Introduction to Point Set Topology, (Part I)
Prof. Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture - 15
Interiors, boundaries etc.

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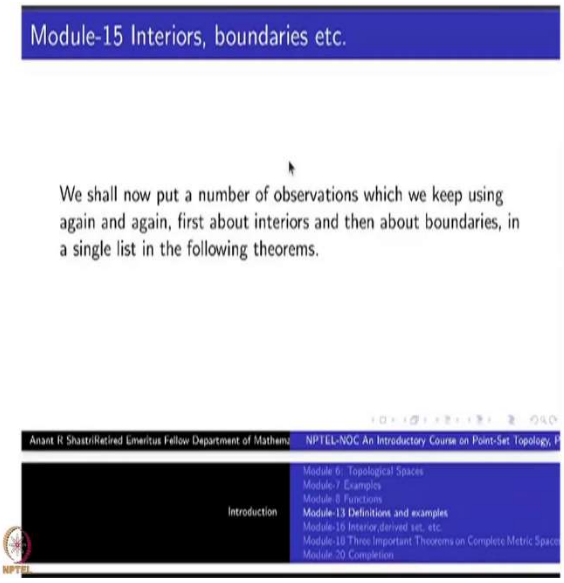

Welcome to module 15 of Point Set Topology Part 1. Last time we did something about the closure of a set and closed sets right. So, similar thing we will try to do for interior of a set and boundaries of a set and such things ok. So, most of them are similar in nature, but you have to be more careful that is all.

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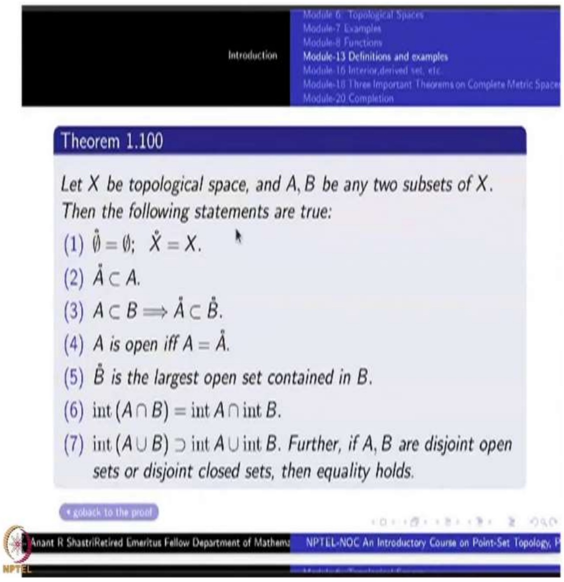

So, start with a topological space again, A and B are any two subsets of X. Then I have these seven properties listed here. The first one is interior of empty is empty. It is similar to closure of empty set is empty and interior of X is the whole of X. Again closure of X was X is similar to that right? So, there is some similarity, but do get carried away. You have to be very careful.

A contained in B implies A° is contained in B°. This is similar to the closure property by the way. So, how do we prove that? A° is all those open sets, union of all those open sets contained in A, but then they will be contained in B also. So, they will be contained in B interior ok, because B° is all those open sets contained in B. So, this is also obvious.

Finally, A is open if and only if A is A°, ok? So, this is similar to A is closed if and only if A is A closure. Here A is open if and only if A is A°, A is equal to A°, ok? So, why? If A is equal to A°, A° is always open set because it is union of open sets. So, A will be open. If A is open this it is the largest open set already. All other open sets are already there. So, I have to take this also that will be interior right. So, A will be equal to A°.

B° is the largest open set contained in B. So, this is somewhat you know complementary to similar, but complementary to a closure. Closure was what? The closure of a set is the smallest closed set containing that set containing that B. So, it is exactly you like applying De Morgan law. Exactly the opposite property. B° is the largest open set contained in B; yeah I could have put A here, but deliberately I have used B here you will see. So, B° is open set we know. Suppose you have another open set contained in B, but then B° is the union of all such things. So, it will contain B, it is contained in B°. So, B° is the largest open set. So, these are all just restatements of whatever we have done, they are not very difficult, slowly. You have to be bit a careful.

Interior of $A \cap B$ is $A^\circ \cap B^\circ$. Here again interior of anything is an open set right and it is contained in the $A \cap B$ here, therefore, is contained in A. Therefore, it is contained in A°, but same argument for interior B also. So, it is contained in the intersection. So, this this side is contained in the this side, LHS contained in RHS is obvious. Now, how to show this is contained here? Look at this one.

Interior of A is an open set, interior of B is an open set, intersection is an open set, A° is contained in A, ok? So, this intersection is also contained in $A.B^\circ$ is contained inside B. So, its intersection is contained inside B, but it is an open set. So, it is contained in the $(A \cap B)^\circ$, ok?

So, this last one here $(A \cup B)^\circ$ contains the $A^\circ \cup B^\circ$. So, there is no equality assertion here, ok? It is only one way and that is obvious because A° is subset of A right? So, it is a subset of $A \cup B$. Similarly, B° is subset of $A \cup B$. So, this whole thing is a subset of $A \cup B$, but this is an open set being the union of two open sets. Therefore, it is contained the interior. So, one way is clear.

Further if A and B are disjoint open sets or disjoint closed sets. Disjointness is common; both of them are open or both of them are closed that is what we have to assume ok? Then equality holds. So, let us see this one. So, all these things I have written down here, ok?

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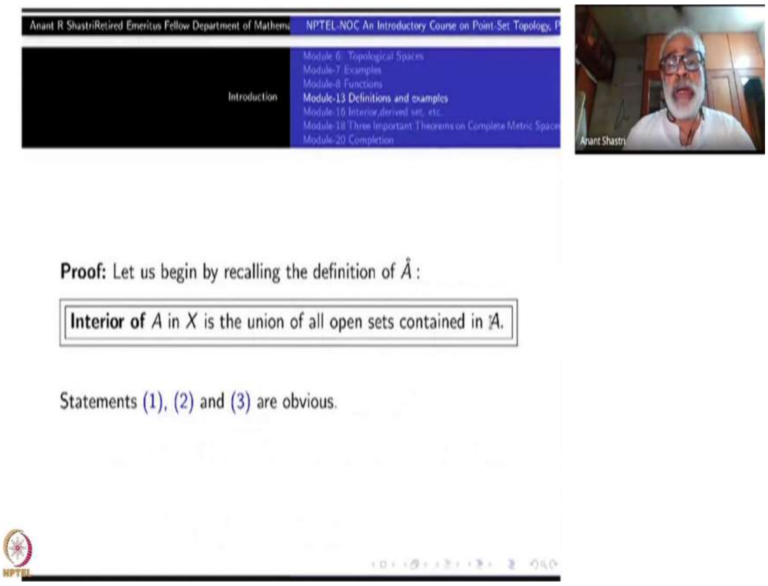

Starting with the definition, what is the definition now? This one, that A° is a union of all open sets contained in A, this was the definition. So, you can take different definitions then your proofs may be somewhat slightly different ok? That is all. I have taken this definition, I have done it ok?

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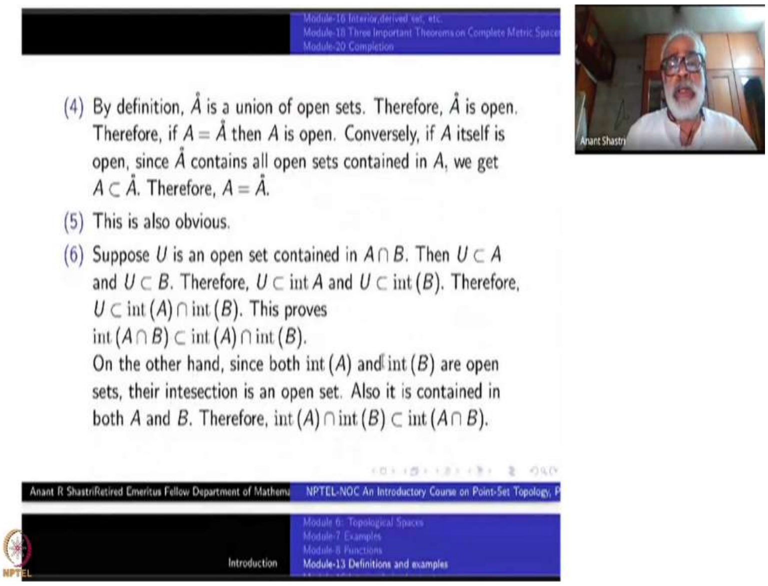

So, I have gone through 1, 2, 3, 4, 5, 6, 7 and so on up to here, interior of A intersection interior of B is contained.

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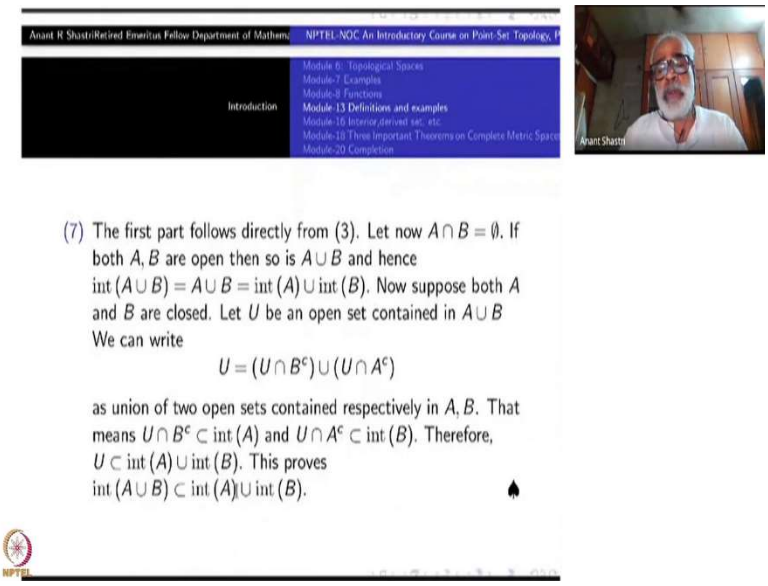

So, I have gone through $1, 2, 3, 4, 5, 6, 7$ and so on up to here, $A^\circ \cap B^\circ$ is contained. 7th one, one part we saw ok? First part directly from 3. Now, I want to show the second part here. Namely, suppose $A \cap B$ is empty, ok. If both A and B are open then the union is also open. Therefore, interior of A union B, $A \cup B$ is already open. So, you know $(A \cup B)^\circ$ is equal to $A \cup B$, but A is A°, B° is B because A and B are open. So, equality is obvious alright.

So, perhaps you have not used the intersection is empty here at all right. Now, suppose both A and B are closed, this time you may have to use that alright? Suppose both A and B are closed subsets and I assume $A \cap B$ is empty. Let U be an open set contained in the union, ok?

Suppose U is an open subset contained in $A \cup B$. If I have to show that interior of this one is equal to $A^\circ \cup B^\circ$, I must show that this open set first of all is contained in the $A^\circ \cup B^\circ$ separately. So, first of all you can write this U as union of two subsets here. One is $U \cap B^c$ and the other $U \cap A^c$. This is possible only because $A \cap B$ is empty.

$A \cap B$ is empty would mean that $A^c \cup B^c$ will be the whole space. Therefore, every set is contained in the whole space. So, it is $U \cap B^c, U \cap A^c$ take the union. This is true for all subsets now. Because $B^c \cup A^c$ is the whole space ok? Whole set, it is just a set theoretic thing right now. So, for that I have to use $A \cap B$ is empty.

Now, how do I use this one? U is now written as union of two sets here, as union of two open sets. U is open, B is closed, B^c is open so intersection is open, A^c is open, intersection is open. So, actually this one gives you that U is the union of two open sets. Respectively, this one is inside A now, because its it B^c is there.

And $A \cap B$ is empty ok. So, one of them must be inside A, the other one must be inside B, ok. So, that means that $U \cap B^c$ is contained in A° and $U \cap A^c$ is contained in B°. Therefore, U is subset of $A^\circ \cup B^\circ$. This proves that one way namely $(A \cup B)^\circ$ is contained in $A^\circ \cup B^\circ$, ok? The other way we have already seen, ok.

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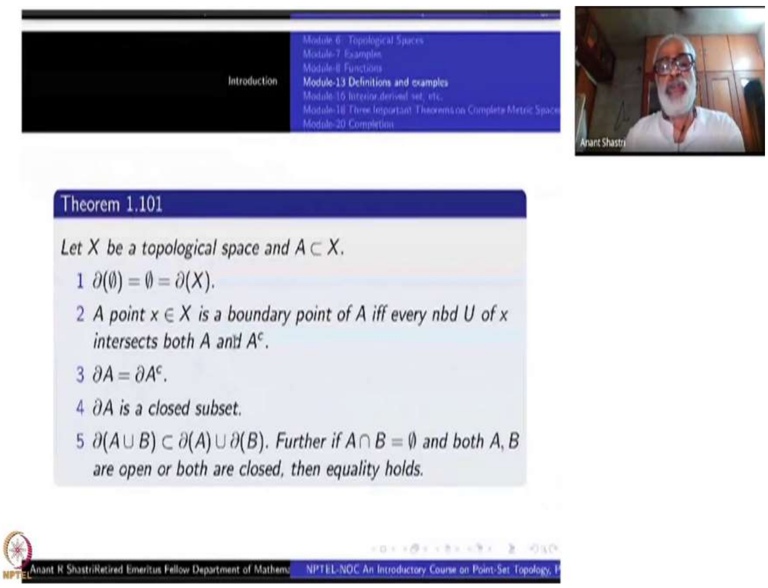

So, similar thing we will do for the boundary operator. Take any subset of a topological space boundary of the empty set is empty boundary of the whole space is also empty, not the whole space X, so this is a difference now ok. So, recall what is the boundary.

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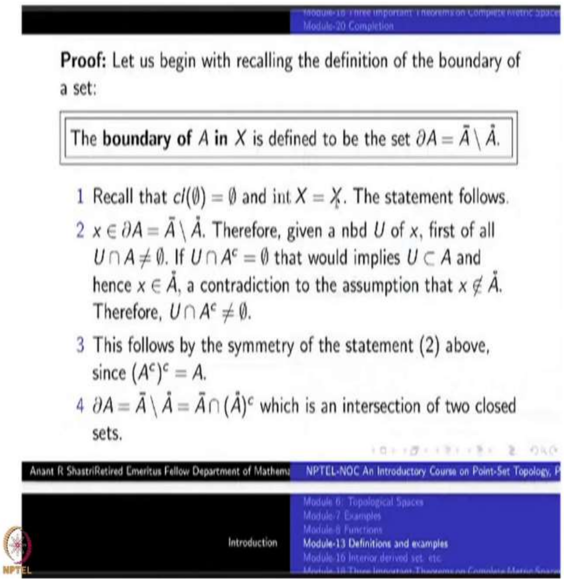

Boundary of a set is the closure of $\bar{A} \setminus A^{\circ}$. Throw away all the interior points from the $\bar{A}$. Keep that that is boundary of A, ok. So, closure is the whole space. If A is X then $\bar{X} = X$, interior is also X. So, $X \setminus X$ is empty set. So, that is what the first thing says here.

Second one says that a point x is a boundary point of A, if and only if every neighborhood U of X intersects both A and A^c, ok? Take a neighborhood of a point alright? I have to show it intersect both of them. If it does not intersect the complement means what? That neighborhood is completely contained inside A, right? Which means that that point is an interior point, but I have thrown the interior points from boundary right?

A point is is boundary point iff it is in the complement, first of all, of the interior points. No interior point is the boundary point alright. Similarly, this argument is similar here. If the neighbourhood does not intersect A then it will be contained in A^c, right? If it is contained in A^c, then it is in the interior of the complement right. Therefore, it is not a boundary point at all because every point should be a closure point of A. So, this is this argument is similar here.

Similarly, now interchanging A and A^c, this says boundary of A is equal to boundary of a complement also. So, this statement is a symmetric. So, this is also. Boundary of A is always equal to boundary of the complement alright, ok.

The 4th statement is boundary of A is a closed subset. That is also equally easy because what is boundary of set? It is $\bar{A} \setminus A^{\circ}$ which is a same thing as writing $\bar{A} \setminus (A^{\circ})^c$. A° is open. So, its complement is closed. Then you are taking intersection with $\bar{A}$, that is also closed. So, intersection of two closed sets is closed that is what we wanted to show, ok.

So, each boundary is a closed set. Finally, boundary of A union B is contained in $\partial(A) \cup \partial(B)$. This thing I will leave to you as an exercise because I have done so many things you have to do something on your own to get the feeling what is going on. So, you have to start with the ... I will just tell you how to do it.

Take a point here. You have to show it is either here or here. What you have to do? Take a point here which is not here then it must be here. This is what you have to do. This kind of argument also we have used already right. So, write the full detail as an exercise. The second part I will do it here.

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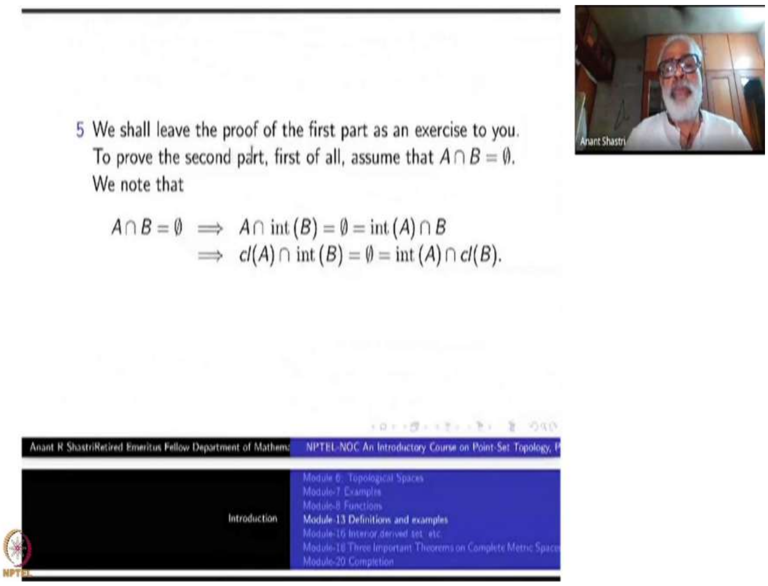

If A intersection B is empty and both A and B are open or both are closed it is similar to our interior part then the equality holds ok. I can leave this also an exercise, but I will do this one so that you are not feeling that I am cheating ok? So, we am leave the first part as an exercise. For the second part, I have to assume that A and B are disjoint. Once they are disjoint the first thing you have to notice this one. $A \cap B$ is empty implies $A \cap B^{\circ}$ is also empty because B° is smaller subset. Similarly, $A^{\circ} \cap B$ is also empty. So, this also because this is smaller subset than A.

So, this is just set theoretic because interiors are contained in the original sets, but now this implies we have seen this in the closure part, that the $\bar{A} \cap B^{\circ}$ is empty because B° is an open set. Whenever an open set does not intersect a set it does not intersect the closure that is what we have seen.

So, from here to here you get this is also empty, from here to here you get $A^{\circ} \cap \bar{B}$ is empty because A° is open. So, this is all that I can say right from the assumption that $A \cap B = \emptyset$ ok.

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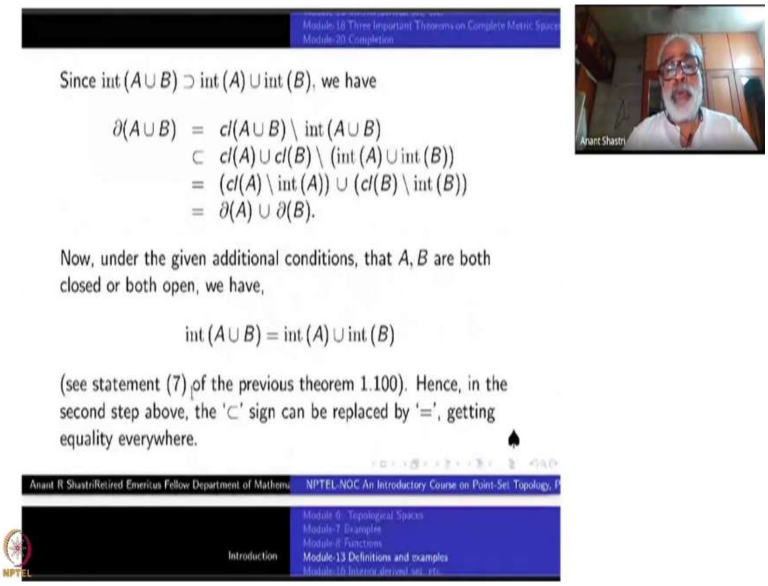

Now, let us start. The first thing is interior of the union contains $A^{\circ} \cup B^{\circ}$, this we have seen ok. So, boundary of $A \cup B$, which is by definition the $\overline{(A \cup B)} \setminus (A \cup B)^{\circ}$, right. Now, closure of the union finite union, here only two sets, is $\bar{A} \cup \bar{B}$ and this one is this is smaller instead of this one is larger than this one.

So, I have throwned away a larger set, here I am throwing a smaller set. So, I have a containment relation here not equality. Instead of throwing this one, we throw away $A^\circ \cup B^\circ$, which is a smaller set ok. So, this is contained inside here so this is contained here because you are taking the complementation and this is the minus, ok. Once you have here look at A°. This is a subset of $\bar{A}$, ok.

And I have just shown that A° does not intersect $\bar{B}$ at all. So, when I am subtracting this one from the union, I am actually subtracting it only from closure of A. So, it is the $\bar{A} \setminus A^\circ$. Similarly, when you are throwing B°, it is only thrown away from here, no point of this one comes there.

This follows because of this relation that we have proved, but this is boundary of A and that is boundary of B. So, what you have what A? $\partial(A \cup B)$ is contained in the $\partial(A) \cup \partial(B)$, ok. Now, now I put one more hypothesis.

What is the hypothesis? That A and B are both open or both closed that is the extra hypothesis here. In that case we have proved in the previous theorem that what we have proved, in the last part of previous theorem, that the $(A \cup B)^\circ$ is actually equal to $A^\circ \cup B^\circ$, right. Remember that. Then equality holds right. So, once equality holds go back to these steps. There are four steps here. The second step was only containment. Why? Because here there was no equality, this was containment.

If there is an equality here I can put equality here and everything else will be done, got it. So, one equality comes extra comes instead of this one provided both A and B are closed or both are open. So, this containment becomes equality here and all other things are equality there alright.

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So, now I will sum up something. Namely, open sets were the foundations for our definition of topology right. The topology was axiomatized on depending upon open sets. There is a parallel theory. As we have already observed because of De Morgan law, you can just convert all these three statements (T), (AU) and (FI). (AU) is arbitrary union, (FI) is finite intersection.

When you take the complements of these by De Morgan law the first statement remains as it is. Empty set and the whole space become whole space and empty set that is all. The second one arbitrary union becomes what arbitrary intersections, open sets become closed sets. And then finite intersection of open sets becomes finite union of closed sets.

So, you could have defined the same way with three axioms for closed closed sets right, but that would be cheating. What Kuratowski did was even more entertaining. At the time when Bourbaki adopted a modern and modified version of the axioms for topology ok? that is the definition that we have been working with ok? The Kuratowaki's result was only the correct definition available.

What I am talking about? The Kuratowski's result. However, for some reason which I do not know what it is, they the Bourbaki's rejected Kuratowski's approach and adopted a modification of Hausdorff's definition. If you directly take Hausdorff definition you do not get all topological spaces as we consider them today, because Hausdorff puts an extra condition, which will amount to what are called Hausdorff spaces.

So, they dropped out that one, but instead of dropping out the whole of Hausdorff, just dropped out that condition, the extra one. So, let us take a look at this Kuratowski's approach here. Maybe it is of some use ok. Gives you some more insight that is all, but we will not change our definition of topology which is based on upon open sets only, alright.

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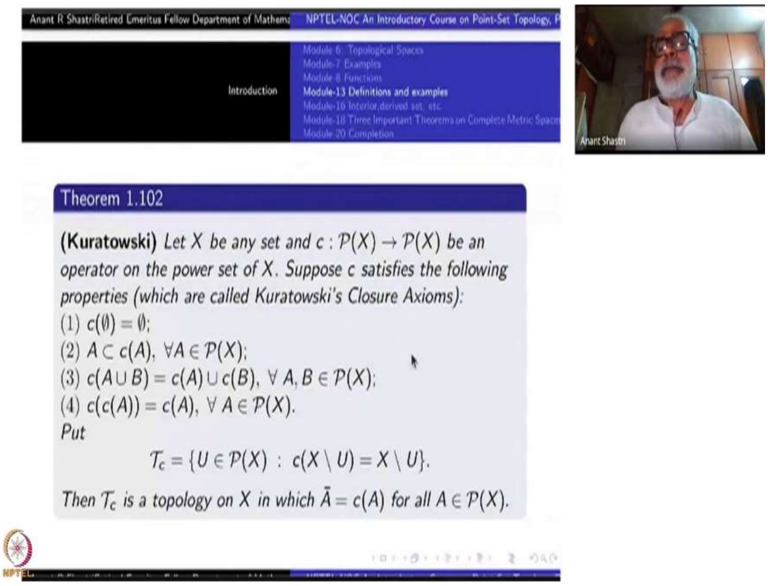


Let us do that. Let X be any set and c from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ be an operator, operator is just a self map ok? Do not worry about the new word. Usually this word operator is used by function analysists, ok? So, $\mathcal{P}(X)$ to $\mathcal{P}(X)$ be an operator on the power set of X. $\mathcal{P}(X)$ is the power set of X, c is a function from power set of X to itself.

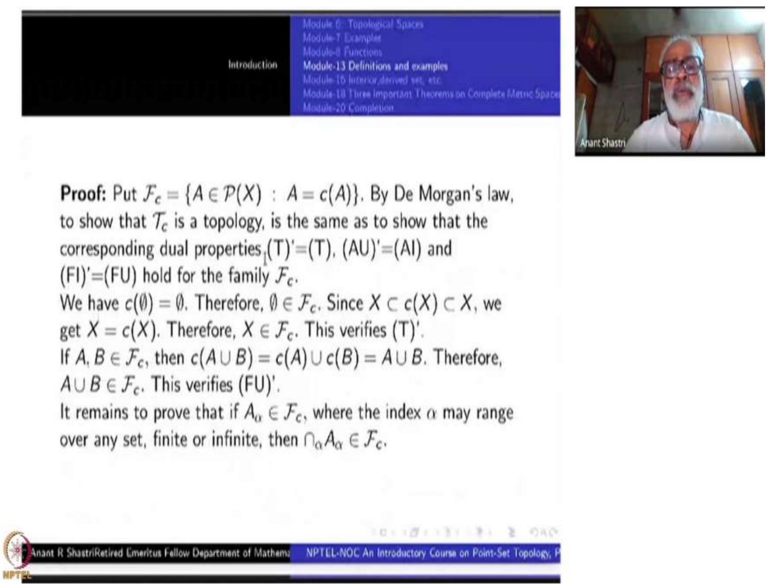
Suppose this operator satisfies the following properties. These properties are called Kuratowski's closure axioms ok? The first thing says that, let me read this c as closure itself now ok? just for just for the sake that how Kuratowski read it, though we have a different meaning for closure right now, but let me just for some time, read c of emptyset as closure of emptyset is empty.

Remember this was a property we have proved ok? So, here it becomes an axiom of Kuratowski. There is no topology here now. There is just this operator. A set is there and look at the all its subsets, on that there is an operator. So, the first thing is closure of $\emptyset$ is $\emptyset$. Whatever you assigned to empty set is empty set ok. Whatever you are assigned to A in general contains $A.A$ is contained inside $c(A)$ for every $A.c(A \cup B)$ is $c(A) \cup c(B)$.

If I read this as the closure what is it? Closure of $A \cup B$ is closure of A union closure of B. We have proved it. This is a theorem there here it is an axiom. This is true for every A, B. $c(c(A))$ is $c(A)$. This also you have proved, right? Closure of a closure is the closure itself. So, this an axiom again here and that is it. The four axioms he has selected he can create the entire topology that is the claim of Kuratowski, ok?

So, how do you do that? Put $\mathcal{T}_c$ instead of $\mathcal{T}$. Take $\mathcal{T}_c$, c corresponds to this closure operator. All those U inside $\mathcal{P}(X)$ ok; that means what? All subsets of X, such that the complement of this one, c of the complement is equal to c itself. So, put all of them, complements of that one, they form a topology on X and this topology has something to do with c. What is that?

The the usual definition of closure A, viz. $\bar{A}$ is equal to $c(A)$ for all A. So, this c becomes the closure that is why it is called closure operator ok. So, when I read it first time I was really thrilled by this one. (Refer Slide Time: 27:59)



So, let us verify this one. It is very straightforward not at all difficult. So, let us write $\mathcal{F}_c$ instead of $\mathcal{T}_c$. $\mathcal{F}_c$ is the collection of all all those A such that A equal to $c(A)$. What is $\mathcal{T}_c$? Rmember, it is a complement of this thing this thing that is $\mathcal{T}_c$. $\mathcal{T}_c$ is a topology we have to verify. Just now I have told youm by De Morgan law instead of (T) I have to verify (T'). Instead of (AU) I have to prove (AU'), which is nothing but (AI); that means, arbitrary intersection and instead of finite intersection I have to use finite union because under De Morgan law intersection becomes union and so on.

So, I have to verify these three axioms for $\mathcal{F}_c$, ok. Then $\mathcal{T}_c$ will verify (T), (AU) and (FU) and that is why its topology. So, first part will be over. So, let us verify these two. The first (T') is nothing but closure of empty set is empty now that is what is already there. Therefore, empty set is there $\mathcal{F}_c$, but X is already contained in $c(X)$. So, $c(X)$ cannot be bigger than X because they are all subsets of X. So, equality holds here. Therefore, X is in $c(X)$, therefore, X is in $\mathcal{F}_c$. So, (T') is verified ok.

If A and B are in $\mathcal{F}_c$, what is $c(A \cup B)$? It is $c(A) \cup c(B)$. Therefore, $c(A)$ is $A, c(B)$ is B because they are in $\mathcal{F}_c$. So, $A \cup B$ is in $\mathcal{F}_c$. So, this verifies finite union.

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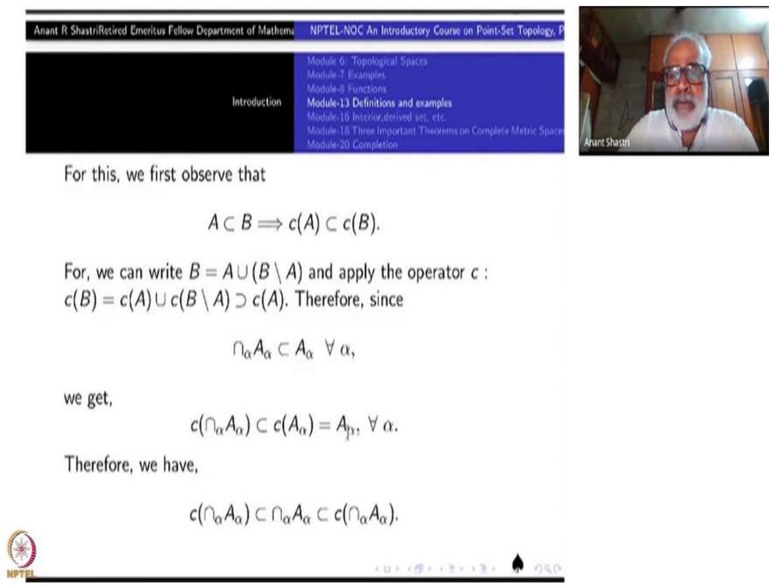

Now, I have to show arbitrary intersection. Take any family of A_α inside $\mathcal{F}_c$. Inside $\mathcal{F}_c$ means what? $c(A_\alpha)$ is equal to A_α for every α, then I have to show the same property for the intersection is what I have to show ok, no problem.

First we use another property. We will derive. This is not a part of the axiom. This we are proving as a consequence. What is that? If A is contained inside B, then $c(A)$ is contained inside $c(B)$. Why? Write B, B is a larger set, write it as $A \cup (B \setminus A)$. If you write this way then apply the operator c, $c(B)$ is $c(A) \cup c(B \setminus A)$, but $c(A)$ is there. So, $c(A)$ is contained inside $c(B)$, that is all. Whatever happens to $c(B \setminus A)$, do not care ok. So, A contained inside B implies $c(A)$ is contained $c(B)$. Now, the intersection is contained in every A_α. Therefore, its closure here is contained in closure of each of them $c(A_\alpha)$, but $c(A_\alpha)$ is A_α right. So, this closure contains here, but this equal to A_α. So, if its it is contained inside A_α for every α it is contained in the intersection.

So, closure of the intersection is contained in the intersection, but intersection is contained is always closure because for intersection for any set its contained in its closure. So, these two are there, so, there must be equality. These two are same right. So, there is equality. So, this proves the this proves the first part. I will leave the second part namely $\bar{A}$ equal to $c(A)$, just a very nice you know you just think about it, it is not difficult at all.

In this topology you have to prove that $\bar{A}$ equal to $c(A)$, for all A. So, that I will leave it as an exercise to you, ok.

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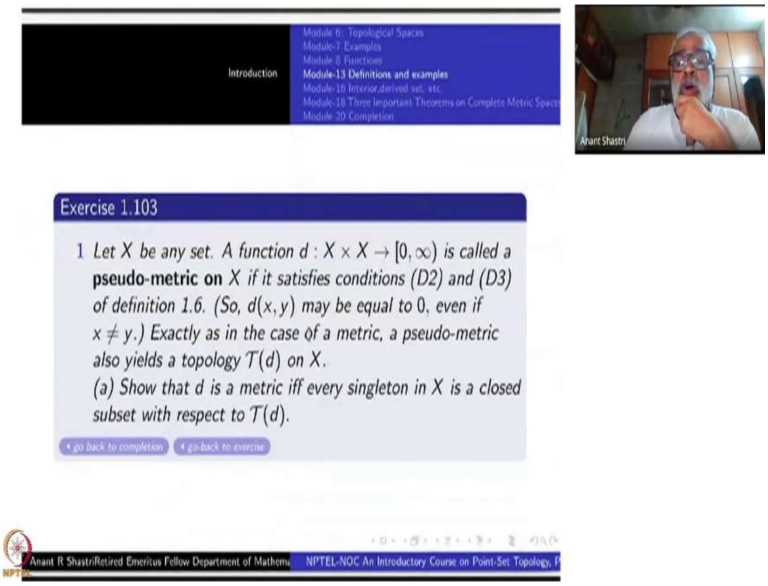

Here are some exercises about pseudo-metric. Just to tell you what is definition ok, D1 is not there that is all. A metric satisfies D1, D2, D3, remember D1 was the positive definiteness that is missing. D2 is symmetry, D3 is triangle inequality, only these these two are there. All other things are working exactly same, no problem ok.

You can put a $\mathcal{T}(d)$, namely the topology associated with these three. What are they? They are unions of balls open balls. Open balls sets are all these things are same, same way you have to carry on ok. So, that becomes a topology, you have to do that it show that d is a metric if and only if in this topology $\mathcal{T}(d)$, ok. Every singleton is closed so that is the first exercise.

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This part is easy you can prove the topology and so on. So, that I am not bothered about. First you have show this observe this one, this is also not difficult, observe this one. The second part says put an equivalence relation $d(x, y)$ equal to 0, then x is related to y. See if it is a metric $d(x, y)$ equals 0 would have implied that $x = y$. Since it is only a pseudo metric you do not know whether x is equal to y, that is why you want to identify them.

So, you put a equivalence relation. You put a relation. Now, you check that is an equivalence relation. For that you know all that you have to do is symmetry D2 you have to use, D3 or to use, it is very easy. Look at the equivalence classes, that is my notation $\hat{X}$ here. Denote the equivalence classes. On each equivalence class I am denoting by this $[x]$, $[y]$ and so on.

So, I am defining a map from $\hat{X} \times \hat{X}$ into $[0, \infty)$ by this formula; $d([x], [y])$ equals to $d(x, y)$. I am just putting this formula, but why this is well defined you have to check. Next you have to show that now this $\hat{d}$ here, there is a hat here ok, $\hat{d}$ becomes a metric on $\hat{X}$, ok.

So, the associated to a pseudo-metric space and a pseudo-metric there is a metric space ok which is the quotient of this X because equivalence relations are there that is the that is the gist of this one. Verify this one. That is all; all these things we have to verify.

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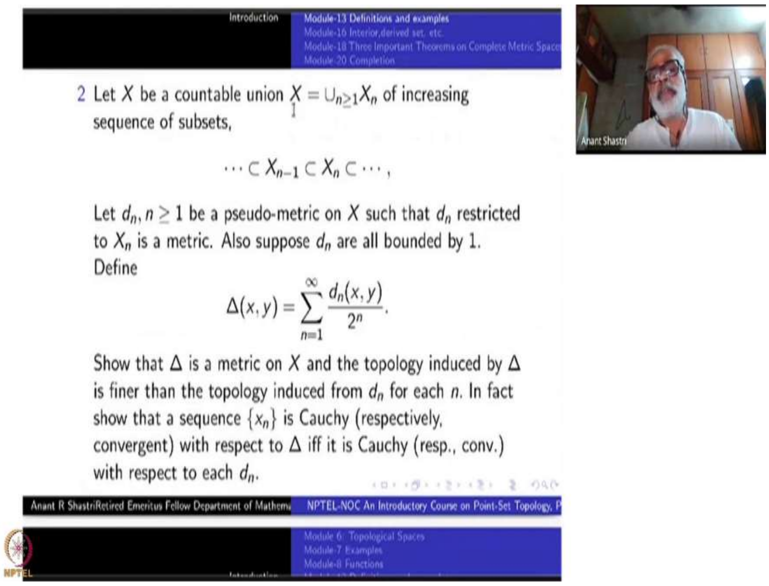

So, there is a similar exercise here about when you have union of increasing sequence of union of subsets X is countable increasing sequence of subsets. X_{n-1} contained in X_n and so on. On each X_n, you have a pseudo-metric d_n ok. Actually sorry, d_n's are metric pseudo-metrics on X on the defined on whole of X, when you restrict it to X_n, it is a metric. Only when x, y are inside X_n, $d_n(x, y) = 0$ implies $x = y$. If you go out of that into X that relation is not there that is all. So, on X they are all pseudo-metrics.

Restricted to X_n, they are all metrics. Suppose that is a case then you do this funny thing if you typical way of summing up infinite things ok, inside a when you have infinitely many real number only thing you have to assume that all the d_n's are bounded by one single number.

I have assumed bounded by 1, does not matter. Bounded by 1 single number it is enough. Define $\delta(x, y)$ equal to this summation of $d_n(x, y)/2^n$. It is bounded by $m, m/2^n$ summation that is convergent so, this also convergent. So, this will be finite number. This has wonderful properties now.

This δ will become a metric on X ok and it has some wonderful properties all these thing you have to verify. Namely, a sequence in (x_n) is Cauchy with respect to this δ if and only if it is

Cauchy with respect to each d_n, ok. So, the topology is very closely related that is what it means ok. So, let us meet next time.

Thank you.