Introduction to Point Set Topology, (Part I)
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Lecture - 12
Topology Equivalence – Similarity

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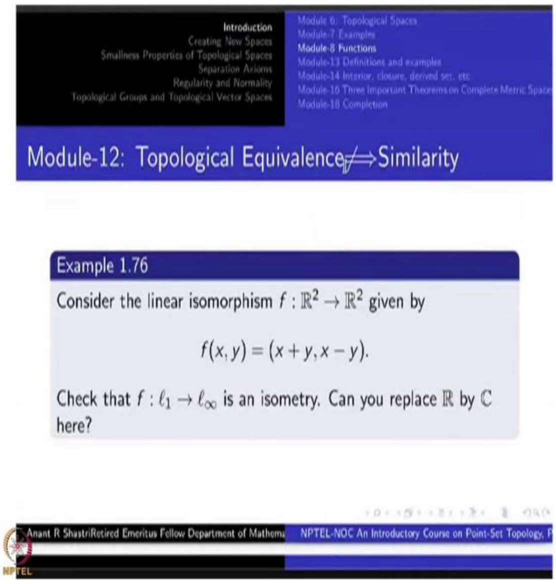

So, last time we studied the implication signs the other way round namely a homeomorphism does not imply similarity and similarity does not imply isometry. This was our topic. But we covered only one of them namely similarity does not imply isometry. So, today let us cover the other one namely, topological equivalence does not imply similarity, homeomorphism does not imply similarity.

Before that, I want to give you another example here of isometry, this time. Remember last time I proved that ℓ_1 and ℓ_2 are not isometric and similarly ℓ_2 and ℓ_∞ are not isometric. But this may come to you as a surprise. When I noticed it, I was really very happy to have noticed it, namely ℓ_1 and ℓ_∞ are isometric on $\mathbb{R}^2$.

All that you have to do is inside $\mathbb{R}^2$ rotate by 45° and scale appropriately. If you do not know how to do that you may say ok that is bit more complicated, ok, you write (x, y) going to $(x + y, x - y)$. That is a linear isomorphism ok? That is a linear isomorphism alright.

What I want to say is that it is an isometry namely you take here in the domain the ℓ_1 norm and on this side take ℓ_2 norm, if you take the same norm this is not an isometry ok take ℓ_1 norm here and ℓ_∞ norm here. What is the ℓ_1 norm? $|x| + |y|$ right? If both x and y are positive what do you get? Maximum of that will be $|x| + |y|$ ok. So, there is some ℓ_∞ norm. If one of them is positive and the other one is negative then $|x| - |y|$ will be maximum. If both of them are negative then the again this will be maximum and and so on. So, maximum norm is equal to $|x| + |y|$ always right. So, this is an isometry alright.

The smallest norm ℓ_1 and the largest norm whatever ℓ_∞, they are isometric on $\mathbb{R}^2$, but can you do it over complex number? The definition is the same ok? Think about it can you do it for $\mathbb{R}^3, \mathbb{R}^4, \mathbb{R}^5$ and so, on? Think about it. I do not want to make any more comments. I want to go to this topic now namely topological equivalence does not imply similarity.

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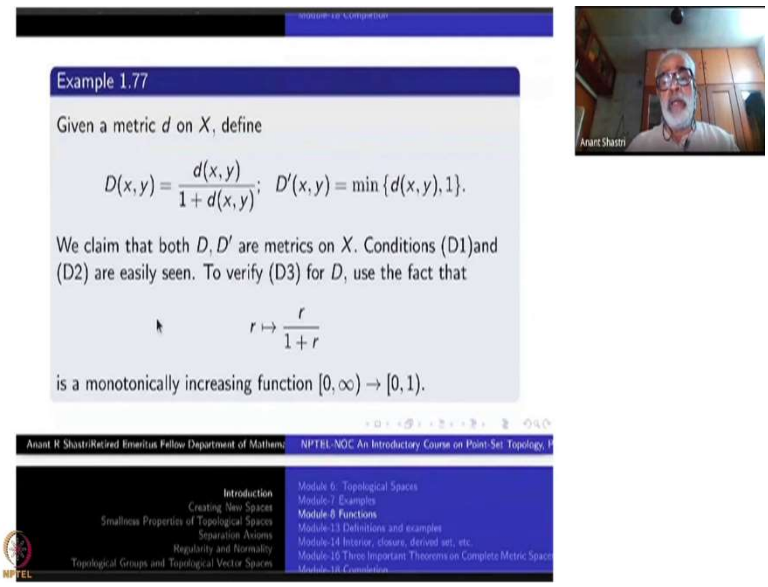

So, come here, start with a metric d on some non-empty set X ok. Now define another metric which I am denoting by capital D, $D(x, y)$ is equal to $d(x, y)/(1 + d(x, y))$ is non negative. So, $1 + d(x, y)$ is never 0, therefore, I can divide out by this.

So, the definition makes sense ok? The point is you have to verify that this is a metric. Once again condition (D1) is obvious because this is always non-negative and if it is 0 then the numerator must be 0; that means, x equal to y. So, that is fine. In (D2), you have to check for symmetry. If you interchange the x and y here this formula remains the same.

Because d is symmetric, D is also symmetric. So, (D1) and (D2) are easy. (D3) maybe a bit difficult ok. So, for this look at this function which is something like $r/(1+r)$. So, look at that the function. This is a familiar function to you ok? This function is a homeomorphism we have studied earlier from $(0,\infty)$ to $(0,1)$.

In fact, here I could have taken the domain to be $(-\infty,\infty)$ and then I must put a mod here and here $|r|/(1+|r|)$ to get into $(0,1)$. If I do not put a mod then I would get into $(-1,1)$. I am interested in the positive part therefore, I am taking $(0,\infty)$ to $(0,1)$ ok? So, this is not just an arbitrary homeomorphism, it is monotonically increasing function.

So, that you have to check ok? That fact will be used here now. The only property of this map, what is important is that it monotonically increasing function ok? Use this to prove that this capital D satisfies triangle inequality ok?

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A similar thing I am doing in another example. Here $D'(x,y)$ is equal to minimum of $d(x,y)$ and 1. I should not take maximum then it will not be a metric ok? if $d(x,y)$ is 0 then minimum will be also 0, and so, $D'(x,y)$ is 0. Conversive if $D'(x,y)$ is 0 this must the minimum of the two and hence $d(x,y)$ is 0 therefore x is equal to y.

Now, symmetry is obvious here. Once again for verifying the triangle inequality you have to work a little bit ok. For this one, you will have to do some adhoc method, namely look at all the points for which $d(x,y)$ is less than or equal to 1 that is one case. Then this 1 will not be there at all because I am all the time taking minimum of $d(x,y)$ and 1.

So, it will be always $d(x,y)$; $d(x,y), d(y,x), d(x,z)$. So, all of them are less than or equal to 1 then the formula for D' is same thing as formula for $d(x,y)$. So, triangle inequality satisfied. So, first case is obvious. If one of them is bigger namely $d(x,y)$ is bigger than 1 then you see what happens, OK? Irrespective of the value of $d(x,z)$. What happens?

So, you have to break the argument into subcases. As soon as $d(x,y)$ is bigger than 1 by the definition of D', it will be equal to 1 because this is bigger than 1, ok. So, what you will get is something like 1 is less than or equal to 1 plus something 1 is less than or equal to 1 plus 1 or something. So, you verify all these case by case.

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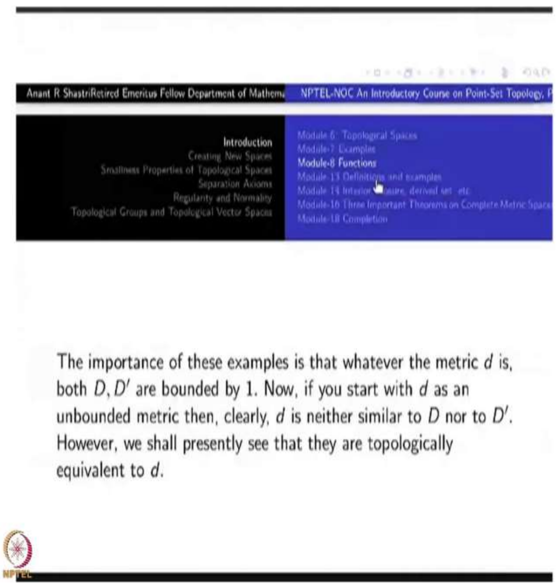

The importance of these examples is that whatever metric d is both D and D' are bounded by 1, you see. The value never goes beyond 1 ok? It will exceed 1 ok? This is bounded by 1. So, this property does not depend upon little d at all; capital D and D' are always bounded by 1 ok?

So, to each metric you can associate another metric in this method there are many many methods actually, but this has some charm. So, let us see what is that it is bounded by 1. Now, if you start with d as an unbounded metric then clearly d is neither similar to D nor to D' because if something is unbounded then similarity will preserve that.

So, that will be also unbounded right. Therefore, these two are not similar I mean D is not similar to little d, nor D' is. I can use one of them to produce perhaps a counter example now. These are not similar ok. My aim was to get a metric which will give the same topology as well.

So, the claim is that the topologies associated with d, D' and D are all the same, viz., $\mathcal{T}(d)$ is equal to $\mathcal{T}(D)$ equal to $\mathcal{T}(D')$ they are all the same on the underlying set X ok? Remember here the underlying set has not changed at all. So, perhaps we do not even need some function, the identity map itself will give you the homeomorphism,

which is the same as sying that the topologies are the same ok?

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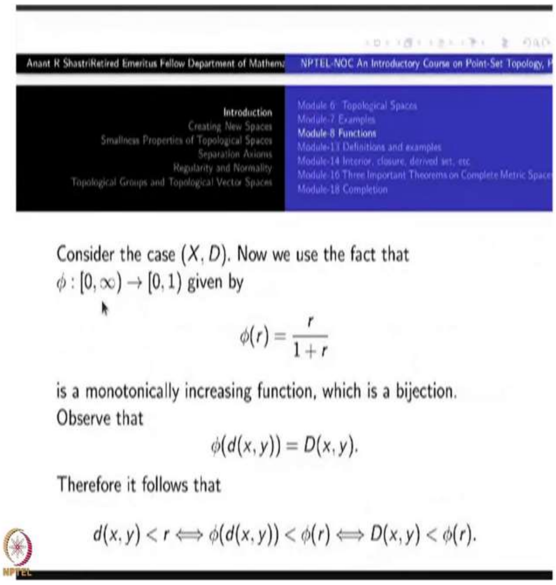

Let us see. So, how to see that? Consider first the case namely when I take this function ok $r/(1+r)$, $d(x,y)/(1+d(x,y)$, that is capital $D(x,y)$. Let us first look at that ok. Now, use the fact that again this $\phi(r)$ equal to $r/(1+r)$ is a monotonically increasing function, ok?

So, it is a bijection is all that you have to observe. $\phi(d(x,y))$ by definition is $D(x,y)$ this is what we have. If you replace r by $d(x,y)$ what you get is capital D ok. Therefore, it follows that $d(x,y)$ is less than r ok then ϕ of this will be less than $\phi(r)$ because ϕ is monotonically increasing ok. Now, $\phi(d(x,y))$ less than $\phi(r)$ or $\phi(d(x,y))$ is capital $D(x,y)$, right.

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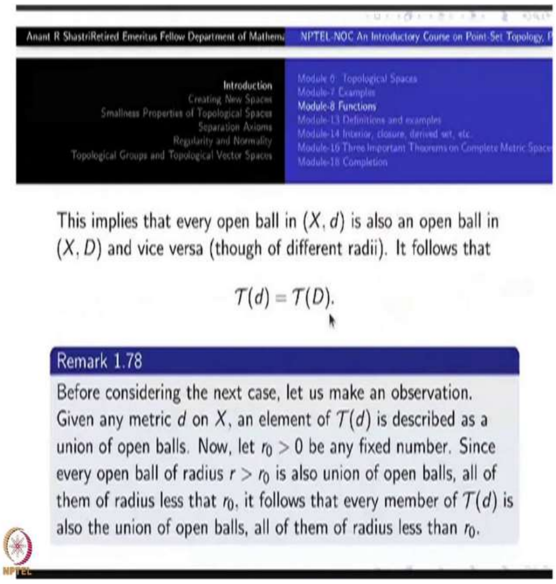

So, capital $D(x, y)$ is less than equal to r and vice versa is going back and forth. This implies that every open ball in X with respect to little d here is also an open ball in X with respect to capital D. Only thing is that radius has changed from r to $\phi(r)$ ok? And vice versa. Take an open ball of radius s in D metric, it will an open ball of radius ϕ inverse of s in d metric ok?

So, every open ball here is also an ball there, the set of open balls on there side is the same. Of course, their radii has changed, but the entire collection is the same. Therefore, when you take arbitrary union etc to produce the topology on one side you can do the same on the other side also. So, for each open set here there would be the open set there ok. The sets $\mathcal{T}(d)$ and $\mathcal{T}(D)$ are the same ok. So, the topology is the same alright.

Now look at D'. For D' you need a different argument altogether, which is easier and hence left to you as an exercise with the follwing remark as a hint. Let me make one of the points very clear to you about the topology $\mathcal{T}(d)$, itself ok? how the topology $\mathcal{T}(d)$ is defined. Given any metric d on X an element of $\mathcal{T}(d)$ is described as a union of open balls in the metric d right? Now let r_0 bigger than equal to 0 be any fixed number ok. Note that every open ball of radius bigger than r_0 is also a union of open balls all of them of radius less than r_0. If you take ball of radius 1 for example, take any point x, inside that, I can find a ball of radius something less than 1 with at x. By taking all such open balls, the entire union is equal to the origin ball.

Similarly, for any r bigger than r_0 ok, I can take balls of radius smaller than r_0 smaller and smaller balls, I do not want to take anything bigger that r_0, it is possible to cover the original ball of radius r, ok? Therefore, it follows that every member of $\mathcal{T}(d)$ is also a union of open balls all of them of radius less than r_0. You do not have to take any bigger balls at all smaller things you cannot control ok is that clear?

Student: Sir, can you please repeat once.

Professor: Let us say for example, r_0 is 1 ok? or let us say it is half, maybe 1 or maybe $1/2$. Can you write the entire real line as a union of intervals of length less than half?

Student: Yes.

Professor: Can you take any open interval now, not the whole of $\mathbb{R}$ ok? Can you write it also as a union of intervals of length less than half?

Student: Yes.

Professor: Yeah. So, now, question is: you take any open set not necessarily an interval, can you do the same thing? I do not want you to take any intervals of lenght less than r. So, it is like a measurement you know your scale is only a 6 inch long; you are not taking one yard but you have to measure all the way from here to say Delhi is it possible or not? I am asking?

Student: Yes.

Professor: Yes. That is precisely what it is happening here. Suppose your measurement is total measurement is something 5, but what you have is say 10 foot roller. Can you measure that smaller one it may be 5, it may be 4, you do not know right?

Student: No.

Professor: But the otherway is always possible that is the only thing that is ignored here ok. So, take any r_0 ok any fixed number. Less than that I can always take and all the things which are less than that can be measured for all the balls now. See your r_0 maybe too large say it is 50, but your metric space itself is bounded by 1, then is there a contradiction? I can take anything less than 50 means I can take less than 1 also right, I can take half I can take one-third ...

So, all those smaller numbers are always there, there is no restriction there right. It should be less than r_0, all of them balls which are of radius less than r naught ok. Each ball which you take must be of radius less than r_0, never take any ball bigger than r_0 or equal to r_0 even. Can you write any ball, first of all as a union of such things is a question ok?

Student: Yes.

Professor: Once you find one balls contained inside another ball, center is the same. Center whatever you wanted to choose, you have chosen ok original ball center is different. You take a point and that is the center now inside that ball, you have find an open ball inside that one ok. After that you can take smaller smaller smaller balls. So, I want it to be smaller than r_0. So, you take the minimum also ok.

Student: Yes.

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Yeah. So, now, you use that here all that you have to do. So, I have put D' as minimum of $d(x, y)$ and 1 right. Suppose I take only balls of radius less than 1. Then whether I take D' or d it is the same thing. Look at a ball of radius less than 1 in the metric d. When you measure the distance all of them will be less than 1, right.

Therefore, D' will be also equal to d. The same thing because D' is a minimum of $d(x, y)$ and 1, ok. So, you put this r_0 here equal to 1 here then what we get is any ball here ok whatever you have small ball you have taken they are sufficient here also and vice versa. So, $\mathcal{T}(d)$ will be equal to $\mathcal{T}(D')$ ok.

In this part no ball will be of radius bigger than 1. Here there are. So, that is why you start with bigger balls here it does not matter ok, but then first thing is you write them as union of balls of radius less than 1. After that they are the same thing as this one alright. So, the two topologies are the same ok.

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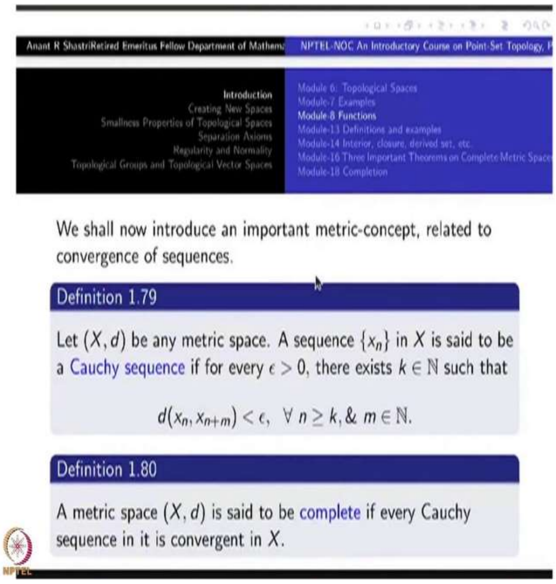

Now, we shall introduce an important metric concept related to convergence of sequences. Once again I say I am introducing, but these are all in elementary analysis ok. Copied down; definition is copy down except the modulus is replaced by d that is all go back and check your definitions, here is the same thing ok.

Let (X, d) be a metric space. A sequence is said to be a Cauchy sequence if for every $\epsilon > 0$, there exists a natural number k such that if I take $n \geq k, m$ could be any arbitrary; I have written like this x_n, and x_{n+m}. So, both of them are bigger than equal to k that is all, you can write it as say x_i, x_j, i and j both bigger than equal to k, then the distance between such points inside a sequence must be less than ϵ.

So, that is the definition of a Cauchy sequence ok. A metric space (X, d) is said to be complete if every Cauchy sequence in it is convergent. And elementary analysis says that every convergent sequence is always Cauchy, but a Cauchy sequence may not be convergent very easy examples. Take a we take a sequence, which convergent to a point and remove that point, in the resulting space the Cauchy sequence will remain, but the convergence point you have removed, therefore, it is no longer convergent, ok?

That is the way you can produce Cauchy sequences in $\mathbb{R}$ minus any single point inside $\mathbb{R}$. In side $\mathbb{R}$ itself, you cannot because every Cauchy sequence in $\mathbb{R}$ is convergent. So, this you must have studied namely completeness of $\mathbb{R}$. So, I am going to use that one here. I am not going to prove that $\mathbb{R}$ is complete here right now ok. So, if (X, d) is a complete metric space then every Cauchy sequence is convergent ok.

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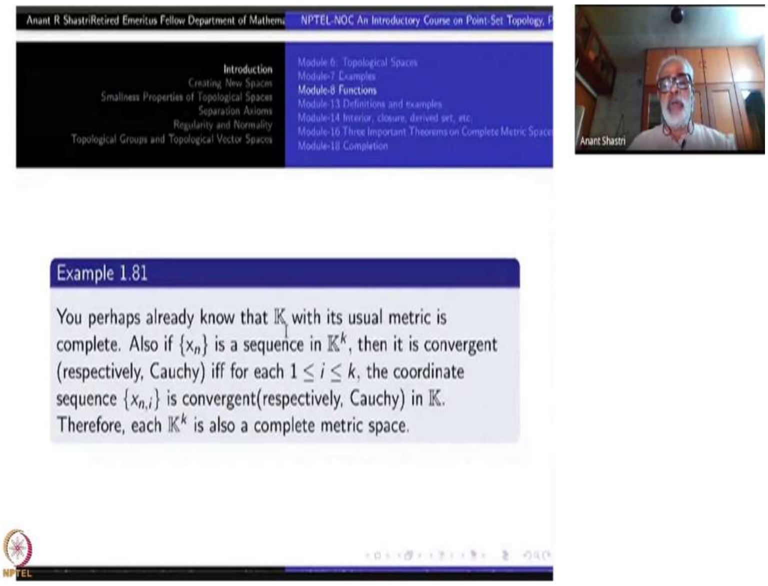

So, I am assuming that you know that $\mathbb{K}$, which is either $\mathbb{R}$ or $\mathbb{C}$ with the usual metric namely our modulus metric, then it is complete. So, you can take $\mathbb{K}^k$, $\mathbb{K}$ copies of k; $\mathbb{K} \times \mathbb{K} \times \ldots \mathbb{K}$. When is a sequence convergent? or respectively Cauchy?--- if and only if each coordinate sequence $x_{n,i}$, ok I am writing $x_{n,i}$ for each $i = 1, 2, ;k$, $x_{n,1}, x_{n,2}, \ldots x_{n,k}$ there are k coordinate sequences here each of them must be Cauchy. Then x_n will be Cauchy. If x_n is convergent each of them will be convergent and conversely alright? So, this much we have seen already actually. So, therefore, each $\mathbb{K}^k$ is also complete metric space, alright.

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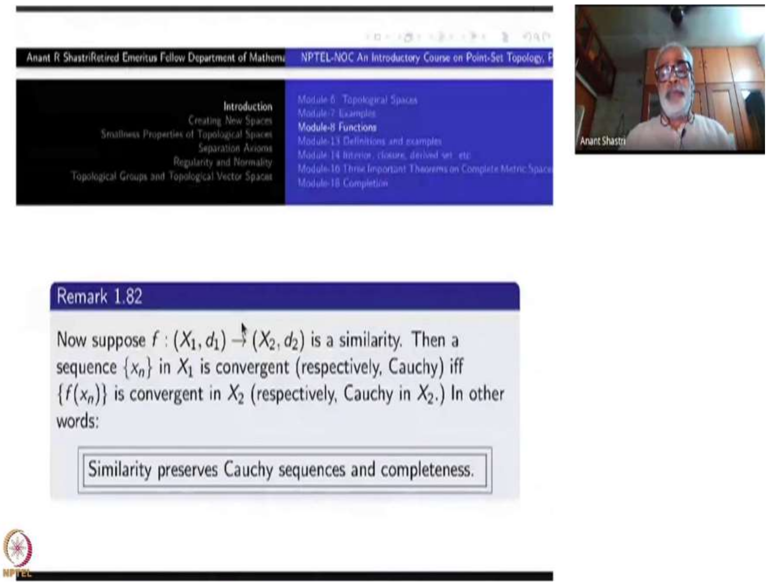

Now, comes the point namely see our title for today's talk is the study of similarities and isometries and so on right? Suppose you have a similarity f between $(X_1, d_1), (X_2, d_2)$. Then a sequence x_n in a X_1 is convergent or Cauchy if and only if $f(x_n)$ is convergent in X_2 or Cauchy respectively. Cauchy implies $f(x_n)$ is Cauchy, convergent imply $f(x_n)$ is convergent.

In fact, if x_n converges to x, then $f(x_n)$ will converge to $f(x)$. ok? So, these things are elementary ok. So, you should remember it like this. Similarity preserves Cauchy sequence and completeness. You do not need the strong hypothesis namely isometry, isometry will also preserve because any isometry is a similarity also, ok?

So, similarity preserves these things alright. So, next question is whether homeomorphism will preserve this one right? We are going to see that that is not the case. So, these things are metric dependent, these notions are metric dependent, ok?

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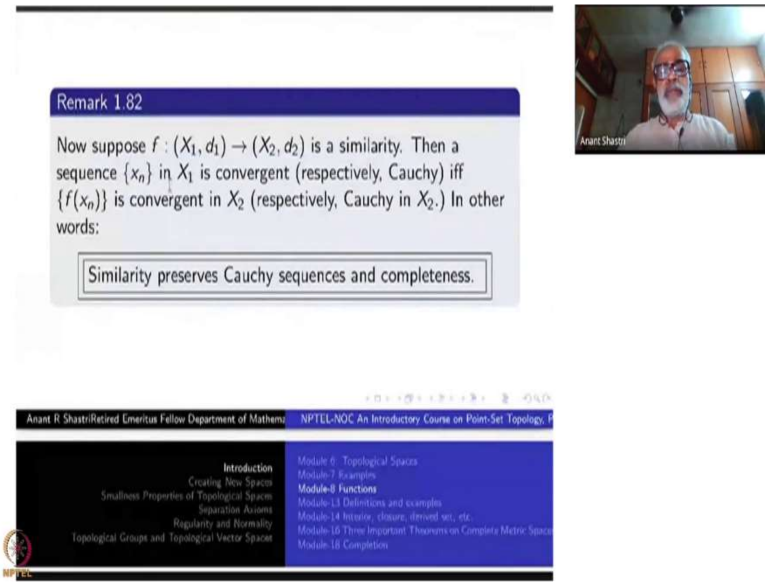

So, how to verify such a thing? If and only if one is one is convergence another is Cauchy there is if and only if there are four statements here right? I mean four sub statements of one single statement you can break it into four different thing right. Yes or no? See there is if and only if part. So, once x_n convergence implies $f(x_n)$ is convergent and next $f(x_n)$ convergent implies x_n is convergent; I have to prove.

Similarly this one Cauchy should imply that one Cauchy and vice versa right? But look at the similarity. If f is a similarity f inverse is also similarity. Therefore, if I prove one way here then the other way also gets proved right?

Therefore, instead of four statement you have to prove only two statements, but even those two statements are very much similar here. All that you have to do is for every ϵ there exist some k blah blah blah exactly similar things you have to do, in both cases. Therefore, I will prove one of them. Say I will prove that x_n is Cauchy implies $f(x_n)$ is Cauchy ok? Just to remind you what are this Cauchy is and so, on ok.

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So, instead of four statements I will prove one of them x_n Cauchy implies $f(x_n)$ Cauchy. So, recall that f is a symmetry, f is a similarity implies this one half condition I am taking namely, $d_2(f(x), f(y))$ is less than or equal to some positive constant $c_2 d_1(x, y)$; this was the definition. Actually there is another inequality here $d_1(x, y), c_1 d_1(x, y)$ is less than or equal to this one is another part, which I do not need now. That can be used for f inverse ok, you use this one ok?

Now, I am assuming that this x_n converges right. So, given ϵ choose a k such that $d_1(x_n, x_{n+1})$ is less than instead of ϵ put ϵ/c_2, that the c_2 factor is there divided by c_2. So, I am assuming x_n is a cauchy sequence, not necessarily convergent. Suppose x_n is a Cauchy sequence right? Cauchy implies this one a Cauchy.

Then that is this a condition you get; where is $n?n$ is greater than or equal to k there is such a k and m is any number, does not matter. Now put this x equal to x_n and y equal to x_{n+m} here that c_2 cancels out. What you get? $d_2(f(x_n, f(y_{n+m}))$ that is what I am going to put here $f(x_n), f(x_{n+m})$, instead of $f(x)$ and $f(y), y$ equal to x_{n+m} right the distance is less than or equal to the c_2 cancels, so is less than epsilon ok. This whole thing is less than ϵ.

So, other three things also you can write down. Write it down for your practice, if I keep doing it you will not get a practice, you have to do that.

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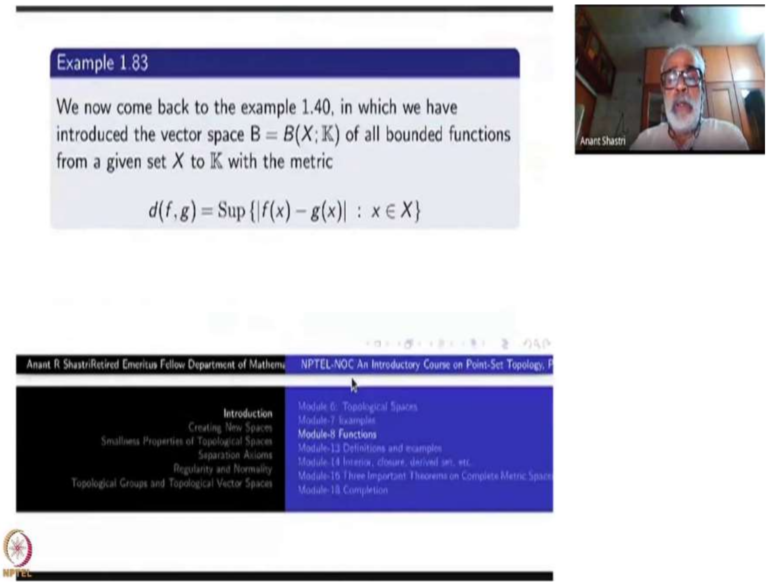

Now, I will take this opportunity to complete one of the little more knowledge about one of our you know favorite example namely take X to be any set and take all functions from X to $\mathbb{K}$. So, that was vector space that was a ring and so on, inside that take the set of all bounded functions that was denoted by B.

On B we had a metric namely $d(f, g)$ is equal to supremum of $|f(x) - g(x)|$. Since each of f and g is assumed to be bounded their difference should be also bounded. So, this supremum make sense and we have verified that this is a metric ok.

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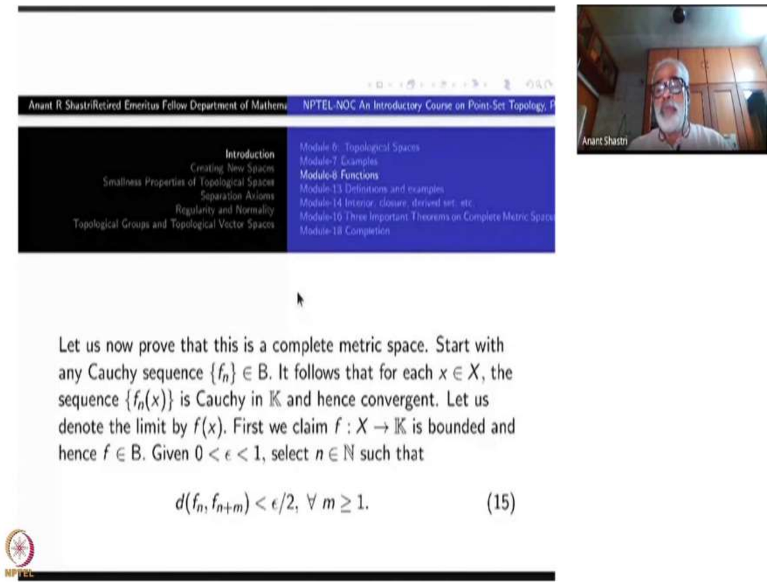

So, look at this metric space, what I want to prove that is that this metric is a complete metric space, every Cauchy sequence here converges. So, this you might not have seen. So, pay attention and see that it is not at all difficult, it uses the completeness of $\mathbb{K}$ and the supremum of metric ok? Convergence with supremum metric. You can talk about this as if it is a uniform convergence ok. So, that is it. That is the key here ok. So, start with a Cauchy sequence f_n inside B, these are functions which are bounded, that is the meaning and a sequence is Cauchy. For each fixed x belonging to X, just one x then look at the sequence $f_n(x)$, this will be also Cauchy. Why? Because when f_n is Cauchy inside B, it is distance between this distance between f and g, the supremum is less than something right; $f_n(x)$ for all x supremum is less than ϵ. So, each of them will be also less than that ϵ that is all.

When the supremum is bounded the each point is also bounded that is what it is here. So, each of them is a Cauchy sequence ok. So, where are the Cauchy sequences. They are inside $\mathbb{K}$ because this f_n's are $\mathbb{K}$-valued functions right. Therefore, they are convergent for each x, you get a convergent sequence. Let us denote the limit by $f(x)$. So, this way I have already cooked up a function ok.

Remember a Cauchy sequence if it is convergent any convergent sequence, there is only a unique limit point ok. So, I can call that as $f(x)$ that is happening inside $\mathbb{K}$ ok. So, first we claim that this f itself is bounded. See we do not know whether this function is inside B, for that I have to show that it is bounded, so that it is inside B ok? How do you show it is bounded? Take any ϵ less than $1, 0 < \epsilon < 1$.

According to this Cauchy condition there will some n such that $d(f_n, f_{n+m})$ is less than ϵ. So, instead of ϵ, I made it $\epsilon/2$ now and I have cleverly chosen ϵ less than 1, any ϵ may do perhaps ok does not matter ok? You have got this thing right. Now, select one such n and let M positive be such that f_n is bounded by M, supremum of f_n is M. So, $|f_n(x)|$ for all x is less than equal to M, ok.

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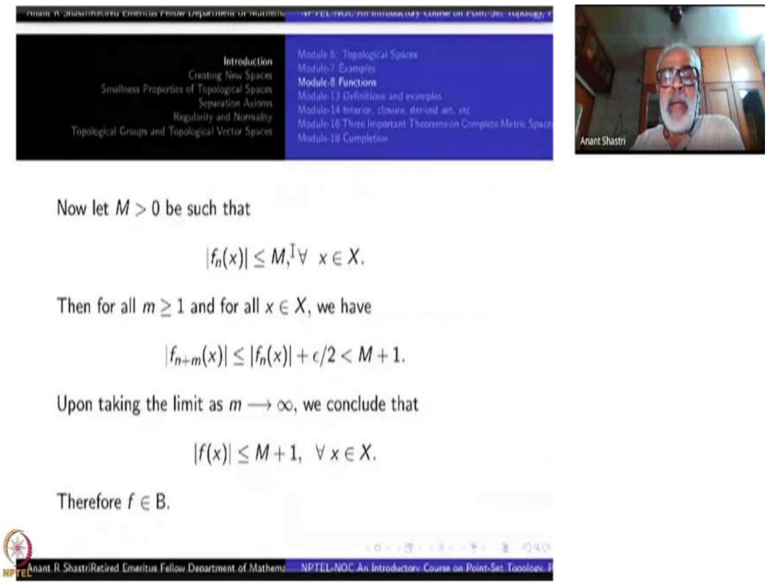

Now, what happens? Distance between $f_n(x)$ and $f_{n+m}(x)$ is less than $\epsilon/2$, right? So, you have bounded set and then these points are away from the bounded set by another $\epsilon/2$, right. By triangle inequality, $|f_{n+m}(x)|$ is less than or equal to $|f_n(x)| + \epsilon/2$, which is less than $M + 1$.

So, here it is modulus will usual metric on mod k ok. So, this is less than $M + 1$. See would have been any epsilon fixed so it does not matter all that I wanted is bounded ok. So, this is true for all $m \geq n$. So, this is a sequence which is bounded by this and it converges. What happens when you take the limit of this as m tends to infinity? The same thing as limit of $f_n(x)$ to $f(x)$.

So, $f(x)$ is less than equal to $M+1$. So, I have shown that whatever x is ok this this M was chosen independent of x right, for all x, this is true. Therefore, this is true for all x. So, f is bounded alright.

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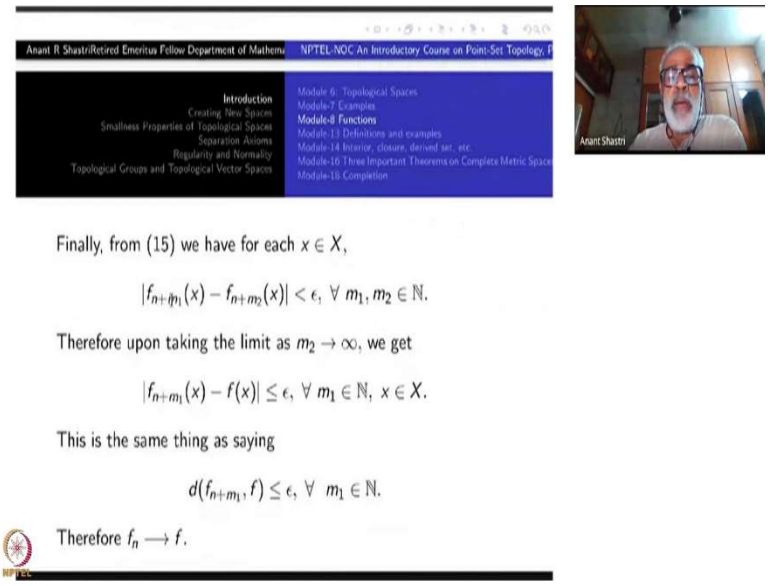

Now, we want to show that this f is the limit of the sequence f n ok? So, look at this look at this equation, which we have already got. We can keep using this one again and again. You do not have to write one more ok. Modulus of $f_{n+m_1}(x) - f_{n+m_2}(x)$ is less than ϵ. So, by triangle inequalities will be less than ϵ now for every m_1 and m_2 bigger than N. Because they are at distance from $f_n(x)$ at $\epsilon/2$, ok, for that reason I have chosen this $\epsilon/2$ there. So, this difference by triangle inequality less than ϵ.

Now, take the limit as m_2 tends to infinity, keep m_1 as it is ok. m_1 is some number here m_2 some other number they are all some positive integers that is all. Take the limit as m_2 tends to infinity, this becomes $f(x)$. So, modulus of $f_{n+m_1} - f$ is less than ot equal to ϵ. Here is strictly less than ϵ, I have to put ϵ less than or equal to ϵ here. ok?

So, this happens for every m_1 ok for every m_1 and every x inside X. So, this is the same thing as saying that the supremum of this is less than equal to ϵ. What that mean? Supremum means distance between f_{n+m} and f ok. So, this is precisely the meaning of that this sequence f_n converges to f ok. So, you do not need the first n terms here you see.

If the sequence converges after n terms ok, you truncate that one that is enough. This is same thing as the whole sequence converges to f.

So, we have proved that the this space B is a Banach space yeah, complete non-linear space is a Banach space ok that is a name for that celebrating name for that one.

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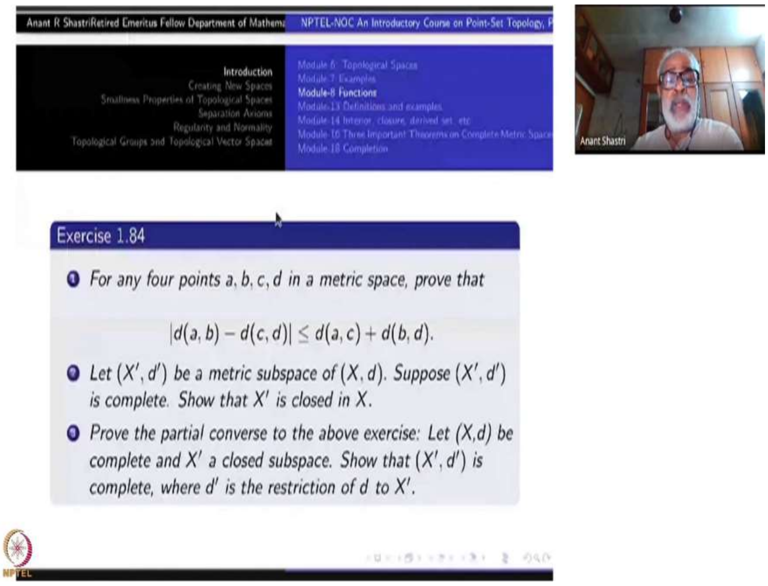

So, I want to do something more about this one, I will do later on, but right now we will stop here and just look at sequence of of exercises here for you ok.

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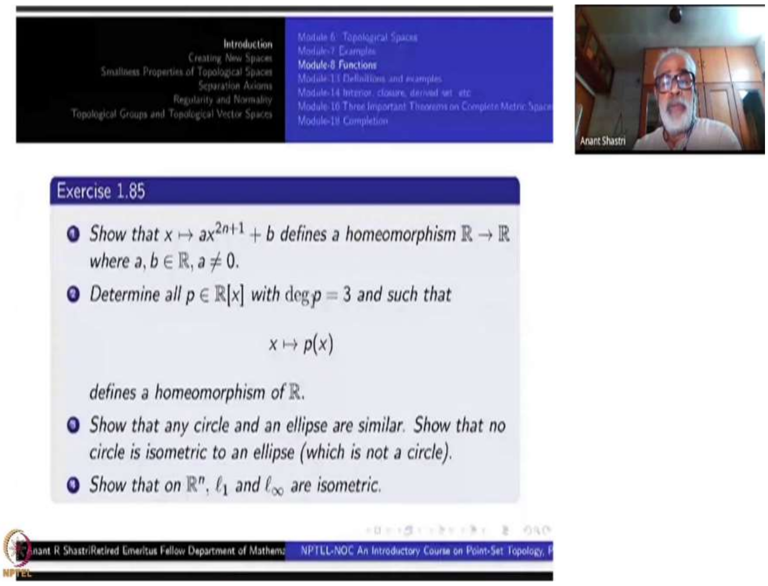

So, one of the exercise here is to take a polynomial in $\mathbb{R}[x]$ ok of degree 3, real coefficients and one variable. Evaluate at each x, x going to $p(x)$ that gives you map from $\mathbb{R}$ to $\mathbb{R}$. You have to show that it is a homeomorphism. From $\mathbb{R}$ to $\mathbb{R}$, what is the homeomorphism? We have a criterion there.

It must be continuous, it must be on-to and it is monotonically increasing strictly monotonically increasing that is enough ok that is the hint ok. So, nothing here is difficult if you think properly a little bit and use whatever little knowledge you have already ok.

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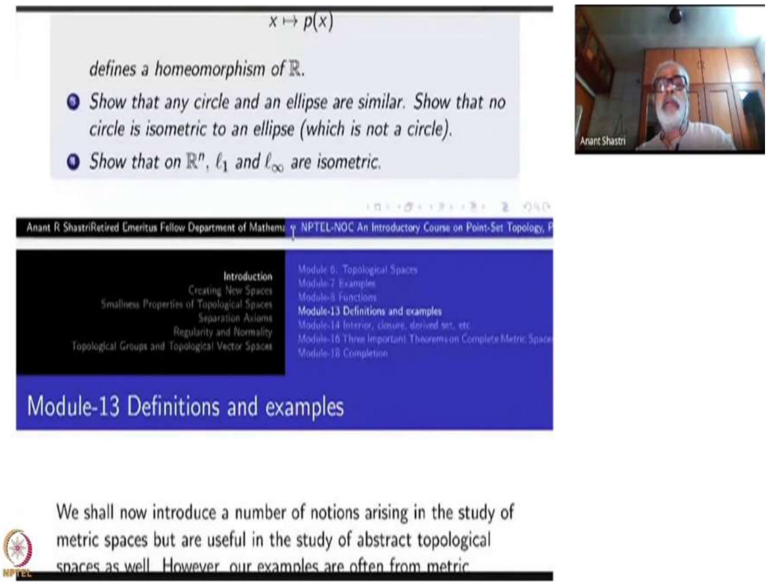
$x \mapsto p(x)$
defines a homeomorphism of $\mathbb{R}$.
Show that any circle and an ellipse are similar. Show that no circle is isometric to an ellipse (which is not a circle).
Show that on $\mathbb{R}^n$, ℓ_1 and ℓ_∞ are isometric.
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Introduction
Creating New Spaces
Smallness Properties of Topological Spaces
Separation Axioms
Regularity and Normality
Topological Groups and Topological Vector Spaces
Module-13 Definitions and examples
Module-13 Definitions and examples
We shall now introduce a number of notions arising in the study of metric spaces but are useful in the study of abstract topological
Anant Shastri

Thank you.