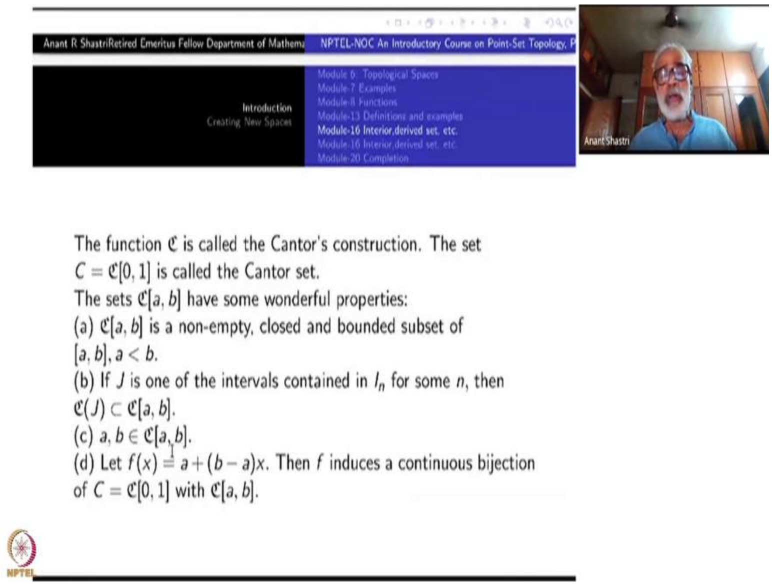

Now I am taking a specialization instead of $[a, b]$ put $[0, 1]$ the standard unit interval. Take the closed interval $[0, 1]$ apply the Cantor operation let us denote that by $\mathfrak{C}$, ok? Usually when someone says Cantor set, it means means this one. But, one actually the word is used in a wider sense, to represent all topological spaces homeomorphis to $\mathfrak{C}$ also. The sets $\mathfrak{C}([a, b])$ so, whenever there is arbitrary $[a, b]$, I will write like this.

This space has some wonderful properties ok?

So, let me list a few of them. Not all these properties may be easy for you. Some of them may need some concepts which you not know. But this is actually a topic in analysis. So, I will just list them right now. I may not explain all of them to you, ok. Quite a few of them I am doing here, the easier ones. or I will explain them right now ok?

The first thing is $\mathfrak{C}([a, b])$ is non-empty I already told you it is closed and bounded that is easy. Why it is non-empty? So, this is because of compactness of $[a, b]$. So, the first thing is very important here it is a non-trivial thing, but this is non-empty. If J is one of the intervals contained in I_n so, what I mean by say? Look at this picture this is I_0, I_1, I_2, I_3 any one of the I_n. Look at one of these intervals the small intervals, ok. The intervals I am talking not arbitrary intervals it is not just a subset of these interval, but one of these intervals, just take that.

If J is one of the intervals contained in I_n for some n, then if you look at $\mathfrak{C}(J)$ that would be also a subset of $\mathfrak{C}([a, b])$. You can just apply $\mathfrak{C}$ of that because $\mathfrak{C}$ is an operator operating on any intervals closed intervals. So, $\mathfrak{C}(J)$ is contained in $\mathfrak{C}([a, b])$ which is contained in $[a, b]$. The endpoints are always inside $\mathfrak{C}([a, b])$. See endpoints are never deleted. These two endpoints this endpoint this point ok. So, beauty is to begin with I can say this here endpoints they are not deleted.

But now this point and this point will never get deleted because from here I could have started and arrived at one. So, this point the end points are never deleted. So, same thing is here endpoints these endpoints are never deleted. So, in the end these endpoints are all there they are never deleted. If something is somewhere in between you will never know whether it will be get in deleted. So, that is the beauty of this.

So, there are all the endpoints of successively cut-off intervals they are all there that is the whole idea. Though I say it only for $[a, b]$ because it is true for all sub intervals also, ok. Now take the map $f(x)$ equal to $a + (b - a)x$. This is a linear map ok? What does it do? Put $x = 0$ you will get a, put $x = 1$ you will get b. So, that is a linear map homeomorphism from $[0, 1]$ to $[a, b]$ right?

It just expands. This one-third becomes $(b - a)/3$ right. So, what happens is the Cantor set $\mathfrak{C}([a, b])$ and cantor set of $\mathfrak{C}([0, 1])$ this is C, they will be isomorphic under this continuous bijection. So, this map preserves the Cantor construction. First of all I have defined it $[0, 1]$ to $[a, b]$. The middle one-thirds correspondingly get deleted. This map is not an isometry because its $[0, 1]$ is expanded to $[a, b]$ right, but it is a similarity.

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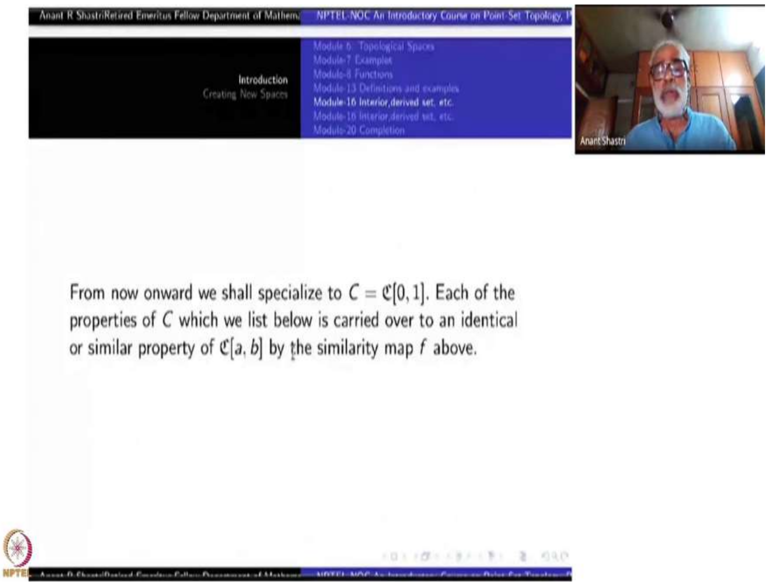

From now onwards, I will just concentrate on the Cantor set $\mathfrak{C}([0,1])$. Each of the properties of $\mathfrak{C}$, which we list below is carried over to an identical or similar property of $\mathfrak{C}([a,b])$ right because of similarity map. So, that is why I do not have to mention it separately for all of them once I say it for C. it will be true for $\mathfrak{C}([a,b])$ for all of them.

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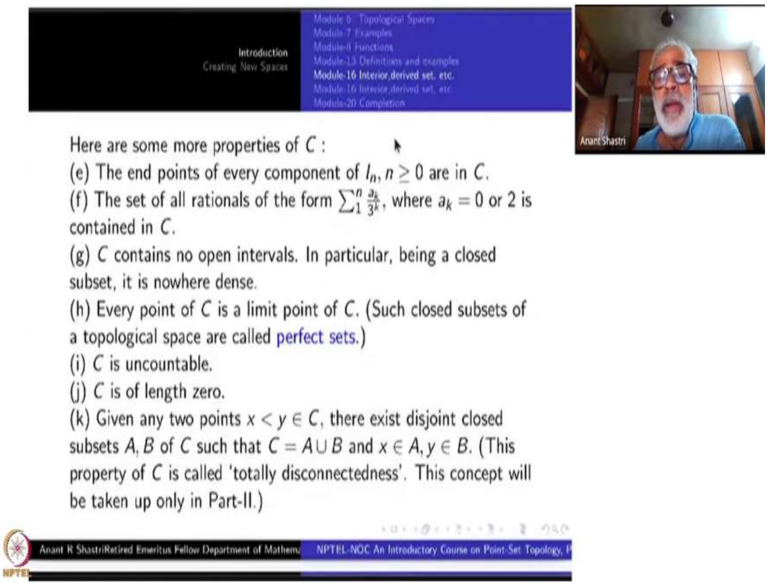

So, here are some more properties of C, ok. These are special in the sense that they are all points between $[0, 1]$ that is the specialty ok. Topologically, when you have a similarity you have to make the corresponding numerical modifications. The endpoints of every component of $I_n, n \geq 0$ they are all in C. This is what we have observed already.

The set of all rationals of the form summation over k equal to 1 to n ok, the finite sum a $k/3^k$ ok? What are a k's? Here k is either 0 or 2, this sum is contained in C. Why this is true? Look at the very first interval in the sequence. Everything from here to here is one-third from here to here one-third plus something, ok, But those point are not there, and finally from here to here, these are two-third plus something that will be always there.

So, elements of these intervals may survive. Accordingly, the first coefficient a_1 will be 0 or 2. The sum will be then $1/9$ of something or two third plus something. So, that is what this means. Ternary expansion is what is being done here. I am writing every element as $a_1/3 + a_2/9 + a_3/27$ and so on right? So, there is that sequence a_k, these a_k; These a_k are generally 0 one or 2 but the claim is that a_k will never be equal to 1, ok? They are either 0 or 2, ok. This is an easy consequence of the construction here. The middle one-third is deleted, ok. All the rationals of this form, ok, they will be there, alright.

Now, C contains no open intervals that is a first observation ok that I am making which is now a serious one. Start with any open interval ok? So, first of all it should be inside $[0, 1]$ right? Say, it is somewhere here. What happens? Go on making one-third, $1/9$ and so on finally, some small portion will lie between the same interval because the interval has positive length and that one-third portion will be get deleted, ok. So, that is easy to see. So, no open interval will be contained inside the Cantor set in the final stage, ok? At any finite stage, they are there are there these are intervals. But at the end there are no open intervals inside C.

What is the meaning of that interior is empty? It is a closed set already. Therefore, it is nowhere dense you see. So, suddenly we have constructed nowhere dense set here ok. So, where are we? Here ok. So, C contains no open intervals. In particular being a closed set it is nowhere dense otherwise, I have to take a closure and then look at the interior.

So, closure of C is C itself. Therefore, interior of that is empty. So, this means its nowhere dense. Every point of C is a limit point of C. Limit point means cluster point. Is that very difficult to see. Take a point of C with a limit point of C take an interval, ok. There will be some other point other than that that point that is all, I have to see right.

So, no point will be isolated, ok. So, for this kind of thing you can just glare keep glaring at this construction one by one you can you can produce that or you can use your arithmetic and compute things and so on just look at every element of this you know this form, ok. So, near that you can take any ϵ here you can produce one more element other than this one, ok. So, that kind of arithmetic will produce proofs also. So, it is not all this difficult. So, what is it? Every point is a cluster point every point is a limit point, ok, such sets are called perfect sets. So, $\ell(C)$ is equal to C.

The fifth property is C is uncountable. So, this follows by looking at all these distinct points. Suppose you have a sequence here ok, you can take infinite sequence also. See this I have said here. Look at all infinite sequences the entries are either 0 or 2. Take a sequence and then form this sum. This will be convergent sequence ok, right.

If you truncate it at a finite level, all these points are there therefore, the limit is also there because it is a closed set right. So, infinite sequence is of this form $0, 2, 0, 2, \ldots$ whatever number here in the Cantor set, but these sequences are themselves uncountable, ok. Now all that you have to see is that if you take two different sequences they will always give you different numbers, ok.

This is similar to what you do with dyadic expansions. So, here we are doing ternary expansion, you could have done it with any number here with decimal expansion also you could have done ok. If it is one-third there is a charm here which is not there with some other number. So, we will see that form come to that one later ok. So, so where were we? Uncountable C is of length 0. Now you may not know what is the meaning of length of these arbitrary subsets here right.

If you do not know that do not worry about that, I will tell you what it is whenever it is needed. The third the last point here I am going to make: given any two points inside C ok so, say assume x is less than y, there exists disjoint closed subsets A, B of C such that C is $A \cup B$, A and B are disjoint closed, x is inside A and y is inside B. So, such a thing is called a separation. Whenever any two points can be separated like this, such a set is called such a topological space is called totally disconnected.

So, do not worry about this one. This is a very profound topological property which we will discuss in part II. But this property I have very nicely defined and this can be proved right directly there is no problem for the Cantor set, ok? For the Cantor set, we will prove this one alright, that is not very difficult.

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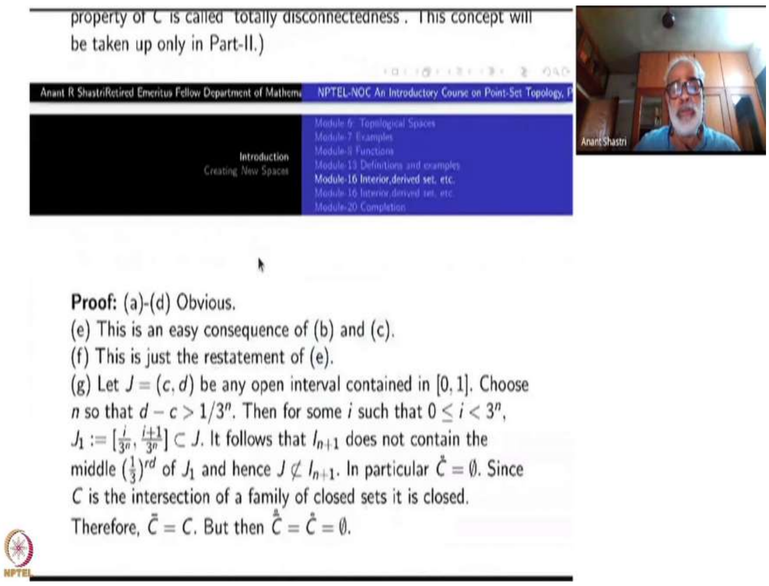

So, in any case I have put all these proofs here also. Let us do the last one first and then come back to all of them.

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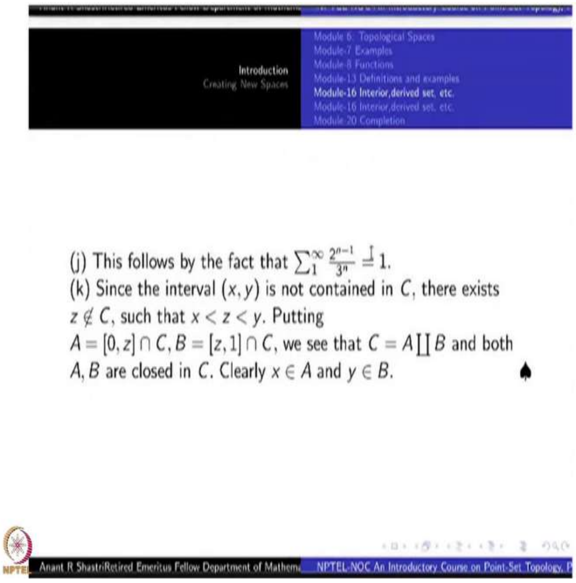

So, look at this one. Take any interval (x, y). This interval is not contained inside C which we have observed earlier. So, take a point z between x and y, strictly between x and y, which is not in C. If it is not in C what happens? Take A equal to the closed intervals $[0, z]$ intersect with C. See this is a closed subset of $\mathbb{R}$, its intersection with C is a closed subset of C. Similarly, B equal to $[z, 1] \cap C$, ok.

$[z, 1]$ contains y, right. So, y is inside B. Similarly, x will be inside A and by the very definition the intersection of $[0, z]$ with $[z, 1]$ will be just $\{z\}$, but z is not there. So, intersection is empty ok. So, both A, B are closed in C, x is inside A, y inside B and $A \cap B$ is empty. So, this is a proof for last property, ok?

Now this is what I am telling you. What is the meaning of length is 0 ok? What am subtracting? First, I am subtracting one-third of the original right? Then I am subtracting $1/9$, right? So I am trying to approve this C is of length 0, ok? You sum up the lengths of all interval subtracted, that is equal to 1.

This totality of total length of all the intervals that I am going to subtract first this is one-third then from whatever remaining I am taking one-third here one-third here is 2 by ninth of the original, then one-third here one-third here one-third one-third one-third that will be 4 by 27 and so on.

So, this is nothing but one third of the summation $2^n/3^n$, sum total is 1. So, whatever remaining is 1 minus 1. So, length will be 0.

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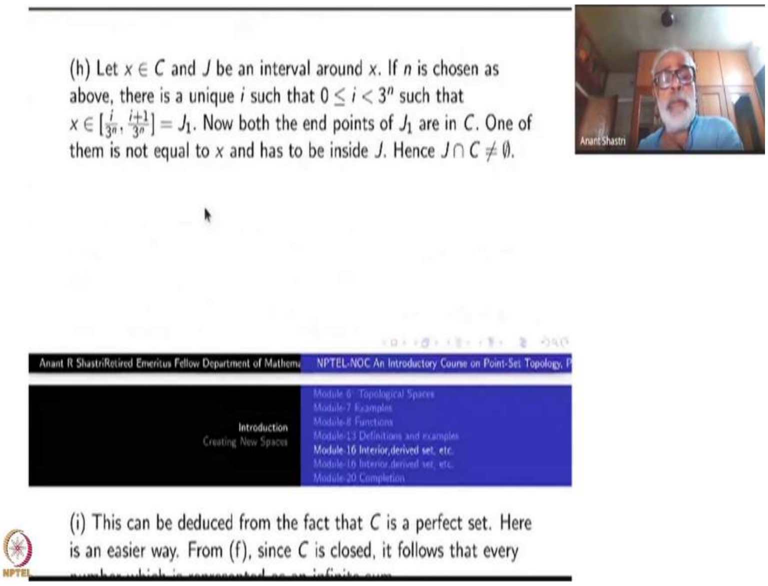

So, I think I have written down all these proofs here more carefully you can go through that you have got the notes in any case. So, why interior is empty I have explained it is completely here. So, this Cantor set is going to play a lot of interesting role in analysis and also in topology ok?

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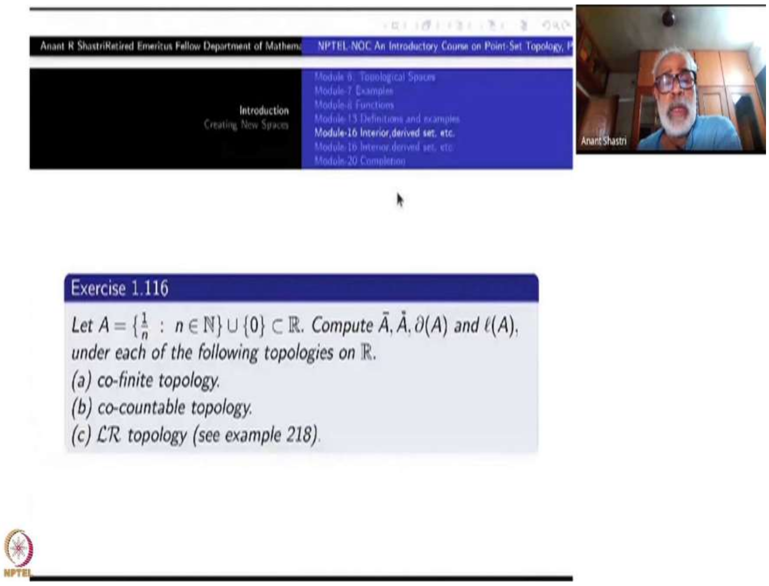

So, once again let me give you some exercises here alright. Start with the set A equal to all $1/n, n$ belonging to $\mathbb{N}$ and include 0. Now compute all these things under the usual topology of $\mathbb{R}$, under these different topologies, co-finite topology, co-countable topology and $\mathcal{LR}$ topology. And the original topology you have done it already. Now you do it for these three topologies.

Once you do it for $\mathcal{LR}$, it is similar to $\mathcal{RR}$. So, it will not be much difficult for you that is similar ok. So, do that exercise.

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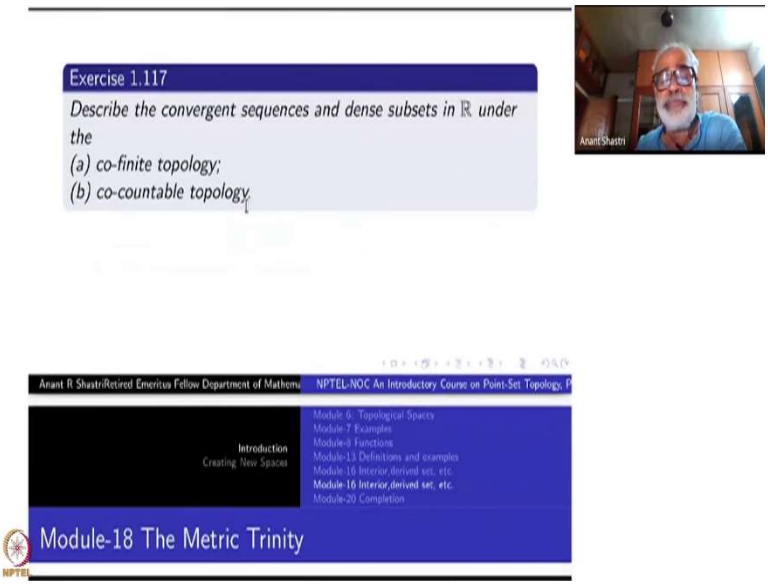

Here there is another exercise about convergent sequences and dense subsets. We have described the co-finite topology and co-countable topology ok? All this does not need many more education. You have been given the definition of a convergent sequence, meaning of dense subset and so on. You have to work out for them for here ok.

A sequence converges to a point if for each given neighbourhood, there must be some n such that $k \geq n$ means x_k is inside that neighbourhood. So, that is the kind of definition. But now you apply it to cleverly chosen neighbourhoods of points inside co-finite topology and see what happens. Similarly for this one see what happens, ok. So, that will be the ah ah material for today, let us stop here.

Thank you.