

**Introduction to Point Set Topology, (Part I)**  
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**Lecture - 14**  
**Closed Sets**

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**Module-14 Closed Sets**

**Theorem 1.93**

Let  $X$  be any topological space and  $A, B$  be any two subsets of  $X$ .  
The following properties hold:

- (i)  $\bar{\emptyset} = \emptyset$ .
- (ii)  $A \subset \bar{A}$ .
- (iii) If  $A \cap B = \emptyset$  and  $B$  is open then  $\bar{A} \cap B = \emptyset$ .
- (iv)  $\bar{A}$  is closed.
- (v) If  $A$  is closed, then  $A = \bar{A}$ .
- (vi)  $A \subset B \implies \bar{A} \subset \bar{B}$ .
- (vii)  $\overline{\bar{A}} = \bar{A}$ .
- (viii)  $\bar{A}$  is the smallest closed subset containing  $A$ .
- (ix)  $\overline{A \cup B} = \bar{A} \cup \bar{B}$ .
- (x)  $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$ .

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Welcome to modules 14 last time we had studied quite a bit about some examples and so on and of various notions like open set, closed sets, interior, boundary, nowhere denseness and so on right? So, today we shall start taking them up one by one, the foremost one is about Closed Sets which are just nothing but complements of open sets right? So, they will be as important as open sets after all.

So, let us start putting various properties in one single place as many as possible. It is not possible to list all of them exclusively ok, but it is possible to do quite a bit of good job here. Start with any topological space. Let  $A$  and  $B$  denote any two subsets of  $X$ , then we have a number of properties 1 2 3 4 5 6 up to 10, I have listed here they are all easy properties let us go through them one by one.

So, 1st one is the closure of empty set is empty. So, that is obvious. Why? Because closure is defined all points which have the property that whenever you take a neighbourhood of that point it should intersect the given set. But the set is empty. So intersection is empty. There therefore, no point will satisfy this property. Therefore the closure is empty.  $A$  is contained in  $\bar{A}$ . Because a neighbourhood of a point of  $A$  will always intersect  $A$ . So, that is also easy.

The 3rd one: if  $A \cap B$  is empty and  $B$  is open then  $\bar{A} \cap B$  is empty. Once again take a point which is in  $B$ , I want to show that it is not in  $\bar{A}$ , then  $\bar{A} \cap B$  will be empty right. So, take a point in  $B$ , then  $B$  is an onbd of that point. We have assumed that  $B \cap A$  is empty therefore, by the very definition of closure the point is not a point of  $\bar{A}$ . So, every point of  $B$ , you take it is not a point of  $\bar{A}$ . So, it is empty. So, you see all these things are very easy one by one. But when you club them together they may become more difficult. That is why we are going through them slowly ok?

The 4th one is:  $\bar{A}$  itself is closed. For proving that  $\bar{A}$  is closed what I should do? I should show that the complement of  $\bar{A}$  is open, ok? Take a point  $x$  in the complement. That means what? It is not a point of  $\bar{A}$  means, the negation a closure point what is the negation? there exists an open neighbourhood  $U$  of  $x$  which does not intersect  $A$ , right?

So, all those points in that open set will have the same property. Once  $y$  belongs to an open set  $U$ ,  $U \cap A$  is empty then  $y$  is also not in  $\bar{A}$ . All the points of  $U$  have that property for the same reason the entire of  $U$  must be in the complement. Therefore, we have shown that the complement is open, ok?

Similarly, now if  $A$  is closed then  $A$  is equal to  $\bar{A}$ . So,  $A$  is already contained in  $\bar{A}$ . I must show that  $\bar{A}$  is contained in  $A$ . Actually what is the meaning of that, every closure point is in  $A$  or conversely what I should that any point which is not in  $A$  is not in the closure that is what I have to show right?

It is not in  $A$  because  $A$  is closed not in  $A$  means what it is in  $A^c$ , but  $A^c$  is now open, clearly  $A^c \cap A$  is empty therefore, we have found a neighbourhood of every point inside the  $A^c$ . What is that neighbourhood?  $A^c$  itself which does not intersect  $A$ . Therefore, they are not points of  $\bar{A}$ . So, fifth is also proved namely  $A$  is closed means  $A$  is  $\bar{A}$ .

The 6th one:  $A$  is contained in  $B$  implies  $\bar{A}$  is contained in  $\bar{B}$ . This is also easy because starting with a point in  $\bar{A}$  and a neighbourhood of that point. By the very definition it intersects  $A$ , but  $B$  is larger. So, it will intersect  $B$  also therefore, the point must be inside  $\bar{B}$  also. So, that is the property 6 here.

Now,  $\bar{\bar{A}}$  is  $\bar{A}$ , that is the statement 7 here. How do you do this one? You can go on taking point wise and so on, but already you have done enough work here. So, this is a consequence of that namely, look at what we have done  $A$  is contained inside  $\bar{A}$  here. therefore, when you take the closure what does it give you this 6th one? It says that  $\bar{A}$  is contained in  $\bar{\bar{A}}$ , ok? One way we have got, but the 5th one says the closure of the closure is already closed. So, it is equal to  $\bar{A}$ . So, therefore, the two are equal ok? So, you can use this one  $\bar{A}$  is closed then  $A$  is equal to  $\bar{A}$ . So,  $\bar{\bar{A}}$  is  $\bar{A}$ , that is directly.

The 7th one says that  $\bar{A}$  is the smallest closed subset containing  $A$ . First of all,  $\bar{A}$  contains  $A$  right? So, it is a closed subset containing  $A$ . Now you take any closed set  $B$  which contains  $A$ . Then you take bar.  $\bar{A}$  itself will be contained  $\bar{B}$ , but  $\bar{B}$  is  $B$  because I take any closed subset which contains  $A$ . So,  $\bar{A}$  is the smallest closed set. So, that is the 7th one.

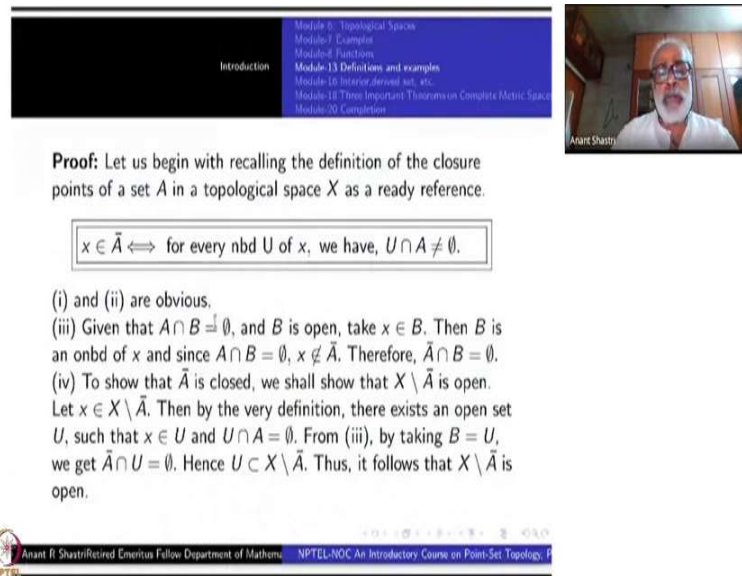
The 9th one and 10th one are quite useful especially this one says what is happening to union under closure. So, this is like you can say that the two operations of taking union and closure they commute each other.

The 9th one and 10th one are quite useful. Especially this one says what is happening to the union under closure. So, this is like you can say that the two operations of taking union and closure they commute each other.

First you take the union and then take the closure. Same thing as first take closures of both of them and then take the union. But in the 10th one when you have infinite union, it only says it is contained in here. You have to be cautious. Equality is not assured here. It may happen, but it is not true in general that is the meaning of this one ok?

Let us work out 9. So, first of all  $A$  is contained in the union therefore, by 6, we get  $\bar{A}$  is contained inside the  $\overline{A \cup B}$ . Similarly  $\bar{B}$  is contained in this. So, one way it is clear namely right hand side is contained in the left hand side ok? You have to show left hand side is contained in the right hand side. For that you have to work a little harder.

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The slide contains a table of contents on the top right, a video feed of Anant Shastri on the top right, and a proof of the definition of closure points in the center.

Introduction	Module 6: Topological Spaces
	Module 7: Examples
	Module 8: Functions
	Module 13: Definitions and examples
	Module 16: Interior, derived set, etc.
	Module 18: Three Important Theorems on Complete Metric Spaces
	Module 20: Completion

**Proof:** Let us begin with recalling the definition of the closure points of a set  $A$  in a topological space  $X$  as a ready reference.

$$x \in \bar{A} \iff \text{for every nbd } U \text{ of } x, \text{ we have, } U \cap A \neq \emptyset.$$

(i) and (ii) are obvious.  
 (iii) Given that  $A \cap B = \emptyset$ , and  $B$  is open, take  $x \in B$ . Then  $B$  is an nbd of  $x$  and since  $A \cap B = \emptyset$ ,  $x \notin \bar{A}$ . Therefore,  $\bar{A} \cap B = \emptyset$ .  
 (iv) To show that  $\bar{A}$  is closed, we shall show that  $X \setminus \bar{A}$  is open. Let  $x \in X \setminus \bar{A}$ . Then by the very definition, there exists an open set  $U$ , such that  $x \in U$  and  $U \cap A = \emptyset$ . From (iii), by taking  $B = U$ , we get  $\bar{A} \cap U = \emptyset$ . Hence  $U \subset X \setminus \bar{A}$ . Thus, it follows that  $X \setminus \bar{A}$  is open.

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So, let me this time, use all these things I have written down carefully you can go through them ok? I have already done. But the 9th one. Let us go to the 9th one now ok?

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Introduction

Module 0: Topological Spaces  
 Module 7: Examples  
 Module 8: Functions  
 Module 13: Definitions and examples  
 Module 16: Interior, derived set, etc.  
 Module 18: Three Important Theorems on Complete Metric Spaces  
 Module 20: Completion

Anant Shastri

(xi) From (vi), it follows that  $\bar{A} \subset \overline{A \cup B}$  and  $\bar{B} \subset \overline{A \cup B}$ .  
 Therefore,  

$$\bar{A} \cup \bar{B} \subset \overline{A \cup B}.$$

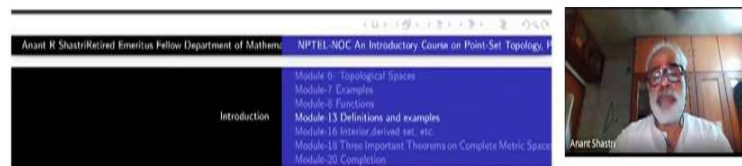
To show the other way inclusion, we have to work a little harder.

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Namely one way you have already done, namely  $\bar{A}$  is contained in the bar of the union because  $A$  is contained in  $A \cup B$ . Similarly,  $\bar{B}$  is contained in  $\overline{A \cup B}$ . So, we want to do the other way round. So, start with a point  $x$  in the closure of the union. We want to show that  $x$  is in  $\bar{A}$  or in  $\bar{B}$ , that is the meaning of the union. If  $x$  is in  $\bar{A}$ , no problem we are done.

So, assume  $x$  is not in  $\bar{A}$ . Then we must show that  $x$  is in  $\bar{B}$ , ok? What is the meaning of  $x$  is not in  $\bar{A}$ ? This means that there is an open subset  $V$  such that  $x$  is inside  $V$ , and  $V \cap A = \emptyset$ , ok? Now, use this fact to show that  $x$  is in  $\bar{B}$ , ok. How? Take any open set  $V$  which contains  $x$  put  $W$  equal to  $U \cap V$ , ok?

(Refer Slide Time: 11:01)



The image shows a presentation slide on the left and a video feed of a speaker on the right. The slide has a blue header with the text 'Anant R Shastri Retired Emeritus Fellow Department of Mathemat...' and 'NPTEL-NOC An Introductory Course on Point-Set Topology, I'. Below the header is a table of contents with 'Introduction' selected. The video feed shows a man with a white beard and glasses, wearing a white shirt, speaking.

Given  $x \in \overline{A \cup B}$ , we want to show that  $x \in \overline{A} \cup \overline{B}$ . If  $x \in \overline{A}$ , then there is nothing to prove. So we assume that  $x \notin \overline{A}$ . This means that there exists an open set  $V$  such that  $x \in V$  and  $V \cap A = \emptyset$ . Now let  $U$  be any open set such that  $x \in U$ . Put  $W = U \cap V$ . Then  $W$  is open and  $x \in W$ . It follows that

$$(A \cup B) \cap W \neq \emptyset.$$

But  $A \cap W \subset A \cap V = \emptyset$ . Therefore, it follows that  $\emptyset \neq B \cap W \subset B \cap U$ . Since this is true for all open sets  $U$  such that  $x \in U$ , we conclude  $x \in \overline{B}$ .



Then  $W$  is an open set and  $x$  belongs to  $W$ . Now, use the fact that  $x$  is in  $\overline{A \cup B}$  that means, that means  $W$  will intersect  $A \cup B$ , ok? What is  $A \cup B$  intersection  $W$ ? It is  $(A \cap W) \cup (B \cap W)$ . One of them must be non-empty because this union is nonempty, ok?

But  $A \cap W$  is already inside  $A \cap V$ . We started with  $A \cap V$  equal to empty here. So, one of them is empty. The other one must be non-empty. What is the other one?  $B \cap W$ , but now  $B \cap W$  is contained inside  $B \cap U$  which is bigger therefore, this is also non-empty. So, that is what we wanted to show. Starting with any open set containing  $x$  we wanted to show that  $U \cap B$  is non-empty. So, that that will prove that  $x$  is in  $\overline{B}$ .

So, what we have done? Assume that  $x$  is in the closure of the union, assume that it is not in one of them closure of one of them then it is in the closure of the other, ok. So, that proves 9.

Student: Actually sir. One question please. Here  $A \cup B$  is subset of  $\overline{A} \cup \overline{B}$ . If I take both sides closure. Then right side is  $\overline{\overline{A} \cup \overline{B}}$  is closed set union of closed sets. What I have started  $A$  is contained inside  $A \cup B$ .

Student: This is done other side other inclusion.

Student:  $A \cup B$  is a subset of  $\bar{A} \cup \bar{B}$ . I take both side closure. But we do not know your  $\bar{A} \cup \bar{B}$  is closed, you have to use yeah.

Student: Haha,  $\bar{A}$  is closed and  $\bar{B}$  is closed; union of closed sets. You are right, hum. Thank you sir.

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The screenshot shows a slide with the following text:
   
(x) Let  $x \in \overline{A \cap B}$ . If  $U$  is a nbd of  $x$ , then we have  $U \cap (A \cap B) \neq \emptyset$ . This implies both  $U \cap A$  and  $U \cap B$  are non-empty. Therefore,  $x \in \bar{A} \cap \bar{B}$ .
   
 Below the slide, there is a navigation bar for the course:
   
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 The course menu includes:
 

- Module 6: Topological Spaces
- Module 7: Examples
- Module 8: Functions
- Module 13: Definitions and examples
- Module 16: Interior, derived set, etc.
- Module 18: Three Important Theorems on Complete Metric Spaces
- Module 20: Completion

 The NPTEL logo is visible at the bottom left of the slide area.

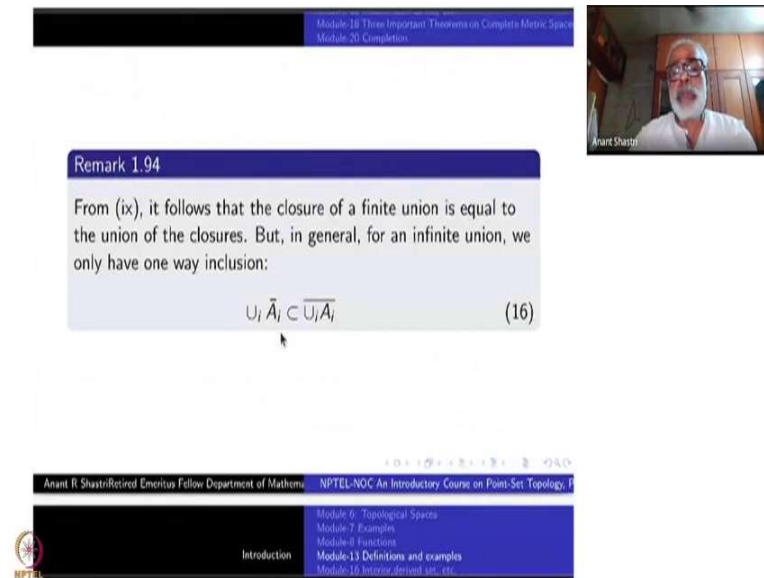
Yeah, you can do that way also very good yeah, ok. So, let us come to the intersection take a point  $x$  in the intersection. If  $U$  is a neighbourhood of  $x$ , then we have  $U \cap (A \cap B)$  is non-empty because  $x$  is inside the closure, but if this intersection is non-empty both  $U \cap A$  and  $U \cap B$  must be non-empty therefore, for every neighbourhood of  $x$  both intersection is non-empty means what;  $x$  is in both  $\bar{A}$  as well as  $\bar{B}$ .

So, the closure of the intersection is contained in intersection of the closures ok, but you have this you have this equality in inclusion map here ok. By repeated application of this you can always get a finite union right? Actually 9th one, let us go to 9th one here it is in equality it can take finite union ok and there will be equality.

But if you take infinite union will this work? equality? Usually that is what is going to be a problem because just now as one of you pointed out infinite union of closed sets may not be

closed. So, you cannot come back ok? It may happen that they are equal, but in general that is not true right?

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The screenshot shows a video lecture interface. At the top, a blue header contains the text "Module-18: Three Important Theorems on Complete Metric Spaces" and "Module-20: Completion". On the right side, there is a small video window showing a man with a white beard and glasses, identified as "Anant Shrivastava". The main content area features a blue box with the text "Remark 1.94" and the following text: "From (ix), it follows that the closure of a finite union is equal to the union of the closures. But, in general, for an infinite union, we only have one way inclusion:". Below this text is the mathematical equation 
$$\bigcup_j \bar{A}_j \subset \overline{\bigcup_j A_j} \quad (16)$$
. At the bottom of the slide, there is a navigation bar with the text "Anant R. Shrivastava, Retired Emeritus Fellow, Department of Mathematics, NPTEL-NOC An Introductory Course on Point-Set Topology, P" and a list of modules: "Introduction", "Module-6: Topological Spaces", "Module-7: Examples", "Module-8: Functions", "Module-13: Definitions and examples", and "Module-16: Interior, derived set, etc.". The NPTEL logo is visible in the bottom left corner.

So, if you take arbitrary union the closure of the union definitely contains the union of the closures, but if you take first separately closures and then take the union it will be contained in. That is all, but may not be equal ok? We have plenty of examples for instance, here is an example.



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Introduction	Module 0: Topological Spaces Module 7: Examples Module 8: Functions Module 13: Definitions and examples Module 16: Interior, derived set, etc. Module 18: Some Important Theorems on Complete Metric Spaces Module 20: Completion
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**Remark 1.95**  
It is not difficult to see that equality does not hold for infinite union in (16). For instance, you can take  $X = \mathbb{R}$  and  $\{A_r\}_{r \in \mathbb{Q}}$  as the family of singleton sets, viz., for all rational numbers,  $A_r = \{r\}$ . Then the RHS of (16) is the whole of  $\mathbb{R}$ , whereas the LHS is  $\mathbb{Q}$ .

**Remark 1.96**  
Similarly, if we take  $A = \mathbb{Q}$  and  $B = \mathbb{R} \setminus A$ , then the LHS of (x) is the empty-set whereas the RHS is the whole of  $\mathbb{R}$ .

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Take each singleton where the point varies over  $\mathbb{Q}$ , ok? Or it is  $A_r$  is just one point  $A_r$  is singleton  $r$ , then you take this collection the union of all these  $A_r$ 's is just  $\mathbb{Q}$ , right? And what is the closure of each point, points are closed inside a metric space.

So, the closure is  $A_r$  itself. So the union of closure is just  $\mathbb{Q}$ , right? But first you take the union that is  $\mathbb{Q}$  then take the closure it is the whole of  $\mathbb{R}$ , ok? So, here I am taking each  $A_r$  as a singleton rational number its closure is that singleton rational number itself. When you come here it is the whole of  $\mathbb{Q}$ , its closure we know is the whole of  $\mathbb{R}$  ok, We know from elementary real analysis that  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , alright?

Similarly, the 10th one here ok. What is the 10th one?  $\overline{A \cap B}$  is contained in  $\bar{A} \cap \bar{B}$ . Equality may not be true here even when there are only two sets. For that what do I do? I take  $A$  equal to  $\mathbb{Q}$  and  $B$  equal to  $\mathbb{Q}^c$ ; rational numbers and irrational numbers intersection is empty here. So, closure is also empty. But what is the closure of  $A$ ? It is the whole of  $\mathbb{R}$ . What is the closure of  $B$ ? It is the whole of  $\mathbb{R}$  so intersection is  $\mathbb{R}$ . So, this is empty and that is the whole of  $\mathbb{R}$ .

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The screenshot shows a presentation slide with a table of contents at the top, a main content area, and a video inset. The table of contents lists: Introduction, Module 6: Topological Spaces, Module 7: Example, Module 8: Functions, Module 13: Definitions and examples, Module 16: Inverse image, etc., Module 18: Three important Theorems on Complex Metric Space, and Module 20: Completion. The main content area is titled 'Example 1.97' and contains the following text: 'Any hyperplane  $L$  in  $\mathbb{K}^n$  is given by a linear equation  $a_1x_1 + \dots + a_nx_n = 0$ . If you write  $f(x_1, \dots, x_n) = a_1x_1 + \dots + a_nx_n$  then  $f : \mathbb{K}^n \rightarrow \mathbb{K}$  is a linear map and hence continuous. We know that  $\{0\}$  is a closed subset of  $\mathbb{K}$ . Therefore, it follows that  $L$  is a closed subset of  $\mathbb{K}^n$ . Since every vector-subspace of  $\mathbb{K}^n$  is a finite intersection of hyperplanes, this shows that union of finitely many vector-subspaces is a closed subspace in  $\mathbb{K}^n$ .' The video inset shows a man with glasses and a white shirt, identified as Anant Shastri.

So, that gives you that equality need not hold alright. So, that is roughly all familiar properties with closed subsets ok. So, let us now go to some more examples here about closed sets. A hyperplane  $L$  in  $\mathbb{K}^n$ , you can talk of  $\mathbb{R}^n$  or  $\mathbb{C}^n$ , does not matter.

So, think of  $\mathbb{R}^n$ , no problem when every coefficients etc will be real numbers that is all. So, hyperplane is given by a linear equation like this ok? This is actually if you put equal to 0, that is also called hyperplane this is are called hyper surface if you put equal to  $r$ , some real number or complex number here that will be the hyper surface. Hyper means what just one dimension lower subspace it is linear subspace, ok.

So, if you write  $f$  as  $f(x_1, \dots, x_n)$  is this linear map  $a_1x_1 + \dots + a_nx_n$ , this is nothing but a linear map from  $\mathbb{K}^n$  to  $\mathbb{K}$ , ok? We have seen that such a linear map is continuous because adding two continuous function is continuous. Scalar multiplication by a continuous function is continuous you use it iteratively, but you know 0 is a closed set inside  $\mathbb{K}$ . So, inverse image of a closed set is closed under a continuous map because inverse image of an open set is open and under a continuous map. So, this is what we have already seen right?

Therefore, what it amounts to saying is that each such  $L$  is a closed subset of  $\mathbb{K}^n$ . Easy examples. Now, you can use finite intersection of closed sets is closed to get that if you have

finitely many equations like this the common 0's will be also closed. So, those are all the vector subspaces of  $\mathbb{K}^n$ , ok, take any vector subset of  $\mathbb{K}^n$ , suppose it is of dimension say  $m$  then you can write it as  $n - m$  equations common solutions of  $n - m$  equations only finitely many of them anyway.

So, they are intersections of co dimension one subspaces ok. So, they will be also closed ok.

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The slide contains the following text:

**Example 1.98**  
 The entire of the above observation applies to half-spaces given by

$$a_1x_1 + \dots + a_nx_n \geq r$$

inside  $\mathbb{R}^n$  because  $[r, \infty)$  is a closed subset of  $\mathbb{R}$ .

The navigation menu at the bottom lists the following modules:

- Module 6: Topological Spaces
- Module 7: Examples
- Module 8: Functions
- Module 13: Definitions and examples
- Module 15: Interior, derived set, etc.
- Module 16: Three Important Theorems on Complete Metric Spaces

These are elementary examples, but they have to be properly understood, same explanation will hold for these half spaces like you can say  $x_1$  is positive  $x_1$  is greater than or equal to 0 or  $y_1$  is greater than equal to 0 and so on. So, here  $a_1x_1 + \dots + a_nx_n$  is greater than equal to  $r$  greater than or equal to  $r$ ; that means, what it is the inverse image of the closed interval  $[r, \infty)$  open, inverse image will be a closed set.

So, such half spaces are also closed. So, intersections of these half spaces is also closed ok. So, for example, you can take all points which are bigger than or equal to  $x$  coordinate is bigger than or equal to 0 and  $y$  coordinate also bigger than or equal to 0, bigger than or equal to 0 we have put both of them are closed sets intersection will be closed set. What is it? It is the closed first quadrant right.

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Remark 1.99

For similar reasons, all closed discs  $D_r(x; d)$  and spheres  $S_r(x, d)$  are closed subsets in  $(X, \mathcal{T}(d))$  for any metric  $d$ . The important starting point is that for each fixed  $y \in X$ , the association

$$x \mapsto d(x, y)$$

is a continuous function from  $X$  to  $[0, \infty)$ . Of course, we have already observed these facts, in a different way.

So, like this you can construct a lot of them. Moreover, you can go to other things. Look at the closed disc  $D_r(x; d)$  in any metric space, little  $d$  is metric.  $D_r(x; d)$  set of all points  $y$  such that distance between  $x$  and  $y$  is less than or equal to  $r$ , right? Similarly, the sphere what is  $S_r$ , it is those points wherein, the distance between  $x$  and  $y$  is equal to  $r$ , ok. So, those things will be closed also. Why? This time you have to use this fact namely fixing any  $y$  or fixing any  $x$  whichever one you want to fix.

Fix  $y$ ,  $x \rightarrow d(x, y)$ , this is a continuous function on the topology of the space namely  $\mathcal{T}(d)$  to the real numbers ok? It is a continuous function from  $x$  to  $[0, \infty)$ . What is the topology on this one? It is the metric topology on  $X$  and usual topology on  $\mathbb{R}$  here, ok?

So, these things we have seen that they are continuous function therefore, closed balls, the spheres they are all closed subsets ok any questions. So, let us consolidate these things. Any questions? If you have any questions you should ask now ok?

Student: Hello sir. So, in the hyperplane equation instead of 0 it can be replaced by some real number.

Professor: Yeah. So, definition this is actually this is a hyperplane fine, but this is actually apassing through origin. When they pass through origin they are called hyper subspaces ok.

Student: Ok.

So, I wanted to concentrate on one thing instead of floating  $r$  first. So, if you can float  $r$  singleton  $r$  is also closed. So, same argument holds. So, that I have used in a second part we are going to put  $r$  it is fine ok. So, let us stop here until next time.

Thank you.