Introduction to Point Set Topology, (Part I)
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Lecture - 14
Closed Sets

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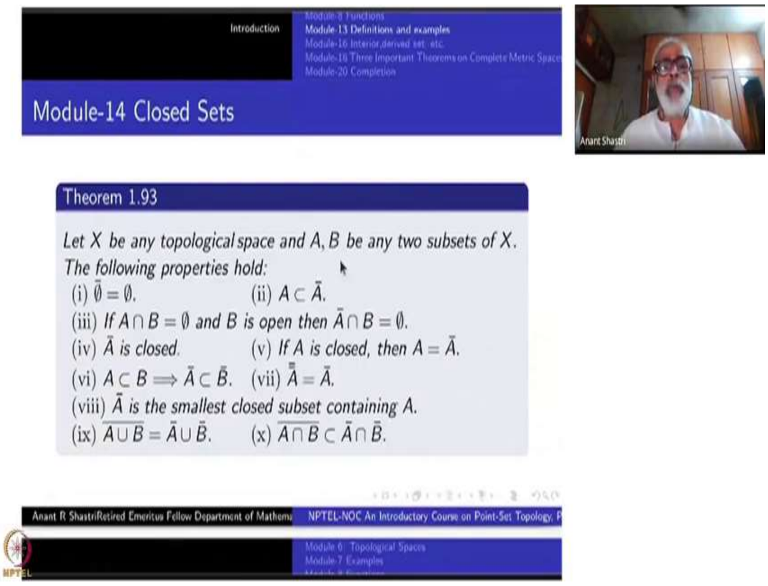

Welcome to modules 14 last time we had studied quite a bit about some examples and so on and of various notions like open set, closed sets, interior, boundary, nowhere denseness and so on right? So, today we shall start taking them up one by one, the foremost one is about Closed Sets which are just nothing but complements of open sets right? So, they will be as important as open sets after all.

So, let us start putting various properties in one single place as many as possible. It is not possible to list all of them exclusively ok, but it is possible to do quite a bit of good job here. Start with any topological space. Let A and B denote any two subsets of X, then we have a number of properties 1 2 3 4 5 6 up to 10, I have listed here they are all easy properties let us go through them one by one.

So, 1st one is the closure of empty set is empty. So, that is obvious. Why? Because closure is defined all points which have the property that whenever you take a neighbourhood of that point it should intersect the given set. But the set is empty. So intersection is empty. There therefore, no point will satisfy this property. Therefore the closure is empty. A is contained in $\bar{A}$. Because a neighbourhood of a point of A will always intersect A. So, that is also easy.

The 3rd one: if $A \cap B$ is empty and B is open then $\bar{A} \cap B$ is empty. Once again take a point which is in B, I want to show that it is not in $\bar{A}$, then $\bar{A} \cap B$ will be empty right. So, take a point in B, then B is an onbd of that point. We have assumed that $B \cap A$ is empty therefore, by the very definition of closure the point is not a point of $\bar{A}$. So, every point of B, you take it is not a point of $\bar{A}$. So, it is empty. So, you see all these things are very easy one by one. But when you club them together they may become more difficult. That is why we are going through them slowly ok?

The 4th one is: $\bar{A}$ itself is closed. For proving that $\bar{A}$ is closed what I should do? I should show that the complement of $\bar{A}$ is open, ok? Take a point x in the complement. That means what? It is not a point of $\bar{A}$ means, the negation a closure point what is the negation? there exists an open neighbourhood U of x which does not intersect A, right?

So, all those points in that open set will have the same property. Once y belongs to an open set $U, U \cap A$ is empty then y is also not in $\bar{A}$. All the points of U have that property for the same reason the entire of U must be in the complement. Therefore, we have shown that the complement is open, ok?

Similarly, now if A is closed then A is equal to $\bar{A}$. So, A is already contained in $\bar{A}$. I must show that $\bar{A}$ is contained in A. Actually what is the meaning of that, every closure point is in A or conversely what I should that any point which is not in A is not in the closure that is what I have to show right?

It is not in A because A is closed not in A means what it is in A^c, but A^c is now open, clearly $A^c \cap A$ is empty therefore, we have found a neighbourhood of every point inside the A^c. What is that neighbourhood? A^c itself which does not intersect A. Therefore, they are not points of $\bar{A}$. So, fifth is also proved namely A is closed means A is $\bar{A}$.

The 6th one: A is contained in B implies $\bar{A}$ is contained in $\bar{B}$. This is also easy because starting with a point in $\bar{A}$ and a neighbourhood of that point. By the very definition it intersects A, but B is larger. So, it will intersect B also therefore, the point must be inside $\bar{B}$ also. So, that is the property 6 here.

Now, A bar of bar of A, closure of the closure is the closure itself $\bar{\bar{A}}$ is $\bar{A}$, that is the statement 7 here. How do you do this one? You can go on taking point wise and so on, but already you have done enough work here. So, this is a consequence of that namely, look at what we have done A is contained inside $\bar{A}$ here. therefore, when you take the closure what does it give you this 6th one? It says that $\bar{A}$ is contained in $\bar{\bar{A}}$, ok? One way we have got, but the 5th one says the closure of the closure is already closed. So, it is equal to $\bar{A}$. So, therefore, the two are equal ok? So, you can use this one $\bar{A}$ is closed then A is equal to $\bar{A}$. So, $\bar{\bar{A}}$ is $\bar{A}$, that is directly.

The 7th one says that $\bar{A}$ is the smallest closed subset containing A. First of all, $\bar{A}$ contains A right? So, it is a closed subset containing A. Now you take any closed set B which contains A . Then you take bar. $\bar{A}$ itself will be contained $\bar{B}$, but $\bar{B}$ is B because I take any closed subset which contains $\bar{A}$. So, $\bar{A}$ is the smallest closed set. So, that is the 7th one.

The 9th one and 10th one are quite useful especially this one says what is happening to union under closure. So, this is like you can say that the two operations of taking union and closure they commute each other.

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First you take the union and then take the closure. Same thing as first take closures of both of them and then take the union. But in the 10th one when you have infinite union, it only says it is contained in here. You have to be cautious. Equality is not assured here. It may happen, but it is not true in general that is the meaning of this one ok?

Let us work out 9. So, first of all A is contained in the union therefore, by 6, we get $\bar{A}$ is contained inside the $\overline{A \cup B}$. Similarly $\bar{B}$ is contained in this. So, one way it is clear namely right hand side is contained in the left hand side ok? You have to show left hand side is contained in the right hand side. For that you have to work a little harder.

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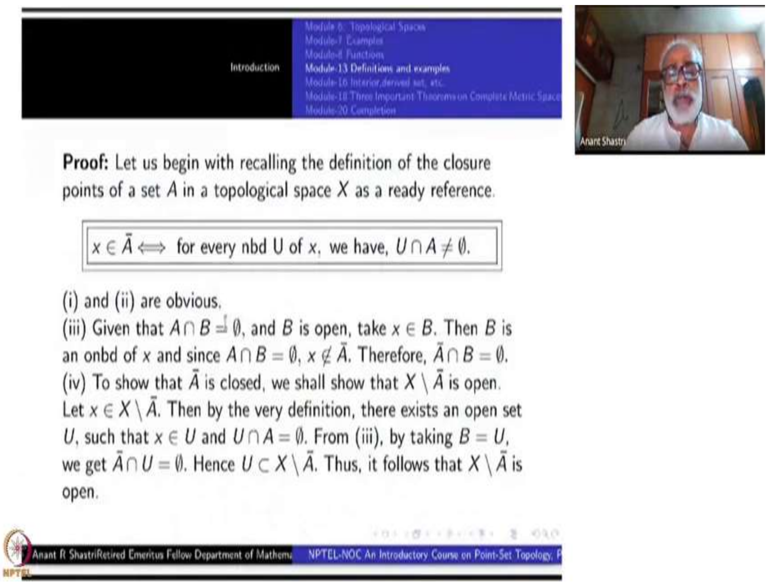

So, let me this time, use all these things I have written down carefully you can go through them ok? I have already done. But the 9th one. Let us go to the 9th one now ok?

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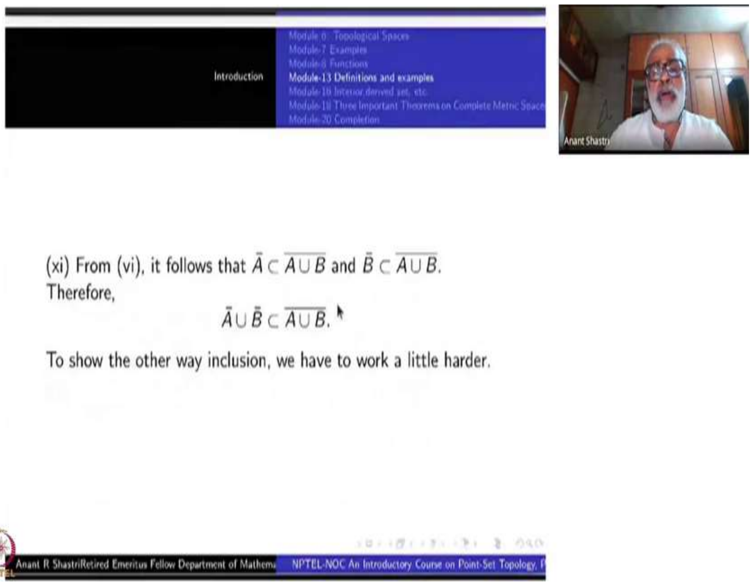

Namely one way you have already done, namely $\bar{A}$ is contained in the bar of the union because A is contained in $A \cup B$. Similarly, $\bar{B}$ is contained in $\overline{A \cup B}$. So, we want to do the other way round. So, start with a point x in the closure of the union. We want to show that x is in $\bar{A}$ or in $\bar{B}$, that is the meaning of the union. If x is in $\bar{A}$, no problem we are done.

So, assume x is not in $\bar{A}$. Then we must show that x is in $\bar{B}$, ok? What is the meaning of x is not in $\bar{A}$? This means that there is an open subset V such that x is inside V, and $V \cap A = \emptyset$, ok? Now, use this fact to show that x is in $\bar{B}$, ok. How? Take any open set V which contains x put W equal to $U \cap V$, ok?

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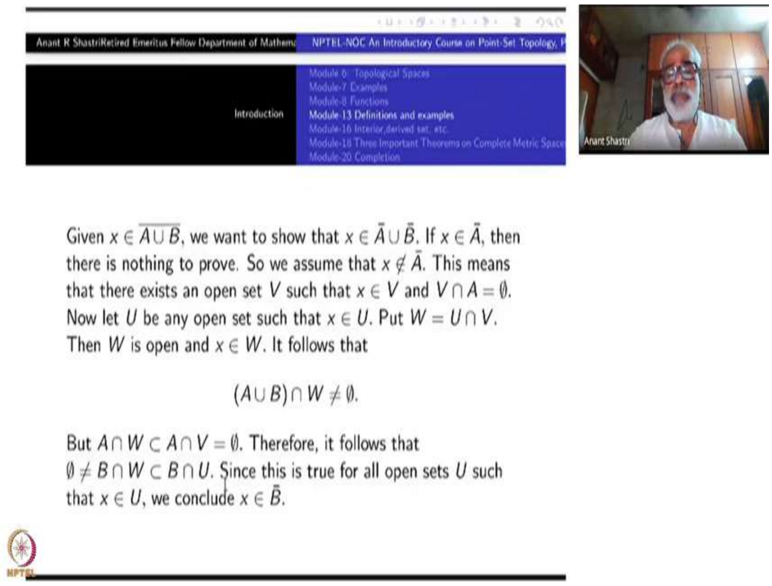

Then W is an open set and x belongs to W. Now, use the fact that x is in $\overline{A \cup B}$ that means, that means W will intersect $A \cup B$, ok? What is $A \cup B$ intersection W? It is $(A \cap W) \cup (B \cap W)$. One of them must be non-empty because this union is nonempty, ok?

But $A \cap W$ is already inside $A \cap V$. We started with $A \cap V$ equal to empty here. So, one of them is empty. The other one must be non-empty. What is the other one? $B \cap W$, but now $B \cap W$ is contained inside $B \cap U$ which is bigger therefore, this is also non-empty. So, that is what we wanted to show. Starting with any open set containing x we wanted to show that $U \cap B$ is non-empty. So, that that will prove that x is in $\bar{B}$.

So, what we have done? Assume that x is in the closure of the union, assume that it is not in one of them closure of one of them then it is in the closure of the other, ok. So, that proves 9.

Student: Actually sir. One question please. Here $A \cup B$ is subset of $\bar{A} \cup \bar{B}$. If I take both sides closuer. Then right side is $\bar{A} \cup \bar{B}$ is closed set union of closed sets. What I have started A is contained inside $A \cup B$.

Student: This is done other side other inclusion.

Student: $A \cup B$ is a subset of $\bar{A} \cup \bar{B}$. I take both side closure. But we do not know your $\bar{A} \cup \bar{B}$ is closed, you have to use yeah.

Student: Haha, $\bar{A}$ is closed and $\bar{B}$ is closed; union of closed sets. You are right, hum. Thank you sir.

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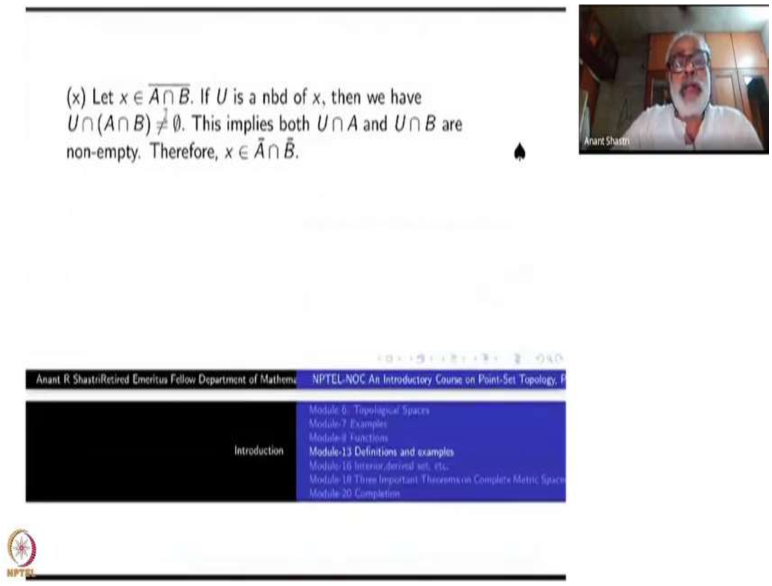

Yeah, you can do that way also very good yeah, ok. So, let us come to the intersection take a point x in the intersection. If U is a neighbourhood of x, then we have $U \cap (A \cap B)$ is non-empty because x is inside the closure, but if this intersection is non-empty both $U \cap A$ and $U \cap B$ must be non-empty therefore, for every neighbourhood of x both intersection is non-empty means what; x is in both $\bar{A}$ as well as $\bar{B}$.

So, the closure of the intersection is contained in intersection of the closures ok, but you have this you have this equality in inclusion map here ok. By repeated application of this you can always get a finite union right? Actually 9th one, let us go to 9th one here it is in equality it can take finite union ok and there will be equality.

But if you take infinite union will this work? equality? Usually that is what is going to be a problem because just now as one of you pointed out infinite union of closed sets may not be closed. So, you cannot come back ok? It may happen that they are equal, but in general that is not true right?

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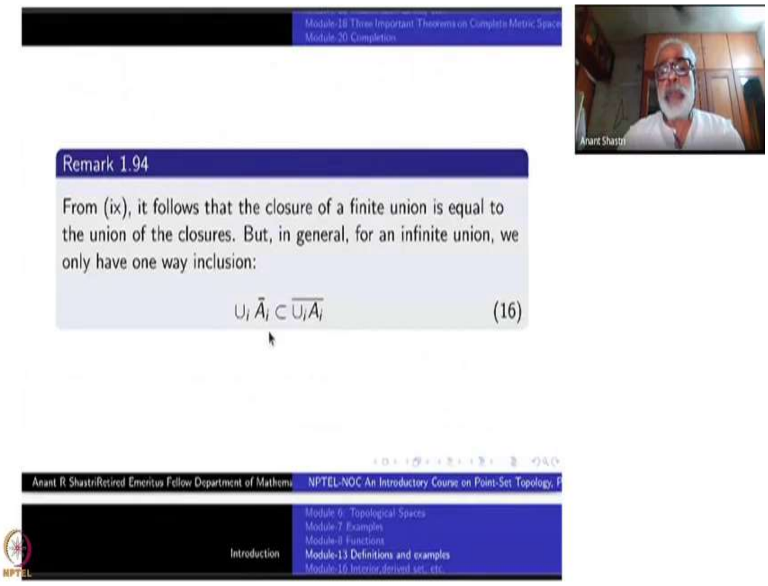

So, if you take arbitrary union the closure of the union definitely contains the union of the closures, but if you take first separately closures and then take the union it will be contained in. That is all, but may not be equal ok? We have plenty of examples for instance, here is an example.

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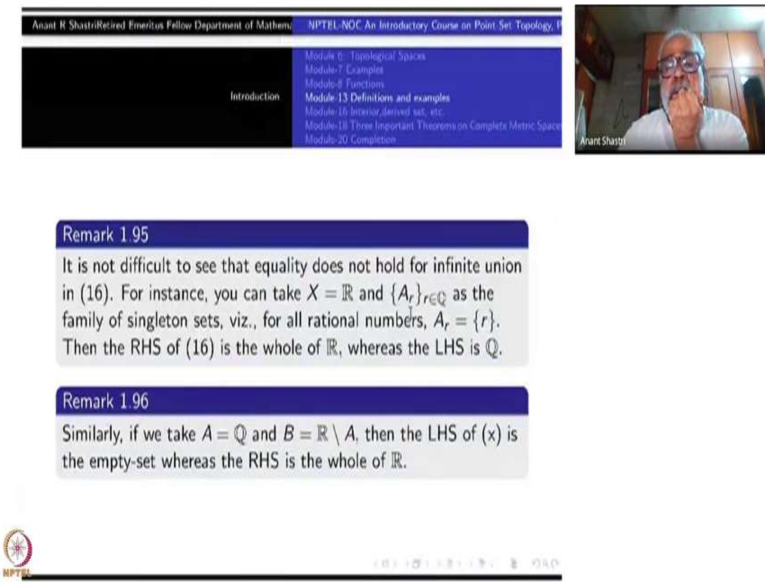

Take each singleton where the point varies over $\mathbb{Q}$, ok? Or it is A_r is just one point A_r is singleton r, then you take this collection the union of all these A_r's is just $\mathbb{Q}$, right? And what is the closure of each point, points are closed inside a metric space.

So, the closure is A_r itself. So the union of closure is just $\mathbb{Q}$, right? But first you take the union that is $\mathbb{Q}$ then take the closure it is the whole of $\mathbb{R}$, ok? So, here I am taking each A_r as a singleton rational number its closure is that singleton rational number itself. When you come here it is the whole of $\mathbb{Q}$, its closure we know is the whole of $\mathbb{R}$ ok, We know from elementary real analysis that $\mathbb{Q}$ is dense in $\mathbb{R}$, alright?

Similarly, the 10th one here ok. What is the 10th one? $\overline{A \cap B}$ is contained in $\bar{A} \cap \bar{B}$. Equality may not be true here even when there are only two sets. For that what do I do? I take A equal to $\mathbb{Q}$ and B equal to $\mathbb{Q}^c$; rational numbers and irrational numbers intersection is empty here. So, closure is also empty. But what is the closure of A? It is the whole of $\mathbb{R}$. What is the closure of B? It is the whole of $\mathbb{R}$ so intersection is $\mathbb{R}$. So, this is empty and that is the whole of $\mathbb{R}$.

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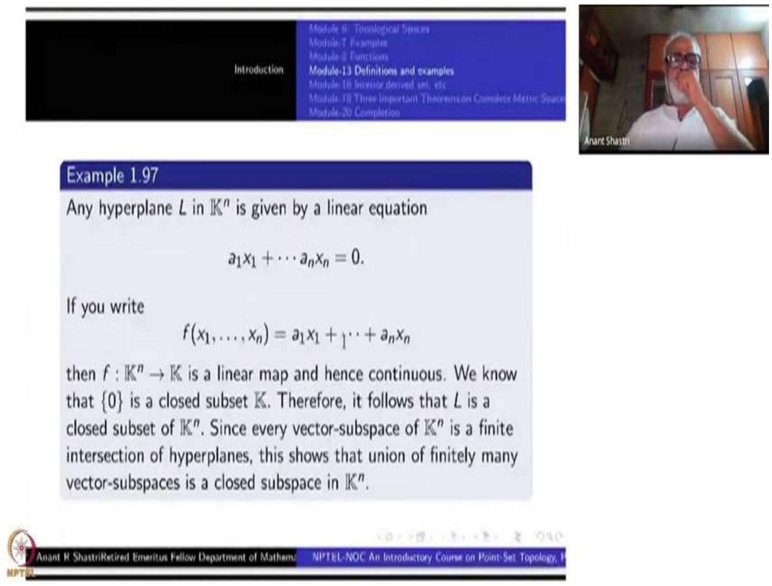

So, that gives you that equality need not hold alright. So, that is roughly all familiar properties with closed subsets ok. So, let us now go to some more examples here about closed sets. A hyperplane L in $\mathbb{K}^n$, you can talk of $\mathbb{R}^n$ or $\mathbb{C}^n$, does not matter.

So, think of $\mathbb{R}^n$, no problem when every coefficients etc will be real numbers that is all. So, hyperplane is given by a linear equation like this ok? This is actually if you put equal to 0, that is also called hyperplane this is are called hyper surface if you put equal to r, some real number or complex number here that will is the hyper surface. Hyper means what just one dimension lower subspace it is linear subspace, ok.

So, if you write f as $f(x_1, \ldots, x_n)$ is this linear map $a_1x_1 + \cdots + a_nx_n$, this is nothing but a linear map from $\mathbb{K}^n$ to $\mathbb{K}$, ok? We have seen that such a linear map is continuous because adding two continuous function is continuous. Scalar multiplication by a continuous function is continuous you use it iteratively, but you know 0 is a closed set inside $\mathbb{K}$. So, inverse image of a closed set is closed under a continuous map because inverse image of an open set is open and under a continuous map. So, this is what we have already seen right?

Therefore, what it amounts to saying is that each such L is a closed subset of $\mathbb{K}^n$. Easy examples. Now, you can use finite intersection of closed sets is closed to get that if you have finitely many equations like this the common 0's will be also closed. So, those are all the vector subspaces of $\mathbb{K}^n$, ok, take any vector subset of $\mathbb{K}^n$, suppose it is of dimension say m then you can write it as $n-m$ equations common solutions of $n-m$ equations only finitely many of them anyway.

So, they are intersections of co dimension one subspaces ok. So, they will be also closed ok.

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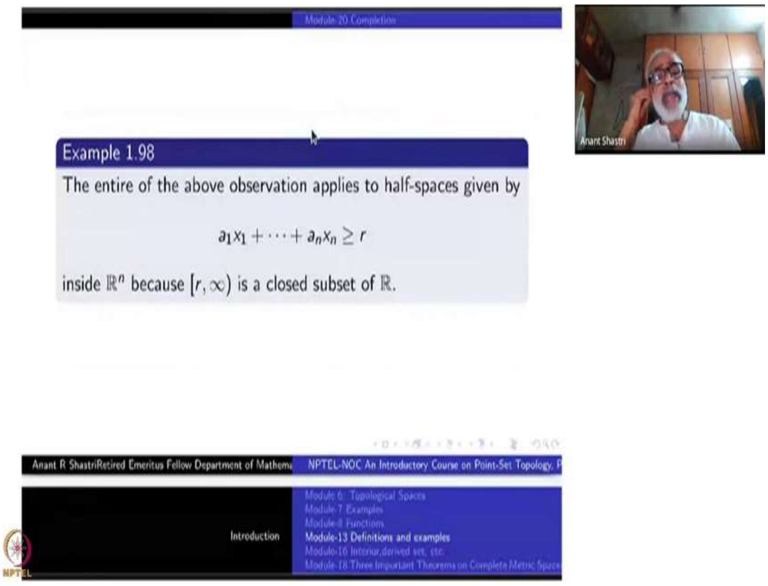

These are elementary examples, but they have to be properly understood, same explanation will hold for these half spaces like you can say x_1 is positive x_1 is greater than or equal to 0 or y_1 is greater than equal to 0 and so on. So, here $a_1x_1 + \cdots + a_nx_n$ is greater than equal to r greater than or equal to r; that means, what it is the inverse image of the closed interval $[r, \infty)$ open, inverse image will be a closed set.

So, such half spaces are also closed. So, intersections of these half spaces is also closed ok. So, for example, you can take all points which are bigger than or equal to x coordinate is bigger than or equal to 0 and y coordinate also bigger than or equal to 0, bigger than or equal to 0 we have put both of them are closed sets intersection will be closed set. What is it? It is the closed first quadrant right.

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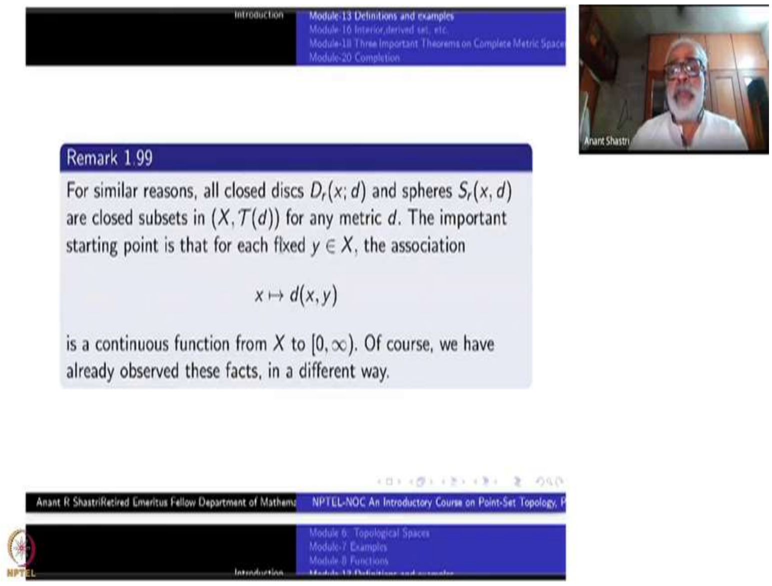

So, like this you can construct a lot of them. Moreover, you can go to other things. Look at the closed disc $D_r(x; d)$ in any metric space, little d is metric. $D_r(x; d)$ set of all points y such that distance between x and y is less than or equal to r, right? Similarly, the sphere what is S_r, it is those points wherein, the distance between x and y is equal to r, ok. So, those things will be closed also. Why? This time you have to use this fact namely fixing any y or fixing any x whichever one you want to fix.

Fix y, $x \to d(x, y)$, this is a continuous function on the topology of the space namely $\mathcal{T}(d)$ to the real numbers ok? It is a continuous function from x to $[0, \infty)$. What is the topology on this one? It is the metric topology on X and usual topology on $\mathbb{R}$ here, ok?

So, these things we have seen that they are continuous function therefore, closed balls, the spheres they are all closed subsets ok any questions. So, let us consolidate these things. Any questions? If you have any questions you should ask now ok?

Student: Hello sir. So, in the hyperplane equation instead of 0 it can be replaced by some real number.

Professor: Yeah. So, definition this is actually this is a hyperplane fine, but this is actually apassing through origin. When they pass through origin they are called hyper subspaces ok.

Student: Ok.

So, I wanted to concentrate on one thing instead of floating r first. So, if you can float r singleton r is also closed. So, same argument holds. So, that I have used in a second part we are going to put r it is fine ok. So, let us stop here until next time.

Thank you.