Introduction to Point Set Topology, (Part I)
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Module - 11
Lecture - 11
Various notions of equivalences Continued

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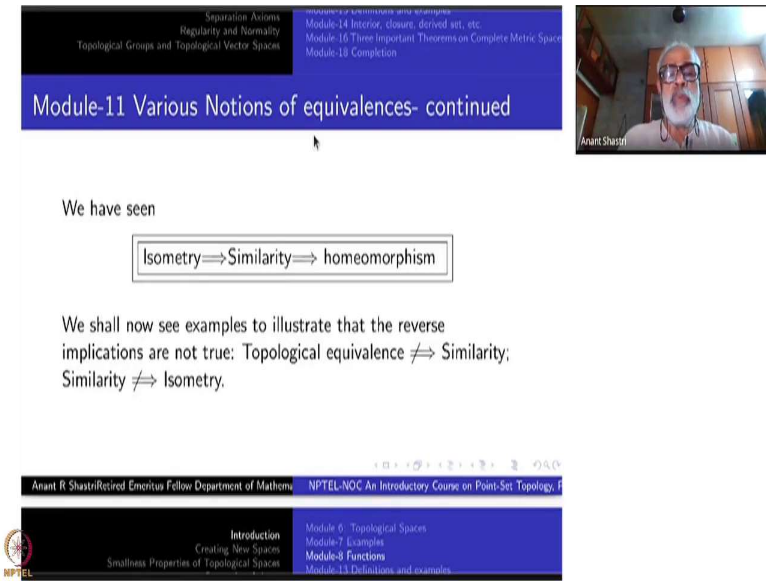

Welcome to module 11 of Point Set Topology course. We have seen, we have introduced the notion of Equivalences amongst metric spaces and topological spaces, Isometry, Similarity and Homeomorphism and you have also seen that isometry implies similarity, similarity implies homeomorphism.

So, today we shall discuss the topic of reversing implications here. Does homeomorphism imply similarity, does similarity imply isometry, the guess is that answer is in the negative and we will see. So, for producing negative answers, you have to produce examples. So, this way the concept of all these isometry, similarity and homeomorphism may be little more clearer to us. So, that is the idea.

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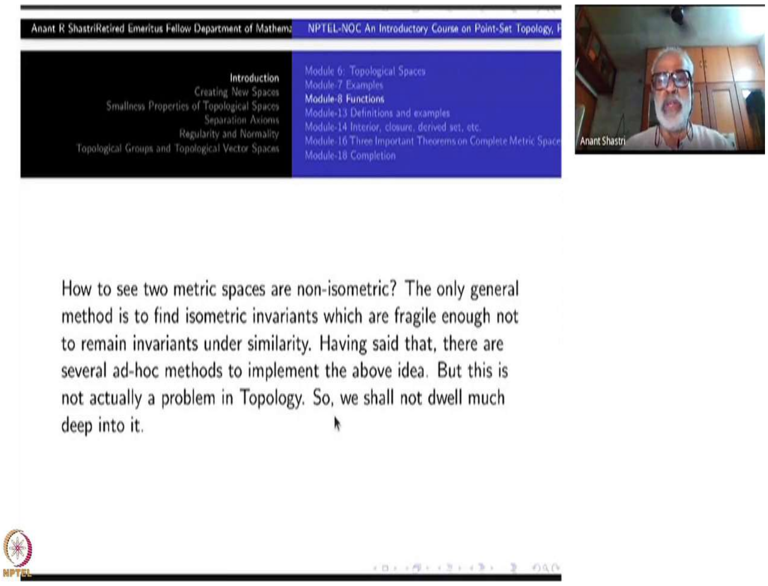

How to see that two metric spaces are non isometric? You know producing an isometry may be very difficult, but how to say given two spaces are non isometric? The only general method is to find some of so called isometric invariants such that one of the space possesses this inavariant and the other space does not ok?

But what we want is, two similar metric spaces which are not isometric. Two spaces, two matrics should be similar. So, if your invariant is preserved under similarity also then it will of no use. So, the invariant should be fragile enough so that when you look at similarity it is not preserved. It is violated ok. So, such a thing we have to hunt around.

So, having said that, there are several ad hoc methods to implement the above idea depending upon the nature of the two spaces you have in front of you ok. This method may not work, that one may work, something which appeals to you, you may just try it out, so that is the kind of approach we have because to begin with, you are not given the two spaces.

So, here are two similar spaces, are they isometric is the question. You do not know. So, that for that you have to understand both the spaces properly, so this is one thing which happens here, and the same does not in the other. This is the way you have to go around ok. So, that is why it is an ad hoc method. You have to quite often do that.

In any case, obviously, we are more concentrating on topology. So, we will not go deeper into this aspect ok? That will be taken up by differential geometers or more surely geometrically minded people and so on ok. So, we will have only few easy example.

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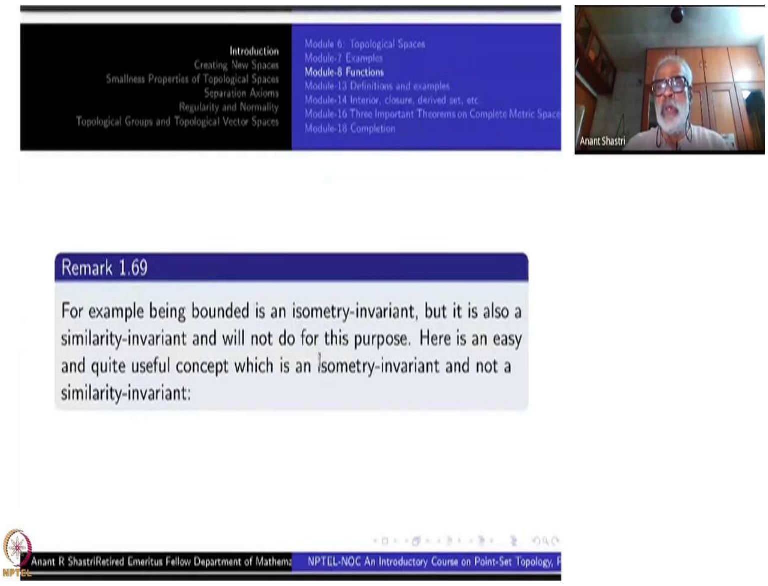

So, let us look this example, the property of being bounded. We see have seen this is an isometric invariant ok? Suppose one metric space is bounded. On the same topological space, on the same set you may have another metric which is not bounded. Then immediately you can say that these two are not isometric, but can they be similar also? maybe they can be similar also. So, that is the whole idea ok?

So, being bounded is an isometry invariant, but it is also a similarity invariant ok. So, this will not work ok. Here is an example, easy way, quite useful concept also. This is an isometric invariant, but not a similarity invariant. So, I am trying to cook up something like that OK?

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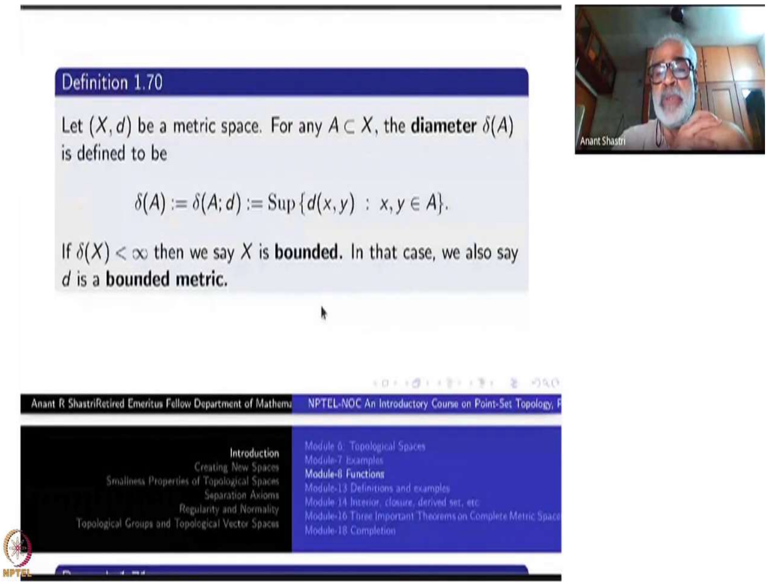

So, this is the new concept that we have here which is very canonical and very geometrical in nature. Take a metric space. Take any subset, can be empty also do not worry about that. The diameter of that subset A is defined as follows: I have two notations here, $\delta(A)$ wherein I have ignored the metric, but if the metric has to be mentioned when there are two different ones then the that has to mentioned, then you have use the other notation $\delta(A, d)$, ok?

This is nothing but look at all the numbers, distance between x and y where x and y are arbitrary points of A and take the supremum ok? This supremum may be infinite, I do not care. If A is empty, this will be minus infinity also, makes sense. Supremum of any set of real number if we include $(-\infty, \infty)$, it makes sense ok.

If this set is bounded above, then the real number $\delta(X)$ is finite ok? In that case, we say X is bounded ok? including the case $\delta(X)$ equal $-\infty$. That will not happen because our metric spaces are usually non empty. If you take A to be empty, $\delta(A)$ may be $-\infty$. Empty set is bounded, that is ok no problem.

So, $\delta(X)$ finite, we say X is bounded, in that case we also say that the metric itself is bounded ok because I have taken the entire X here ok. The metric d may not be bounded, but a subset may be bounded, like any finite set you know is bounded ok. So, all these concepts are there in $\mathbb{R}^n$ also, in $\mathbb{R}$ also. So, it is nothing new as such. So, we were just testing the concept by just looking at a definition, supremum of $d(x, y)$ where x and y range over all points of A ok.

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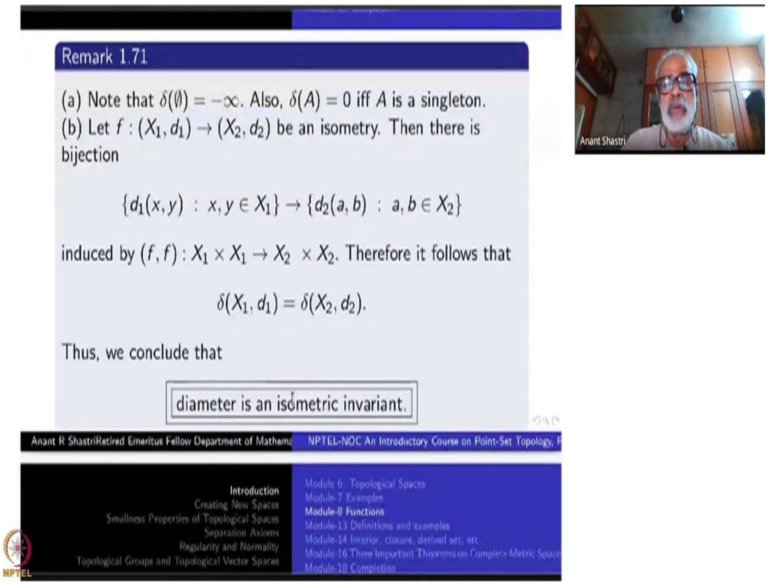

So, I am making this remark here, δ of emptyset is $-\infty$ and $\delta(A)$ is 0, if and only if A is a singleton. As soon as there are two points, the distance between them we know is positive. So, supremum will be positive ok. Only if A is a singleton, then $\delta(A)$ is, is 0 ok.

So, all these things may not be very useful, but it will make the definition clear alright. Now look at an isometry from (X_1, d_1) to (X_2, d_2) ok, then look at all the real number $d_1(x, y)$ where x and y range over X_1 and now on the other side $d_2(a, b)$ were a and b range over X_2.

So, these two are two sets of real numbers. I want to say that there is a bijection between them induced by this isometry f. viz. $f \times f$ from $X_1 \times X_1$ to $X_2 \times X_2$ ok, namely put a equal to $f(x)$ and b equal to $f(y)$, that will give for each point $d_1(x, y)$, $d_2(f(x), f(y))$ which is equal to $d_1(x, y)$. So, the same number will come here.

So, if we can use the reverse: f inverse and you go here. So, what happens is, this set as totally, totality on this set is equal to totality of this set, therefore, supremum over this one is same thing as supremum over this because there sets are same, but supremum on this one is $\delta(X_1, d_1)$ and here it is $\delta(X_2, d_2)$.

So, what we have prove is that under isometry the diameter is preserved ok. Diameter is an isometric invariant ok.

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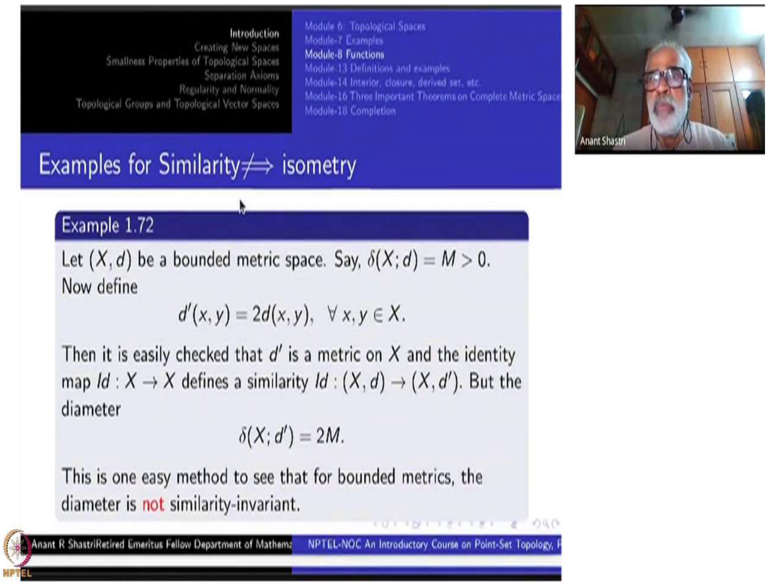

Now we can make this one, we can we use this one to show that similarity does not imply isometry, not just boundedness, but more fragile thing namely diameter right? (X, d) be a bounded metric, now start with a bounded metric so that diameter is finite, that is a definition right.

So, put $\delta(X, d)$ equal to M ok. I am putting this positive because I do not want take X as an nonempty, an empty set that is all. Now define $d'(x, y)$ equal to $2d(x, y)$. So, it is definition of d' again on $X \times X$, ok.

For every x, y belongs to X we can easily check that this is also a metric. So, I am producing another metric by just multiplying by 2. You could have taken any real number here nonzero that is all instead of 2. So, this d' is also a metric. Moreover look at the identity map, that defines a similarity now ok.

So, that is the similarity relation right and there is no further function it is identity function itself. So, identity is a similarity between (X, d) and (X, d'), but the diameter with respective to d' will be exactly $2M$ ok. For each number here in the definition of this one, there will be a twice that number in the other one here, the other one right.

So, there are two set like this. So, supremum of this one be twice the supremum of twice supremum of this one will be twice supremum of this one. So, that is what is $\delta(X, d')$ here is $2M$, ok. I could have put any number r not equal to 1, not equal to 0 of course, then also it would have worked.

So, this was an easy method you see, easy method to see that similarity does not imply isometry. Only thing is I have assumed is that the metric is bounded, without which the this finite number does not makes sense then even multiplying by infinity goes to infinity there is no contradiction.

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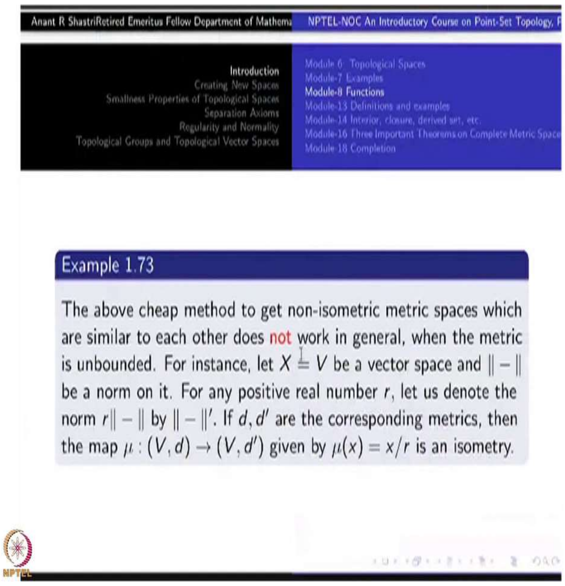

So, the above cheap method to get non isometric metric spaces which are similar to each other does not work in general. Namely when the metric is unbounded. For instance look at a vector space V ok and take any norm on it, for any positive real number r just like we did it for 2 here, you can take r times the norm. You can take 2 times the norm alright by and call it as norm prime.

So, you have two different matrices here d and d', corresponding metric ok, then look at the vector space V to V ok here, here I have put d, here I have put d', here the norm here is the original norm, here you see the norm prime which is r times the to a norm.

Now, you take the function $\mu(x)$ equal to x/r ok? All that you need is r must be again as usual not 0 not equal to 1 then it will not be much of his choice here ok. For here I should have taken r not equal to 0 that is all, r equal to 1, it is a same, they are the same. So, not equal to 1 is obvious choice ok?

You can define like this, there is no problem, but if you want to get something more, like an isometry when you come here, you have to put that condition ok. So, this is an isometry ok? you have check that. So, norm of x is suppose its 1 here, in the second norm, the norm prime the same vector would be r times that, but I am taking x/r. So, r and cancels out. So, it will be how much.

So, I am just giving an example, but that is not needed, just norm of x here will become x by a norm, x it could be equal to that one because because there is an r here in the denominator, r here in the numerator, that is alright ok.

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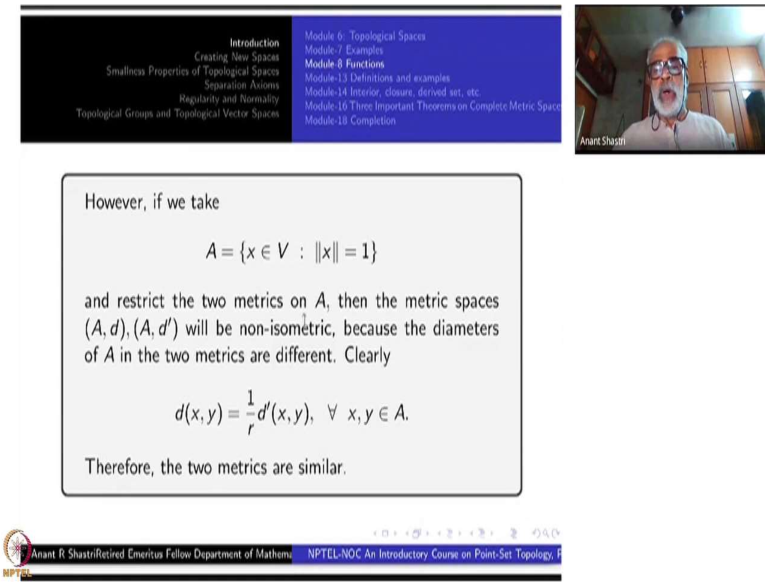

So, multiplication by some number may not produce the necessary non isometry in the case of an entire vector space ok? In the same thing you restrict take subset V any finite subset may do, but take something nice, namely all points is set norm of x equal to 1. Lets say unit here ok and look at the two matrics on this set. I am taking the first norm here ok. So, this set now you take the two different matrics and restrict it to this A ok.

When you restrict a metric to a subset, that is another metric space, that is what we have seen. So, then the metric spaces (A, d), and (A, d') will not be isometric, why? Because the diameter of the first one is 1, whereas the diameter of the other one is r, ok.

Clearly the two metrics are similar by the very definition, every x, y inside A, we have $d'(x, y)$ is equal to $rd(x, y)$. r will be playing the role of c_1 and c_2 both. So, only when you go to the whole V, it does not work, but this words for all nice things namely bounded things.

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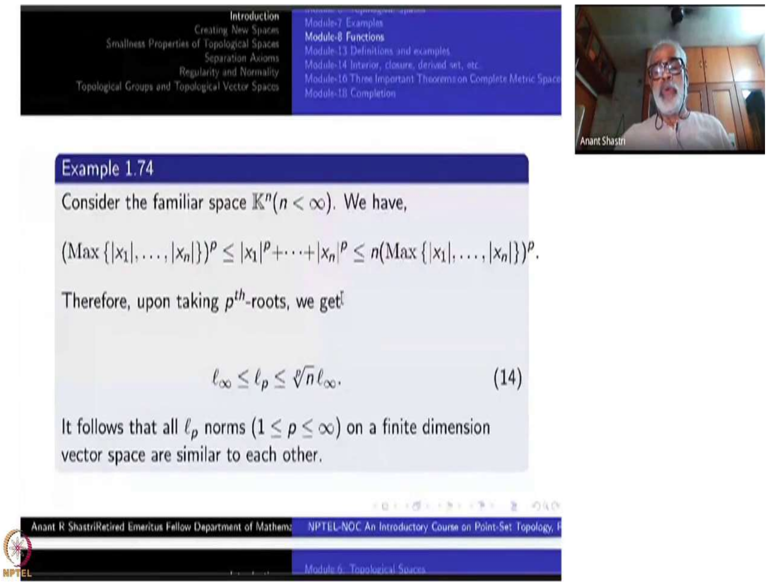

Now, let us come to again our favorite examples namely $\mathbb{R}^n$ and $\mathbb{C}^n$, so, $\mathbb{K}^n$ ok. Start with this elementary inequality, which says that if you take $|x_1|, |x_2|, \ldots, |x_n|$ take the maximum and then take p^{th} power, it is less than or equal to $|x_1|^p, |x_2|^p, \ldots, |x_n|^p$, all added together ok? Why? Because there are n of them.

If maximum is one of them, say $|x_1|$, that will be already there, here because all n of them are there and the rest of them are all non negative. So, if you add something non negative that it should bigger then. So, this is less than equal to this one ok. But again if you take maximum of this ok repeat it n times that will be definitely bigger than this one.

First you can put this p inside and then go outside they are the same. That is first you take the maximum and then take the power p or first you take the power and then take the maximum, they are the same. So, this is an elementary in equality. Take the p^{th} root, what do you get here? You get the l_∞ norm; what do you get here, you get any times l_∞ norm, sorry p^{th} root of n times l_∞ norm. In between what you get? You get the ℓ_p norm ok.

Therefore you get l_∞ is less than equal to ℓ_p, less than equal to p^{th} root of n times l_∞, ok? So, this shows ... what does it show? This shows that all ℓ_p are similar to l_∞. That is a similarity relation. Each ℓ_p norm is trapped, between two non zero multiples of this norm l_∞. So, they are similar to each other ok?

So, all of them are, what are these numbers p? $1 \leq p \leq \infty$ of course, infinity is already there. Take anything other than that they are also similar to l_∞, similarity is an equivalence relation. So, all the ℓ_p norms on a finite dimensional vector space, (that must be $\mathbb{K}^n$), they are all equivalent to each other ok. So, very simple idea here namely this inequality you have to use.

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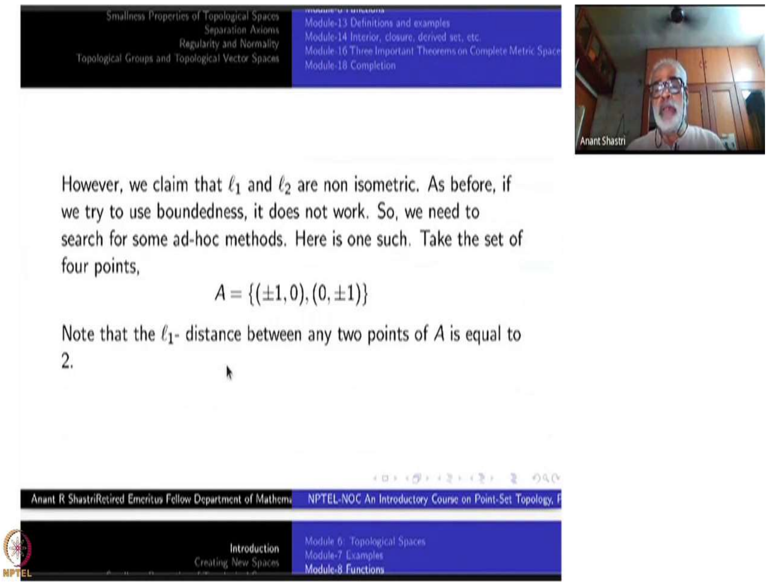

However, we claim that ℓ_1 and ℓ_2 are non isometric. This will be our second example of similarity not implying isometry. So, this is also important to know that ℓ_1 and ℓ_2 are not isometric to each other, but they are similar ok? Since both are unbounded, appealing to diameter does not work. So, we need some other ad hoc method here ok. Here is one such. Take the set A of 4 points whose coordinates are $(\pm 1, 0)$ and $(0, \pm 1)$ ok?

$e_1, e_2, -e_1, -e_2$, all the four are taken ok? In $\mathbb{R} \times \mathbb{R}$, see these are all elements of R 2 So, I am working inside $\mathbb{R}^2$ and then show that the ell 1 norm is not isometric to ℓ_2 norm, ok? What is the ℓ_1 distance between any two elements of A? What is the ℓ_1 distance? That is what, ℓ_1 distance you have to take the difference of the coordinates then take the sum of modulus right? Each of them we have take modulus this minus this modulus so on.

So, it will be always equal to two. Between any two points here ok? So, the some of the distance whatever mod mod of that is equal to 2. If you draw a picture it will be like a diamond shaped thing right. Here these two points, like that. So, distance will be 2, the ℓ_1 distance ok.

Suppose there is an isometry form one space to the other. Look at the image of these four points. You look at image inside again $\mathbb{R}^2$ but with the ℓ_2 metric. We get 4 point which are at distance 2 from each other right? That is what you get. But such a thing is not possible ok.

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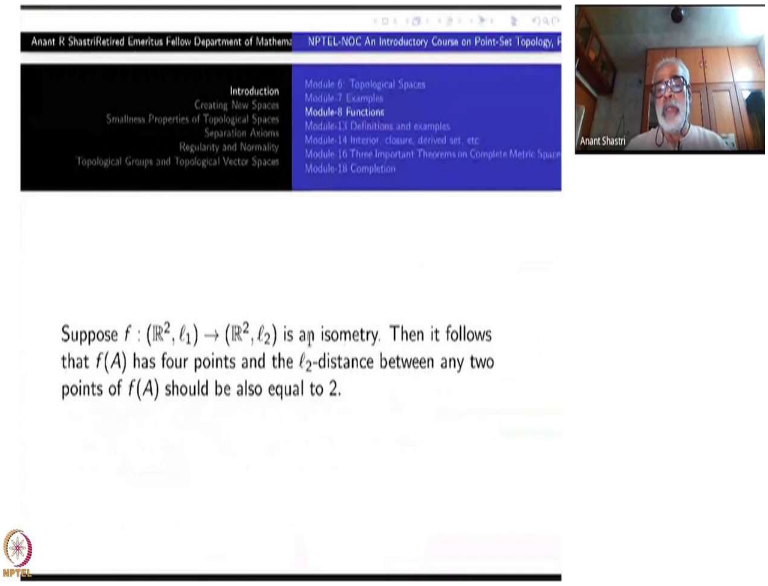

So, what I say is, you have 4 points here in ℓ_1 metric, which are at distance equal to 2 from each other. Suppose you have an isometry f here. Then it follows that $f(A)$ has 4 points and the ℓ_2 distance between any two points of $f(A)$ should be equal to 2. Because f is an isometry, distance preserving.

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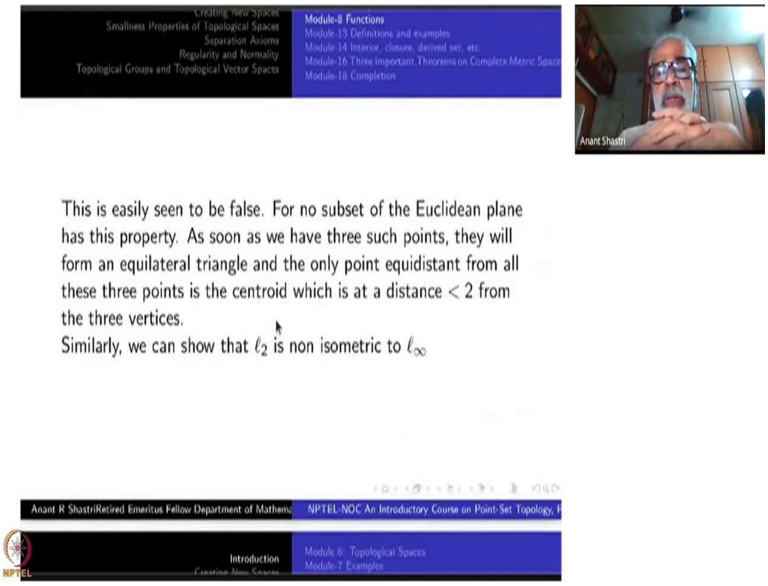

So, this is just false that is very easy from your school geometry. If you have 3 distinct points which are equi-distance from each other, already we have an equilateral triangle right? Now, the fourth point which is equidistant from all of them is the single point namely the orthocenter or the in-center, etc. all of them are same because it is an equilateral triangle. And the center is at a distance much less than the length of the side. You can compute it. I do not care it is much less than the side length 2. So, you cannot have a fourth point which is at a distance equal to 2 from all the three points alright, alright.

So, you see this is just another method. So, there are ways of combining various things and so on so or distance you can an extend this one. One real aspect here is the same argument you can use to show that ℓ_2 is non isometric to ell infinity also. This time if you take the 4 points, all 4 points you have to take, $(1, 1)$, $(1, -1)$, $(-1, -1)$ and $(-1, 1)$. Take for those 4 points, ℓ_∞ distance is always 2 from A from even the diagonally also it will be 2, ok. If there is an isometry to ℓ_2, again you get a contradiction alright.

So, there are such examples and so on. Maybe we will stop here. When we get more examples we will we will have that one.

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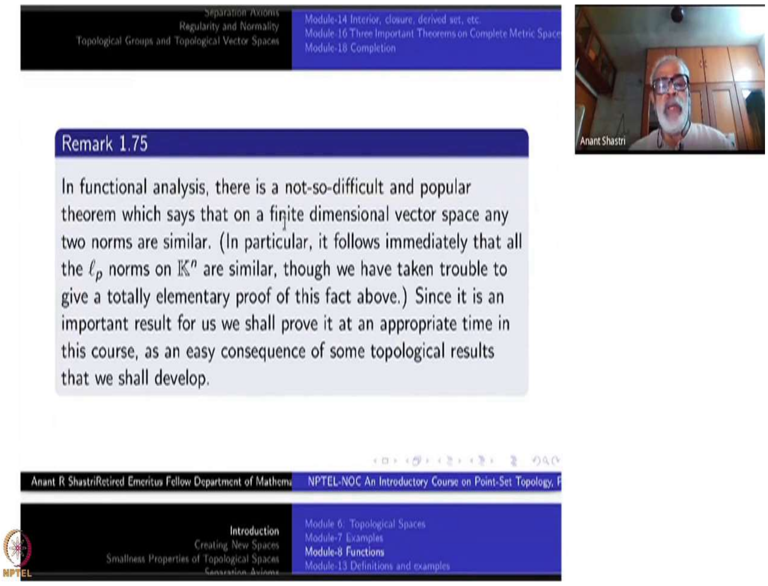

So, here is a comment in function analysis, that is not so difficult and a popular theorem which says that on a finite dimensional vector space any two norms are similar, in particular, it follow that all these ℓ_p norms on $\mathbb{K}^n$ are similar ok? But what we have done, we have already proved, by taking a little bit trouble, but not too much of trouble by starting with this inequality we have, that these ℓ_p norms are all similar ok? However, this theorem says any norm in a finite dimensional vector space, ok, will be equivalent to the Euclidian norm.

So, they are all equivalent ok? So, this is an important result. So, we shall prove it at an appropriate time in this course ok? as an easy consequence of some topological result that we are going to develop, we will not spend time just to prove this theorem ok? On the way we will prove that also alright.

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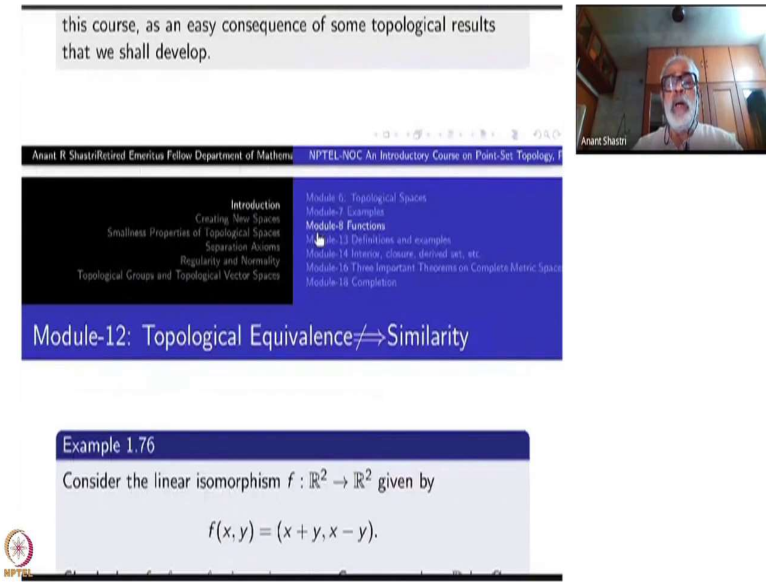

So, let us stop today for this one.

Thank you.