

Introduction to Point Set Topology, (Part I)
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Lecture - 16
Interior, derived set, etc

(Refer Slide Time: 00:16)

Introduction
Creating New Spaces
Smallness Properties of Topological Spaces
Separation Axioms
Topological Groups and Topological Vector Spaces

Module 6: Topological Spaces
Module 7: Examples
Module 8: Functions
Module 11: Definitions and examples
Module 16: Interior, derived set, etc.
Module 18: Three Important Theorems on Complete Metric Spaces
Module 20: An Application in Analysis
Module 21: Completion

Module-16 Interior, derived set, etc.

Here we shall continue the discussion on interiors, closures, derived sets etc.

Theorem 1.104

Let X be any topological space, A, B be any subsets.

- $A \subset B \implies \ell(A) \subset \ell(B)$.
- $\ell(A \cup B) = \ell(A) \cup \ell(B)$.
- $\ell(A \cap B) \subset \ell(A) \cap \ell(B)$.

Remark 1.105

Remark similar to 1.94 holds also for derived set of an infinite union.

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Welcome to module 16 of Point Set Topology course. Today we will discuss once again about interiors derived sets and so on, just like whatever we have done for closures etc last time. Start with a topological space X , A and B be any subsets then the following three statements are true about the derived set. A contained inside B implies $\ell(A)$ is contained inside $\ell(B)$. The derived set of A is contained in the derived set of B . The derived of the union is union of the derived sets.

The derived set of the intersection is contained in the intersection of the derived sets, $\ell(A \cap B)$ is contained in $\ell(A) \cap \ell(B)$. Proofs are all straightforward. Moreover, more or less similar to the corresponding statements for the closures. So, I will leave the proofs to you to write down as an easy exercise.

(Refer Slide Time: 01:52)

The screenshot shows a presentation slide with a table of contents on the left and a video feed of Anant Shastri on the right. The table of contents includes: Introduction, Creating New Spaces, Smallness Properties of Topological Spaces, Separation Axioms, Topological Groups and Topological Vector Spaces, Module 6: Topological Spaces, Module 7: Examples, Module 8: Functions, Module 9: Definitions and examples, **Module 10: Interior, derived set, etc.**, Module 11: Three Important Theorems on Complete Metric Space, Module 20: An Application in Analysis, and Module 21: Completion. The main slide, titled 'Proposition 1.106', states: 'Let A be any subset of a topological space X . The following statements are equivalent: (0) $X \setminus \bar{A}$ is open and dense in X . (i) $\text{int } \bar{A} = \emptyset$. (ii) \bar{A} does not contain any non-empty open set in X . (iii) Each non-empty open set in X has a non-empty open subset disjoint from \bar{A} . (iv) Each non-empty open set in X contains a non-empty open set disjoint from A .' The footer identifies Anant R Shastri as a Retired Emeritus Fellow at the Department of Mathematics, NPTEL, and mentions the course 'NPTEL-NOC An Introductory Course on Point-Set Topology, I'.

There is one more remark just like in the case of closures if you take infinite union then only containment will be there, this number, 2 will be replaced by containment. The derived set of an infinite union of A_i 's is contained in the union of the derived sets of A_i 's ok? So, that can also be done. So, now, let us go to another important auxiliary result about, of course, this is again about interiors and closures and so on, but this time it is about the so called nowhere dense sets.

Later on, we will use this in proving some major theorems about metric spaces. Right now this is in a general topological space. This theorem gives you 5 equivalent definitions of nowhere dense sets.

The first statement I have included as number (0) here, (0), (i), (ii), (iii) (iv). So, there are 5 statements here. Start with a subset A , $X \setminus \bar{A}$ is dense in X . Of course, \bar{A} being closed always, $X \setminus \bar{A}$ is open that part is easy. So, the first statement here is $X \setminus \bar{A}$ is dense in X .

The second statement is interior of \bar{A} is empty. This was the condition for A being nowhere dense in our definition. The closure of A should have interior empty. Third statement is \bar{A} does not contain any non empty open set in X .

The fourth one which is third here, is, each non empty open set of X has a non empty open subset disjoint from \bar{A} , \bar{A} is given ok. So, these all statements about \bar{A} or A . Each non empty

open set in X contains a non empty open set disjoint from A . So, all these things are equivalent. But we shall prove it in a systematic way, in an economic way by proving (0) implies (i) implies (ii) implies (iii) implies (iv) and then (iv) implies 0 ok?

(Refer Slide Time: 04:53)

Introduction
 Creating New Spaces
 Smallness Properties of Topological Spaces
 Separation Axioms
 Topological Groups and Topological Vector Spaces

Module 6: Topological Spaces
 Module 7: Examples
 Module 8: Functions
 Module 13: Definitions and examples
Module 16: Interior, derived set, etc.
 Module 18: Three Important Theorems on Complete Metric Space
 Module 20: An Application in Analysis
 Module 21: Completion

Proof: (0) \implies (i): $\text{int } \bar{A}$ is an open set contained in \bar{A} and hence disjoint from $X \setminus \bar{A}$. Since $X \setminus \bar{A}$ is given to be dense, it must intersect every nonempty open set. Since $(X \setminus \bar{A}) \cap \text{int } \bar{A} = \emptyset$, it follows that $\text{int } \bar{A}$ itself must be empty.

(i) \implies (ii): Every open subset of \bar{A} is a subset of $\text{int } \bar{A}$.

(ii) \implies (iii): If G is a non-empty open set in X , look at $G_1 = G \cap (X \setminus \bar{A})$. It is an open subset of G . Since $G \not\subset \bar{A}$, G_1 is non-empty.

(iii) \implies (iv): Follows because $G_1 \cap A \subset G_1 \cap \bar{A}$.

(iv) \implies (0): Clearly $X \setminus \bar{A}$ is open in X . We have to show that it is dense in X . Given a non-empty open set G of X , we get a non-empty open set $G_1 \subset G$ such that $G_1 \cap A = \emptyset$. From theorem 1.93(iii), it follows that $G_1 \cap \bar{A} = \emptyset$. Therefore, $G_1 \subset X \setminus \bar{A}$. In particular, this implies $G \cap (X \setminus \bar{A}) \neq \emptyset$.

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So, that is the plan. So, (0) implies (i): statement 0 is what? $X \setminus \bar{A}$ is open and dense in X , ok. That is the statement (0). Look at interior of \bar{A} , ok? Interior of any set is an open set. So, interior of \bar{A} is an open set and it is contained in \bar{A} . Therefore, it is disjoint from $X \setminus \bar{A}$ itself, but then $X \setminus \bar{A}$ will not intersect that open subset ok; that means, it will not be dense unless that open subset itself is empty.

So, some $X \setminus \bar{A}$ is given to be dense implies that interior of \bar{A} which is disjoint from $X \setminus \bar{A}$ must be empty. So, interior of \bar{A} itself is empty ok. Now assume the statement (i). Let us prove (ii) ok? So, every open subset of \bar{A} is a subset of interior of \bar{A} , right? Statement one is interior of \bar{A} is empty. \bar{A} does not contain any non empty open subset in X is what we have to show. If it does that non empty open subset will be inside interior of \bar{A} , which is empty. So, that is proves (ii).

Now the proof of (ii) implies (iii). If G is a non empty open set in X , look at G_1 equal to $G \cap X \setminus \bar{A}$. This is an open set, this is an open set. This will be another open set. It is an open subset of G since G is not contained in \bar{A} , no non empty open set is contained in \bar{A} , right? G_1 must be non empty because if it is contained inside \bar{A} , then this should have been empty, non empty means that is something here ok. So, G_1 is non empty. Now, (iii) follows because G_1 is contained inside G and is disjoint from \bar{A} .

So now (iii) implies (iv). Statement (iv) says each non empty open subset of X contains a non empty open subset disjoint from A . Some set is disjoint from \bar{A} , then it will disjoint from A also ok?

Finally, assuming this statement (iv), we have to show that $X \setminus \bar{A}$ is open and dense in X . Openness is obvious. However, I have to show that $X \setminus \bar{A}$ is dense in X . That means, take any non empty open subset it must intersects $X \setminus \bar{A}$, ok? Something intersects with $X \setminus \bar{A}$ if it is not contained in \bar{A} , ok?

Take any non empty open subset, it will contain a non empty that disjoint from A . So, that portion will not be contained inside A . So, it will not be contained inside \bar{A} either, ok? So, these terminologies are just you know a topological one, but what is the point of doing this one?

Suppose you want to deal with a nowhere dense set then at a particular place you may be using this property, this property, this property or this property, any one of them you can use and sometimes using just this much is easier whereas, using this one will be easier at some place and so on. So, this is this fifth fourth one which looks somewhat tedious one ok. This is what is going to be applied soon. We are going to use it in proving Baire's Category Theorem.

(Refer Slide Time: 09:34)

Introduction Creating New Spaces Smallness Properties of Topological Spaces Separation Axioms Topological Groups and Topological Vector Spaces	Module 6: Topological Spaces Module 7: Examples Module 8: Functions Module 11: Definitions and examples Module 10: Interior, derived set, etc. Module 18: Three Important Theorems on Complete Metric Space Module 20: An Application in Analysis Module 21: Completion
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Theorem 1.107


Let (X, d) be any metric space. A subset A of X is nowhere dense in X iff each non-empty open set in X contains the closure of an open disc disjoint from A . \square

So, we come to the theorem in metric spaces now. Let X be a metric space. A subset A of X is nowhere dense in X , if and only if each non empty open set in X contains the closure of an open disc disjoint from A . Now, because we are working in a metric space now, we can talk about discs and its closures and so on ok. Just now what we saw is that if A is nowhere dense set, every non empty open subset contains another non empty open set which is disjoint from A .

Every non empty open set in a metric space contains a closed ball of some positive radius, any ball of positive radius ok, it is the closure of the open ball right. So, that is what we get now. So, start with a non empty open set ok, which is disjoint from A , inside that open set take a ball ok, such that even its closure also is contained inside that.

That you can do because once you start with an open set, and open ball inside that we can take smaller and smaller closed balls ok. So, this comes very easily, but this is what we are going to use later on.


(Refer Slide Time: 11:20)

<p style="text-align: center; margin: 0;">Introduction</p> <p style="margin: 0;">Creating New Spaces Smallness Properties of Topological Spaces Separation Axioms Topological Groups and Topological Vector Spaces</p>	<p style="margin: 0;">Module 6: Topological Spaces Module 7: Examples Module 8: Functions Module 11: Definitions and examples Module 10: Interior, derived set, etc. Module 10: Three Important Theorems on Complete Metric Space Module 20: An Application in Analysis Module 21: Completion</p>	
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Proof: By statement (iv) of the previous proposition 1.106, if G is a non-empty open set in X , it contains a non-empty open set G_1 disjoint from A . Choose $x \in G_1$ and $r > 0$ such that $B_r(x) \subset G_1$. Then the closure of $B_{r/2}(x)$ is contained in $B_r(x)$ and hence does not intersect A . \blacklozenge

So, I repeat this one. By statement (iv) of the previous proposition, if G is a non empty open set in X , it contains a non empty open set G_1 disjoint from A . Choose an $x \in G_1$ and r positive such that $B_r(x)$ is contained inside G_1 , then $B_{r/2}(x)$ is contained in $B_r(x)$, not only that its closure is also contained in $B_r(x)$ and this $B_r(x)$ does not intersect A , ok.

(Refer Slide Time: 11:57)

<p style="text-align: center; margin: 0;">Introduction</p> <p style="margin: 0;">Creating New Spaces Smallness Properties of Topological Spaces Separation Axioms Topological Groups and Topological Vector Spaces</p>	<p style="margin: 0;">Module 6: Topological Spaces Module 7: Examples Module 8: Functions Module 11: Definitions and examples Module 10: Interior, derived set, etc. Module 10: Three Important Theorems on Complete Metric Space Module 20: An Application in Analysis Module 21: Completion</p>	
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Remark 1.108

Recall that condition (i) of Prop. 1.106 was taken as the definition of a nowhere-dense set. Because of the above theorem, any one of the above five conditions can be taken to play the same role.

So, you just remember that interior of \bar{A} being empty the definition of nowhere dense set because of the above theorem anyone of the five conditions can be taken to play the role of the same thing this is what I have already remarked ok?

(Refer Slide Time: 12:16)

The screenshot shows a presentation slide. At the top left is a table of contents with 'Module 10 Interior, derived set, etc.' highlighted. At the top right is a video feed of Anant R Shastri. The main text on the slide reads: 'Here are a few results which are specific to metric spaces, when you can use convergent sequences.' Below this is 'Theorem 1.109' which states: 'Let (X, d) be metric space, and $A \subset X$ be any subset and $x \in X$ be any point. 1. $x \in \bar{A}$ iff there exists a sequence $\{x_n\}$ of points in A such that $x_n \rightarrow x$. 2. $x \in \ell(A)$ iff there exists a sequence $\{x_n\}$ of distinct points in A such that $x_n \rightarrow x$.' At the bottom, there is a footer with the NPTEL logo and the text 'Anant R Shastri Retired Emeritus Fellow Department of Mathematics NPTEL-NOC An Introductory Course on Point-Set Topology, P'.

Now, coming to the metric spaces, we have a few remarks to do about sequences ok. Start with a metric space ok. Take a subset A of X ok. Take any point. This x will be inside the closure of A , if and only if you can find a sequence (x_n) of points inside A such that the sequence converges to x , ok. The second statement is: the point x is in the derived set of A , (is a limit point) if and only if there exists a sequence (x_n) of distinct points in A , such that (x_n) converges to x , ok.

The difference between these two statements is that in the first part you can take some constant sequence of $(x_n) = (x, x, \dots)$, x_n converging to x . Now, x_n itself is x . So, that is not allowed here in the second one. If x, x, x, x, \dots is a sequence that always converges to x that does not mean that x is inside $\ell(A)$, it may be just inside A that is all ok. So, cluster points have special property. So, there must be a sequence of distinct points such that sequence the converges to x , ok?

(Refer Slide Time: 13:59)

Introduction	Module 6: Topological Spaces
Creating New Spaces	Module 7: Examples
Smallness Properties of Topological Spaces	Module 8: Functions
Separation Axioms	Module 11: Definitions and examples
Topological Groups and Topological Vector Spaces	Module 10: Interior, derived set, etc.
	Module 18: Three Important Theorems on Complete Metric Space
	Module 20: An Application in Analysis
	Module 21: Completion

Proof: (1) Suppose $x \in \bar{A}$. For each $n \in \mathbb{N}$, we know that $B_{1/n}(x) \cap A \neq \emptyset$. Therefore, we can choose $x_n \in B_{1/n}(x) \cap A$. It follows that $x_n \rightarrow x$. Converse is easy.

(2) Suppose $x \in \ell(A)$. Then we know that $(B_r(x) \setminus \{x\}) \cap A \neq \emptyset$ for all $r > 0$. Put $r_0 = 1$. Choose $x_1 \in (B_{r_0}(x) \setminus \{x\}) \cap A$. Put $r_1 = d(x_1, x)/2$. Inductively, having chosen $x_n \in (B_{r_{n-1}}(x) \setminus \{x\}) \cap A$ put $r_n = d(x_n, x)/2$ and choose $x_{n+1} \in (B_{r_n}(x) \setminus \{x\}) \cap A$. Check that the sequence $\{x_n\}$ is as required.

Converse is easy.

Exercise 1.110

Write down a detailed proof of the converse parts of both the statements in the above theorem.

So, let us see the proof of this one. Start with a point in the closure. For each $n \in \mathbb{N}$, we know that if you take $B_{1/n}(x)$, the ball, open ball of radius $1/n$ intersection A must be non empty because x is in the closure. Therefore, we can pick up a point (x_n) inside this $B_{1/n} \cap A$. So, this will be sequence in A , but the distance between x_n and x becomes smaller and smaller $1/n$ right as n tends to infinity this will converge to 0. So, x_n converges to x , ok. The converse that if there is a sequence converging to x and a sequence is in A , then it is inside \bar{A} . This we have seen several times.

Now, the second part. Suppose x is a cluster point, it is a limit point of A , We know that $(B_r(x) \setminus \{x\}) \cap A$ is non empty, for all r positive. Start with r , you know r equal to r_0 equal to 1. Choose $x_1 \in B_{r_0}(x)$ and not equal to x . Take a point x_1 not equal to x , but inside A ok. Look at the distance r_1 namely distance between x_1 and x divide it by 2, r_1 is the distance divided by 2. Inductively, having chosen x_n in $(B_{r_{n-1}} \setminus \{x\}) \cap A$, as soon as you choose x_n look at this number r_n equal to $d(x_n, x)/2$, ok.

Use this r_n to choose the next x_{n+1} and on. Here once you choose x_1 is like this. x_2 will be chosen inside $B_{r_1}(x) \setminus \{x\}$. So, the distance between x_{n+1} and x goes to what? You know each time the distance is less than see r_1 is something whatever, r is something, r_0, r_1 is distance between x is less than $r_0/2$. Next, it will be r_2 will be $r_1/2$ and so on. So, at r_n will be less than $r_0/2^n$. So, that will come down to 0, ok.

Why this sequence consists of distinct points? Because look at this one, x_1 is somewhere, but x_2 will be inside this distance. So, distance between x and x_1 is r_1 , distance between x and x_2 will be $r_1/2$ or small r therefore, they cannot be equal. So, next one which we will choose its distance between x and x_n will be smaller than all the earlier distances. So, this is a disjointed you know this is the distinct sequence of distinct points that is why it is so ok, ok. So, you can write down the converse of this one also it is very easy anyway.

(Refer Slide Time: 17:45)

Introduction
 Creating New Spaces
 Smallness Properties of Topological Spaces
 Separation Axioms
 Topological Groups and Topological Vector Spaces

Module 6: Topological Spaces
 Module 7: Examples
 Module 8: Functions
 Module 9: Definitions and examples
Module 10: Interior, derived set, etc.
 Module 11: Three Important Theorems on Complete Metric Space
 Module 20: An Application in Analysis
 Module 21: Completion

We shall now state and prove a statement about functions between topological spaces, which is as close an analogue to sequential continuity as possible:

Theorem 1.111
 Let $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}')$ be a set theoretic function. The following statements are all equivalent:

(a) f is continuous.
 (b) For every closed subset B of Y , $f^{-1}(B)$ is closed in X .
 (c) For every subset A of X , $f(\text{cl}(A)) \subset \text{cl}(f(A))$.
 (d) For every subset B of Y , $f^{-1}(\text{int } B) \subset \text{int}(f^{-1}(B))$.

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So, let us consider some functions between metric spaces and topological spaces first of all and put them in the proper perspective. Now, start with f from (X, \mathcal{T}) to (Y, \mathcal{T}') , a set theoretic function ok. Then f is continuous, it is statement (a).

The second statement is that for every subset B of Y , which is closed in Y , the $f^{-1}(B)$ is closed in X . Remember what was the definition of continuity, for every open subset U of Y , $f^{-1}(U)$ is open. That was the statement for continuity of a function. That one was statement (a). So, in statement (b), open sets are replaced by closed sets.

The third one is even much better. For every subset A of X , $f(\bar{A})$ is contained in the $\overline{f(A)}$. So, this is a forward statement. (a) and (b) were backwards. Starting from subsets of Y , you get some conclusion subsets of X . Here it is the other way around, ok?

And (d) is also similar, but in the reverse way for every subset B of Y , $f^{-1}(B^\circ)$ is contained in the interior of $f^{-1}(B)$, ok? These are all equivalent just means that you can use any one of them to define continuity of a function from any topological space to another topological space ok, that is the statement.

(Refer Slide Time: 19:54)

Introduction
 Creating New Spaces
 Smallness Properties of Topological Spaces
 Separation Axioms
 Topological Groups and Topological Vector Spaces

Module 6: Topological Spaces
 Module 7: Examples
 Module 8: Functions
 Module 11: Definitions and examples
 Module 18: Interior, derived set, etc.
 Module 18: Three Important Theorems on Complete Metric Space
 Module 20: An Application in Analysis
 Module 21: Completion

Proof: (a) \Leftrightarrow (b): This follows by a simple application of De Morgan's law.
 (b) \Rightarrow (c): $A \subset f^{-1}(f(A)) \subset f^{-1}(cl(f(A)))$ which is closed by (b). Since $cl(A)$ is the smallest closed subset containing A , it follows that $cl(A) \subset f^{-1}(cl(f(A)))$. This implies (c).
 (c) \Rightarrow (b): Let B be a closed subset Y . By (c), we have,

$$f(cl(f^{-1}(B))) \subset cl(f(f^{-1}(B))) \subset cl(B) = B.$$

This implies $cl(f^{-1}(B)) \subset f^{-1}(B)$. But then equality holds and hence $f^{-1}(B)$ is closed.
 Proof of (a) \Leftrightarrow (d) is left as an assignment to you.

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So, let us look at the proof of this one which is not all that difficult. First let us prove (a) and (b) are equivalent. This is just by De Morgan Law. If U is an open subset $X \setminus U$ is a closed set. $f^{-1}(Y \setminus U)$ is same thing as $X \setminus f^{-1}(U)$.

So, inverse of closed subset are closed subsets. So, you just use one way (a) implies (b) implies (a). Because you know that the compliment of a closed set is an open, and f inverse behaves very nicely under complimentation.

Now, let us prove (b) implies (c) ok? (c) is what? Start with any subset A of X , ok. So, what you have to prove that f of the closure of A is contained in the closure of $f(A)$, ok? A is always contained in $f^{-1}(f(A))$, ok? And $f(A)$ is contained in closure of $f(A)$. Therefore, f inverse of that is contained, $f^{-1}(cl(f(A)))$, ok? Now, closure of any set is closed. Condition (b) says f inverse of that is closed. Therefore, A is contained in this closed set. Therefore, \bar{A} is contained in this set, ok? because closure of A is the smallest closed subset containing the

given set A . So, this is a larger closed subset. Closure of A is the smallest closed subset in A . So, it is contained here, but this is same thing as if you put f here, $f(\bar{A})$ is contained in the closure of $f(A)$.

(c) implies (b) is what I want to show, the other way around now ok? So, start with any closed subset B of Y . By (c), we have $f(\overline{f^{-1}(B)})$, you do not know what it is ok, closure of $f^{-1}(B)$, f of that is, by (c) is contained inside closure of $f^{-1}(B)$ which is contained in closure of B . So, it is contained closure of B , this one is contained closure of B ok. But B is closed therefore, closure of B is B .

So, f of the closure of f inverse is contained inside B . This means that this closure of $f^{-1}(B)$ is contained in $f^{-1}(B)$ ok? But then equality holds ok? Because closure of any set contains that set. So, they are equal ok and hence $f^{-1}(B)$ itself is closed because this is the closure of that, ok?

The proof of the last statement is almost similar. Again something like De Morgan Law we have to use. So, I will leave that as an assignment to you namely f inverse of B° is contained in $f(B)^\circ$, ok. You should not take more than two lines. You should write down ok.

(Refer Slide Time: 24:14)

Introduction	Module 6: Topological Spaces
Creating New Spaces	Module 7: Examples
Smallness Properties of Topological Spaces	Module 8: Functions
Separation Axioms	Module 11: Definitions and examples
Topological Groups and Topological Vector Spaces	Module 10: Interior, derived set, etc.
	Module 18: Three Important Theorems on Complete Metric Space
	Module 20: An Application in Analysis
	Module 21: Completion

Exercise 1.112

Let $A = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}$. Under the usual topology on \mathbb{R} , compute \bar{A} , $\text{int } A$, $\partial(A)$ and $\ell(A)$.

So, here are some exercises A equal to the set of all $1/n, n \in \mathbb{N}$ and one extra element 0. Take the subset $1, 1/2, 1/3, 1/4$ etc. Then include 0 also. Under the usual topology from \mathbb{R} , compute $\bar{A}, A^\circ, \partial(A), \ell(A)$, ok. So, this is an easy exercise.

(Refer Slide Time: 24:43)

Introduction	Module 6: Topological Spaces
Creating New Spaces	Module 7: Examples
Smallness Properties of Topological Spaces	Module 8: Functions
Separation Axioms	Module 11: Definitions and examples
Topological Groups and Topological Vector Spaces	Module 10: Interior, derived set, etc.
	Module 18: Three Important Theorems on Complete Metric Space
	Module 20: An Application in Analysis
	Module 21: Completion

Exercise 1.113

Let X be any topological space. Prove the following formulae, for all subsets A of X .

(a) $\text{int}(A) = (cl(A^c))^c$.

(b) $\partial(\partial(A)) = \partial(A)$.

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So, this is an easy exercise.

(Refer Slide Time: 24:48)

Introduction	Module 6: Topological Spaces
Creating New Spaces	Module 7: Examples
Smallness Properties of Topological Spaces	Module 8: Functions
Separation Axioms	Module 11: Definitions and examples
Topological Groups and Topological Vector Spaces	Module 10: Interior, derived set, etc.
	Module 18: Three Important Theorems on Complete Metric Space
	Module 20: An Application in Analysis
	Module 21: Completion

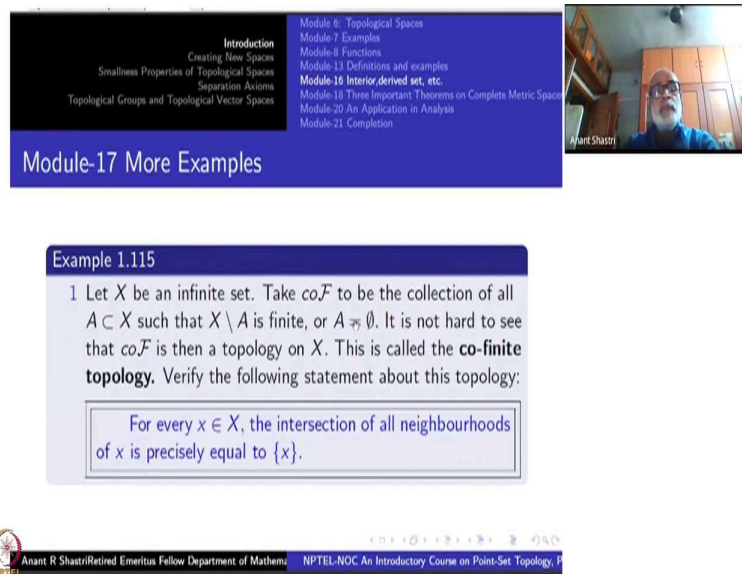
Exercise 1.114

Similar to the Kuratowski's closure axioms, formulate four axioms for the interior operator, obtain a topology associated to it and prove that the operator coincides with the usual operation of taking interiors with respect to this topology.

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Similarly, there are some more exercise here which you can take. Now, there is also this Kuratowski's closure axioms, formulate four axioms for interior operator, obtain a topology associated to it and prove that the operator coincides the usual operation of taking the interiors, similar to Kuratowski's closure axioms ok.

(Refer Slide Time: 25:18)



The screenshot shows a video lecture interface. On the left, a navigation menu lists various modules, with 'Module-17 More Examples' highlighted. On the right, a small video window shows the lecturer, Anant Shastri. The main content area displays 'Example 1.115' with a problem statement and a boxed conclusion.

Module-17 More Examples

Example 1.115

1 Let X be an infinite set. Take $\text{co}\mathcal{F}$ to be the collection of all $A \subset X$ such that $X \setminus A$ is finite, or $A = \emptyset$. It is not hard to see that $\text{co}\mathcal{F}$ is then a topology on X . This is called the **co-finite topology**. Verify the following statement about this topology:

For every $x \in X$, the intersection of all neighbourhoods of x is precisely equal to $\{x\}$.

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So, let us stop here today. Next time we will study more examples.

Thank you.