Introduction to Point Set Topology, (Part I)
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Lecture - 16
Interior, derived set, etc

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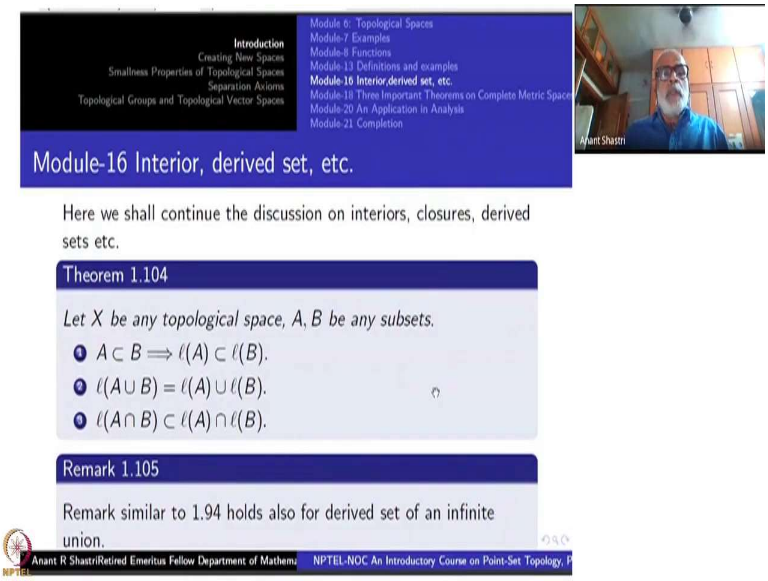

Welcome to module 16 of Point Set Topology course. Today we will discuss once again about interiors derived sets and so on, just like whatever we have done for closures etc last time. Start with a topological space X, A and B be any subsets then the following three statements are true about the derived set. A contained inside B implies $\ell(A)$ is contained inside $\ell(B)$. The derived set of A is contained in the derived set of B. The derived of the union is union of the derived sets.

The derived set of the intersection is contained in the intersection of the derived sets, $\ell(A \cap B)$ is contained in $\ell(A) \cap \ell(B)$. Proofs are all straightforward. Moreover, more or less similar to the corresponding statements for the closures. So, I will leave the proofs to you to write down as an easy exercise.

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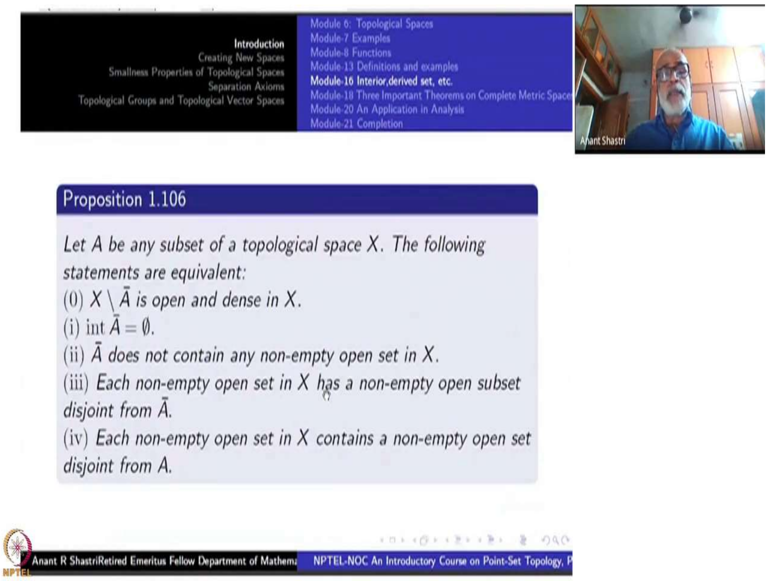

There is one more remark just like in the case of closures if you take infinite union then only containment will be there, this number, 2 will be replaced by containment. The derived set of an infinite union of A_i's is contained in the union of the derived sets of A_i's ok? So, that can also be done. So, now, let us go to another important auxiliary result about, of course, this is again about interiors and closures and so on, but this time it is about the so called nowhere dense sets.

Later on, we will use this in proving some major theorems about metric spaces. Right now this is in a general topological space. This theorem gives you 5 equivalent definitions of nowhere dense sets.

The first statement I have included as number (0) here, (0), (i), (ii), (iii) (iv). So, there are 5 statements here. Start with a subset $A, X \setminus \bar{A}$ is dense in X. Of course, $\bar{A}$ being closed always, $X \setminus \bar{A}$ is open that part is easy. So, the first statement here is $X \setminus \bar{A}$ is dense in X.

The second statement is interior of $\bar{A}$ is empty. This was the condition for A being nowhere dense in our definition. The closure of A should have interior empty. Third statement is $\bar{A}$ does not contain any non empty open set in X.

The fourth one which is third here, is, each non empty open set of X has a non empty open subset disjoint from $\bar{A}, \bar{A}$ is given ok. So, these all statements about $\bar{A}$ or A. Each non empty open set in X contains a non empty open set disjoint from A. So, all these things are equivalent. But we shall prove it in a systematic way, in an economic way by proving (0) implies (i) implies (ii) implies (iii) implies (iv) and then (iv) implies 0 ok?

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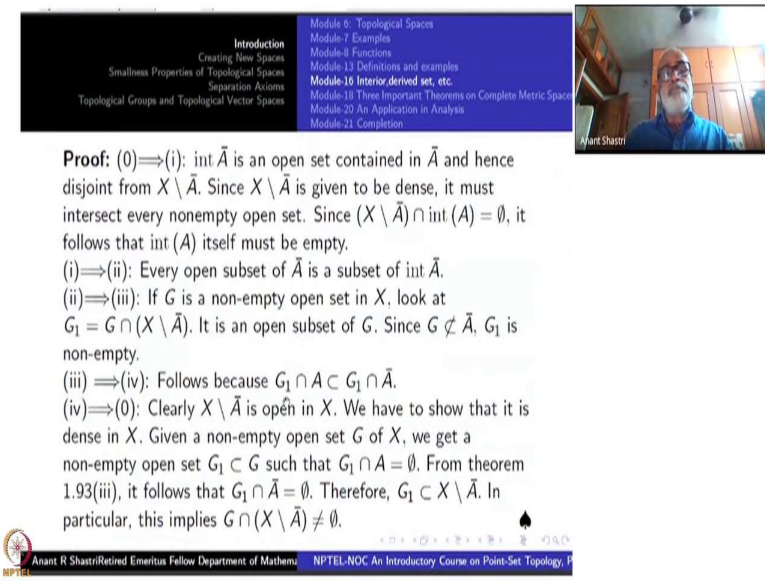

So, that is the plan. So, (0) implies (i): statement 0 is what? $X \setminus \bar{A}$ is open and dense in X, ok. That is the statement (0). Look at interior of $\bar{A}$, ok? Interior of any set is an open set. So, interior of $\bar{A}$ is an open set and it is contained in $\bar{A}$. Therefore, it is disjoint from $X \setminus \bar{A}$ itself, but then $X \setminus \bar{A}$ will not intersect that open subset ok; that means, it will not be dense unless that open subset itself is empty.

So, some $X \setminus \bar{A}$ is given to be dense implies that interior of $\bar{A}$ which is disjoint from $X \setminus \bar{A}$ must be empty. So, interior of $\bar{A}$ itself is empty ok. Now assume the statement (i). Let us prove (ii) ok? So, every open subset of $\bar{A}$ is a subset of interior of $\bar{A}$, right? Statement one is interior of $\bar{A}$ is empty. $\bar{A}$ does not contain any non empty open subset in X is what we have to show. If it does that non empty open subset will be inside interior of $\bar{A}$, which is empty. So, that is proves (ii).

Now the proof of (ii) implies (iii). If G is a non empty open set in X, look at G_1 equal to $G \cap X \setminus \bar{A}$. This is an open set, this is an open set. This will be another open set. It is an open subset of G since G is not contained in $\bar{A}$, no non empty open set is contained in $\bar{A}$, right? G_1 must be non empty because if it is contained inside $\bar{A}$, then this should have been empty, non empty means that is something here ok. So, G_1 is non empty. Now, (iii) follows because G_1 is contained inside G and is disjoint from $\bar{A}$.

So now (iii) implies (iv). Statement (iv) says each non empty open subset of X contains a non empty open subset disjoint from A. Some set is disjoint from $\bar{A}$, then it will disjoint from A also ok?

Finally, assuming this statement (iv), we have to show that $X \setminus \bar{A}$ is open and dense in X. Openness is obvious. However, I have to show that $X \setminus \bar{A}$ is dense in X. That means, take any non empty open subset it must intersects $X \setminus \bar{A}$, ok? Something intersects with $X \setminus \bar{A}$ if it is not contained in $\bar{A}$, ok?

Take any non empty open subset, it will contain a non empty that disjoint from A. So, that portion will not be contained inside A. So, it will not be contained inside $\bar{A}$ either, ok? So, these terminologies are just you know a topological one, but what is the point of doing this one?

Suppose you want to deal with a nowhere dense set then at a particular place you may be using this property, this property, this property or this property, any one of them you can use and sometimes using just this much is easier whereas, using this one will be easier at some place and so on. So, this is this fifth fourth one which looks somewhat tedious one ok. This is what is going to be applied soon. We are going to use it in proving Baire's Category Theorem.

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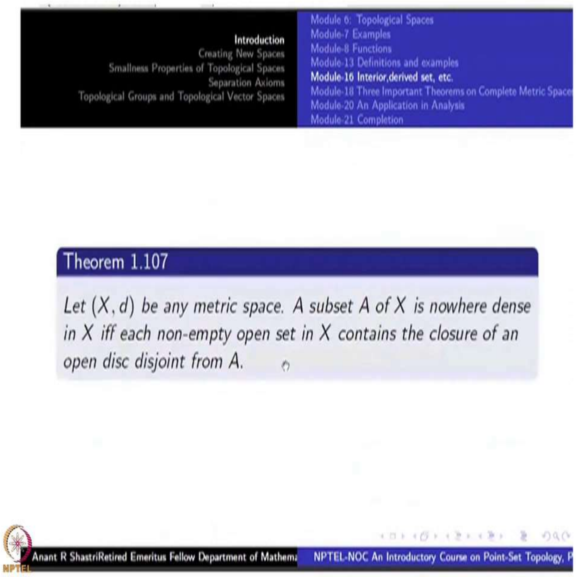

So, we come to the theorem in metric spaces now. Let X be a metric space. A subset A of X is nowhere dense in X, if and only if each non empty open set in X contains the closure of an open disc disjoint from A. Now, because we are working in a metric space now, we can talk about discs and its closures and so on ok. Just now what we saw is that if A is nowhere dense set, every non empty open subset contains another non empty open set which is disjoint from A.

Every non empty open set in a metric space contains a closed ball of some positive radius, any ball of positive radius ok, it is the closure of the open ball right. So, that is what we get now. So, start with a non empty open set ok, which is disjoint from A, inside that open set take a ball ok, such that even its closure also is contained inside that.

That you can do because once you start with an open set, and open ball inside that we can take smaller and smaller closed balls ok. So, this comes very easily, but this is what we are going to use later on.

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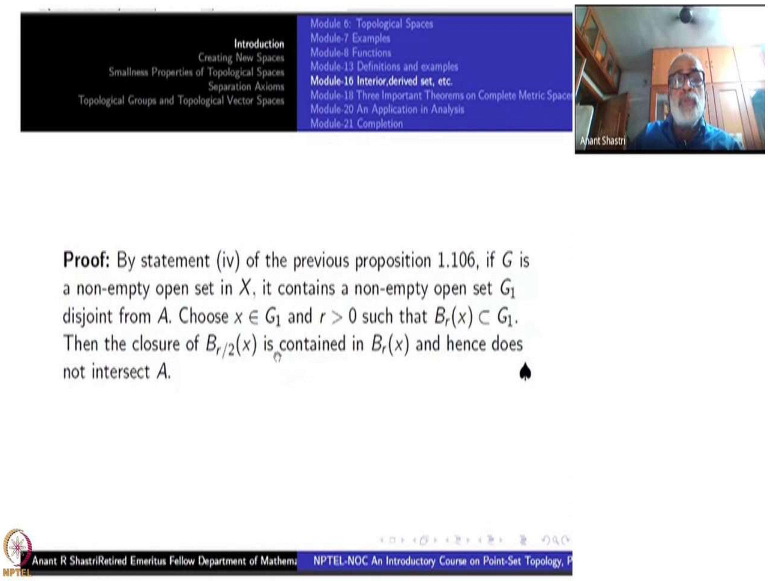

So, I repeat this one. By statement (iv) of the previous proposition, if G is a non empty open set in X, it contains a non empty open set G_1 disjoint from A. Choose an $x \in G_1$ and r positive such that $B_r(x)$ is contained inside G_1, then $B_{r/2}(x)$ is contained in $B_r(x)$, not only that its closure is also contained in $B_r(x)$ and this $B_r(x)$ does not intersect A, ok.

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So, you just remember that interior of $\bar{A}$ being empty the definition of nowhere dense set because of the above theorem anyone of the five conditions can be taken to play the role of the same thing this is what I have already remarked ok?

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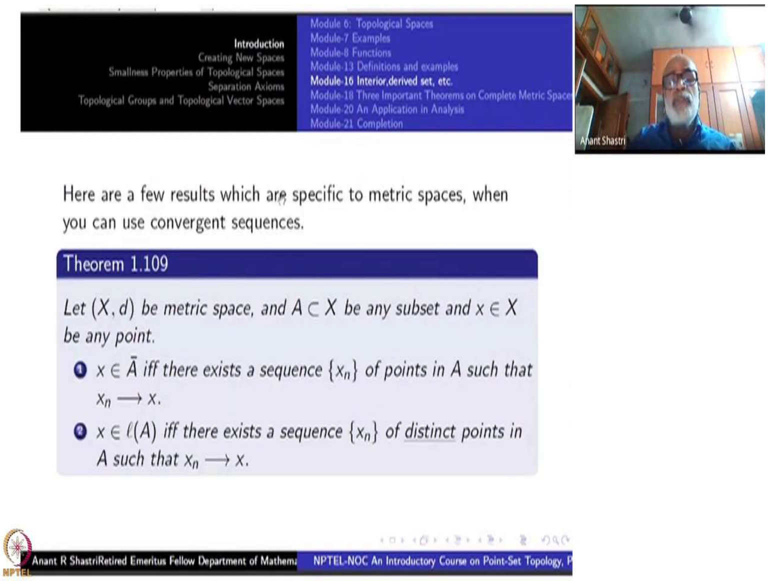

Now, coming to the metric spaces, we have a few remarks to do about sequences ok. Start with a metric space ok. Take a subset A of X ok. Take any point. This x will be inside the closure of A, if and only if you can find a sequence (x_n) of points inside A such that the sequence converges to x, ok. The second statement is: the point x is in the derived set of A, (is a limit point) if and only if there exists a sequence (x_n) of distinct points in A, such that (x_n) converges to x, ok.

The difference between these two statements is that in the first part you can take some constant sequence of $(x_n) = (x, x, \ldots)$, x_n converging to x. Now, x_n itself is x. So, that is not allowed here in the second one. If $x, x, x, x, \ldots$ is a sequence that always converges to x that does not mean that x is inside $\ell(A)$, it may be just inside A that is all ok. So, cluster points have special property. So, there must be a sequence of distinct points such that sequence the converges to x, ok?

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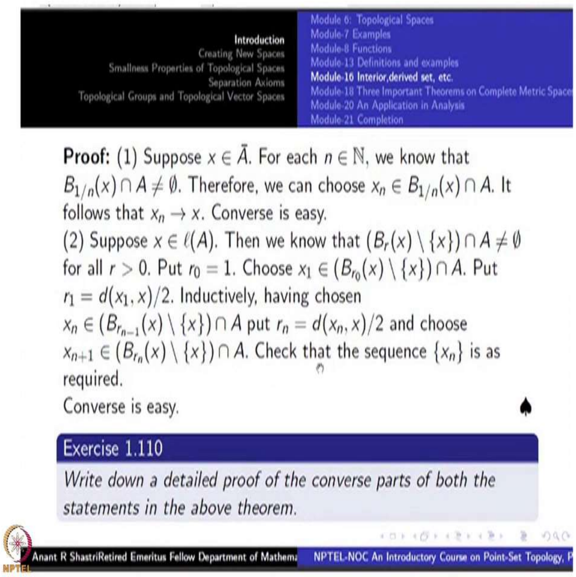

So, let us see the proof of this one. Start with a point in the closure. For each $n \in \mathbb{N}$, we know that if you take $B_{1/n}(x)$, the ball, open ball of radius $1/n$ intersection A must be non empty because x is in the closure. Therefore, we can pick up a point (x_n) inside this $B_{1/n} \cap A$. So, this will be sequence in A, but the distance between x_n and x becomes smaller and smaller $1/n$ right as n tends to infinity this will converge to 0. So, x_n converges to x, ok. The converse that if there is a sequence converging to x and a sequence is in A, then it is inside $\bar{A}$. This we have seen several times.

Now, the second part. Suppose x is a cluster point, it is a limit point of A, We know that $(B_r(x) \setminus \{x\}) \cap A$ is non empty, for all r positive. Start with r, you know r equal to r_0 equal to 1. Choose $x_1 \in B_{r_0}(x)$ and not equal to x. Take a point x_1 not equal to x, but inside A ok. Look at the distance r_1 namely distance between x_1 and x divide it by by 2, r_1 is the distance divided by 2. Inductively, having chosen x_n in $(B_{r_{n-1}} \setminus \{x\}) \cap A$, as soon as you choose x_n look at this number r_n equal to $d(x_n, x)/2$, ok.

Use this r_n to choose the next x_{n+1} and on. Here once you choose x_1 is like this. x_2 will be chosen inside $B_{r_1}(x) \setminus \{x\}$. So, the distance between x_{n+1} and x goes to what? You know each time the distance is less than see r_1 is something whatever, r is something, r_0, r_1 is distance between x is less than $r_0/2$. Next, it will be r_2 will be $r_1/2$ and so on. So, at r_n will be less than $r_0/2^n$. So, that will come down to 0, ok.

Why this sequnce consists of distinct points? Because look at this one, x_1 is somewhere, but x_2 will be inside this distance. So, distance between x and x_1 is r_1, distance between x and x_2 will be $r_1/2$ or small r therefore, they cannot be equal. So, next one which we will choose its distance between x and x_n will be smaller than all the earlier distances. So, this is a disjoined you know this is the distinct sequence of distinct points that is why it is so ok, ok. So, you can write down the converse of this one also it is very easy anyway.

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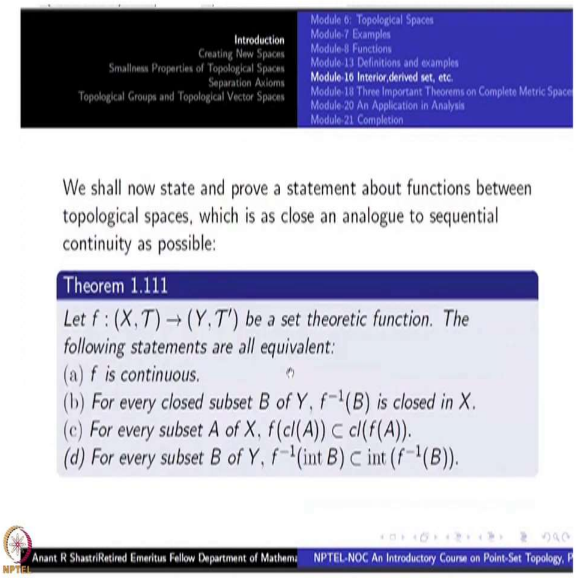

So, let us consider some functions between metric spaces and topological spaces first of all and put the put them in the proper perspective. Now, start with f from $(X, \mathcal{T})$ to $(Y, \mathcal{T}')$, a set theoretic function ok. Then f is continuous, it is statement (a).

The second statement is that for every subset B of Y, which is closed in Y, the $f^{-1}(B)$ is closed in X. Remember what was the definition of continuity, for every open subset U of $Y, f^{-1}(U)$ is open. That was the statement for continuity of a function. That one was statement (a). So, in statement (b), open sets are replaced by closed sets.

The third one is even much better. For every subset A of $X, f(\bar{A})$ is contained in the $\overline{f(A)}$. So, this is a forward statement. (a) and (b) were backwards. Starting from subsets of Y, you get some conclusion subsets of X. Here it is the other way around, ok?

And (d) is also similar, but in the reverse way for every subset B of Y, $f^{-1}(B^\circ)$ is contained in the interior of $f^{-1}(B)$, ok? These are all equivalent just means that you can use any one of them to define continuity of a function from any topological space to another topological space ok, that is the statement.

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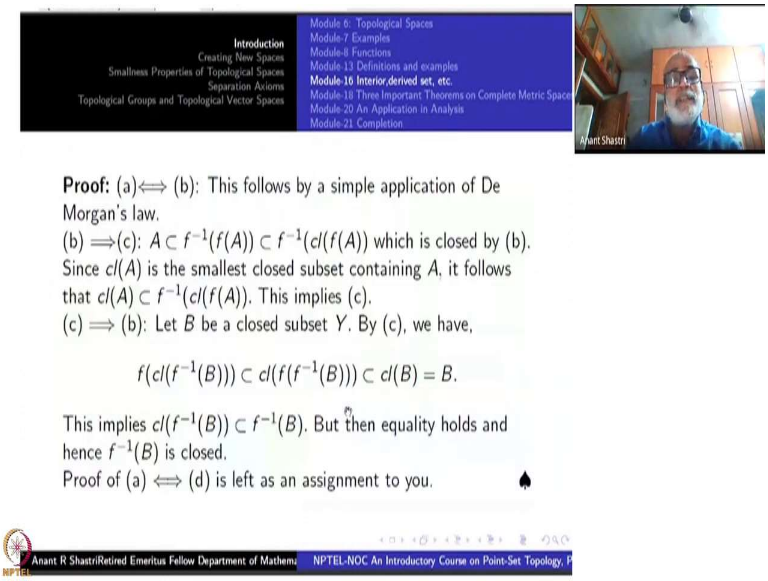

So, let us look at the proof of this one which is not all that difficult. First let us prove (a) and (b) are equivalent. This is just by De Morgan Law. If U is an open subset $X \setminus U$ is a closed set. $f^{-1}(Y \setminus U)$ is same thing as $X \setminus f^{-1}(U)$.

So, inverse of closed subset are closed subsets. So, you just use one way (a) implies (b) implies (a). Because you know that the compliment of a closed set is an open, and f inverse behaves very nicely under complimentation.

Now, let us prove (b) implies (c) ok? (c) is what? Start with any subset A of X, ok. So, what you have to prove that f of the closure of A is contained in the closure of $f(A)$, ok? A is always contained in $f^{-1}(f(A))$, ok? And $f(A)$ is contained in closure of $f(A)$. Therefore, f inverse of that is contained, $f^{-1}(\overline{f(A)}$, ok? Now, closure of any set is closed. Condition (b) says f inverse of that is closed. Therefore, A is contained in this closed set. Therefore, $\bar{A}$ is contained in this set, ok? because closure of A is the smallest closed subset containing the given set A. So, this is a larger closed subset. Closure of A is the smallest closed subset in A. So, it is contained here, but this is same thing as if you put f here, $f(\bar{A})$ is contained in the closure of $f(A)$.

(c) implies (b) is what I want to show, the other way around now ok? So, start with any closed subset B of Y. By (c), we have $f(\overline{f^{-1}(B)})$, you do not know what it is ok, closure of $f^{-1}(B)$, f of that is, by (c) is contained inside closure of $f^{-1}(B)$ which is contained in closure of B. So, it is contained closure of B, this one is contained closure of B ok. But B is closed therefore, closure of B is B.

So, f of the closure of f inverse is contained inside B. This means that this closure of $f^{-1}(B)$ is contained in $f^{-1}(B)$ ok? But then equality holds ok? Because closure of any set contains that set. So, they are equal ok and hence $f^{-1}(B)$ itself is closed because this is the closure of that, ok?

The proof of the last statement is almost similar. Again something like De Morgan Law we have to use. So, I will leave that as an assignment to you namely f inverse of B° is contained in $f(B)^{\circ}$, ok. You should not take more than two lines. You should write down ok.

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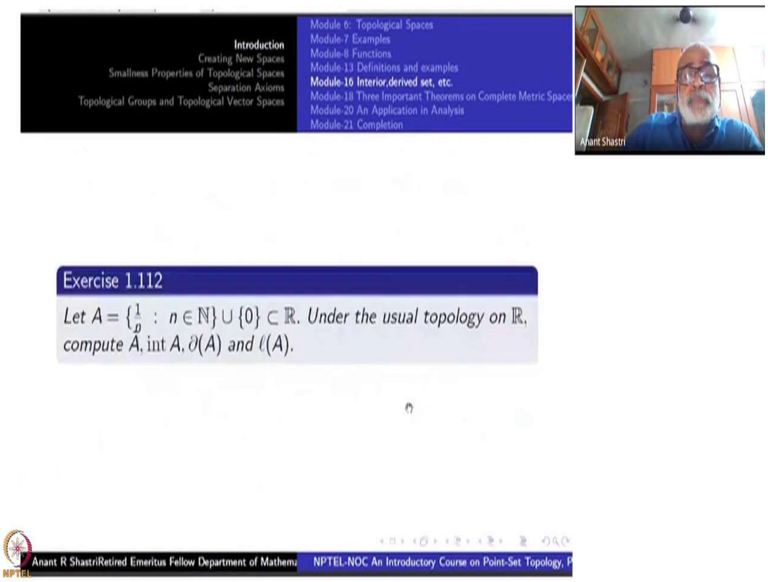

So, here are some exercises A equal to the set of all $1/n, n \in \mathbb{N}$ and one extra element 0. Take the subset $1, 1/2, 1/3, 1/4$ etc. Then include 0 also. Under the usual topology from $\mathbb{R}$, compute $\bar{A}$, $A^{\circ}, \partial(A), \ell(A)$, ok. So, this is an easy exercise.

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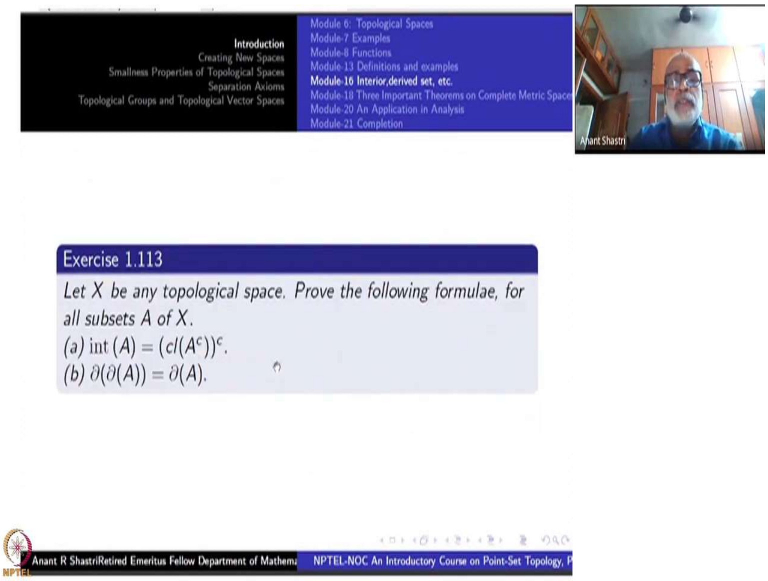

So, this is an easy exercise.

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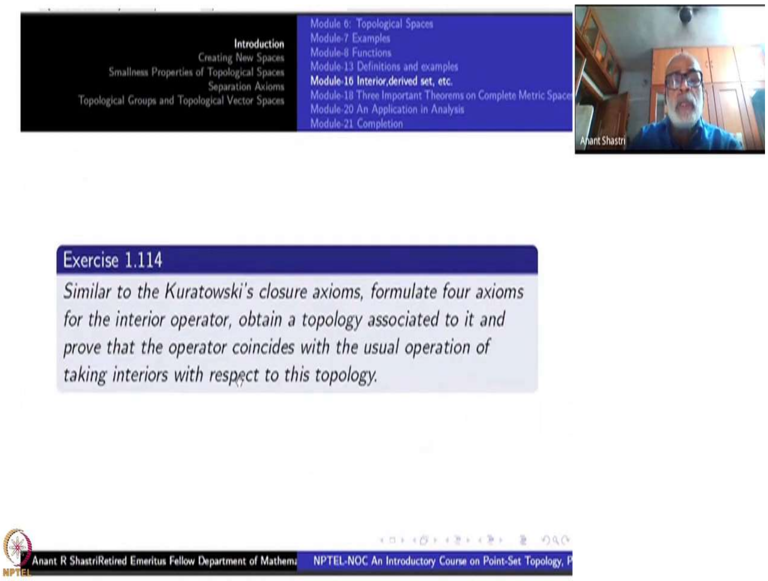

Similarly, there are some more exercise here which you can take. Now, there is also this Kuratowski's closure axioms, formulate four axioms for interior operator, obtain a topology associated to it and prove that the operator coincides the usual operation of taking the interiors, similar to Kuratowski's closure axioms ok.

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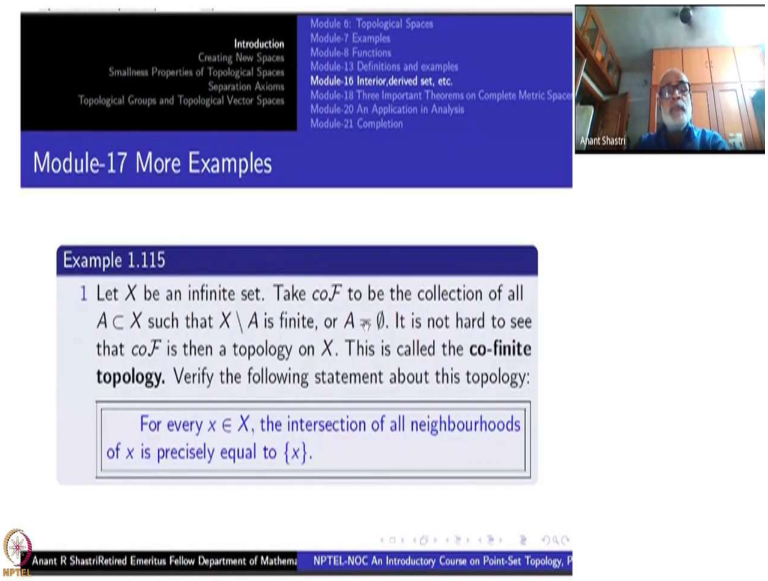

So, let us stop here today. Next time we will study more examples.

Thank you.