Introduction to Point Set Topology, (Part I)
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Module - 10
Lecture - 10
Various notions of equivalences

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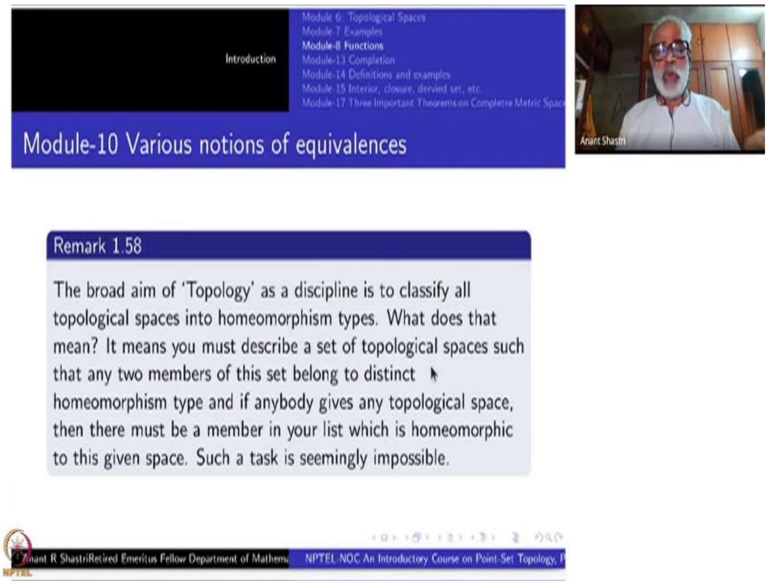

Welcome to module 10. Having introduced the notion of a homeomorphism and having studied some subsets of $\mathbb{R}^n$, let us now do various types of equivalences coming out of the metric. But we will do it in general now.

First of all the broad aim of topology as a discipline is to classify all topological spaces into homeomorphism types, what does that mean? It means that you must describe a set of topological spaces such that any two members of this set belong to distinct homeomorphism type, that means, they are not homeomorphic to each other. And second thing is if anybody gives you any topological space then there must be a member in your list which is homeomorphic to that given space. That is the meaning of classification classification of topological spaces up to homeomorphism.

However, such a task seems to be, you know, it is seemingly impossible. How can you list, you know. all possible homeomorphism types? ‘How can you?’ is not the answer anyway. Lot of very difficult things have been achieved by people right?

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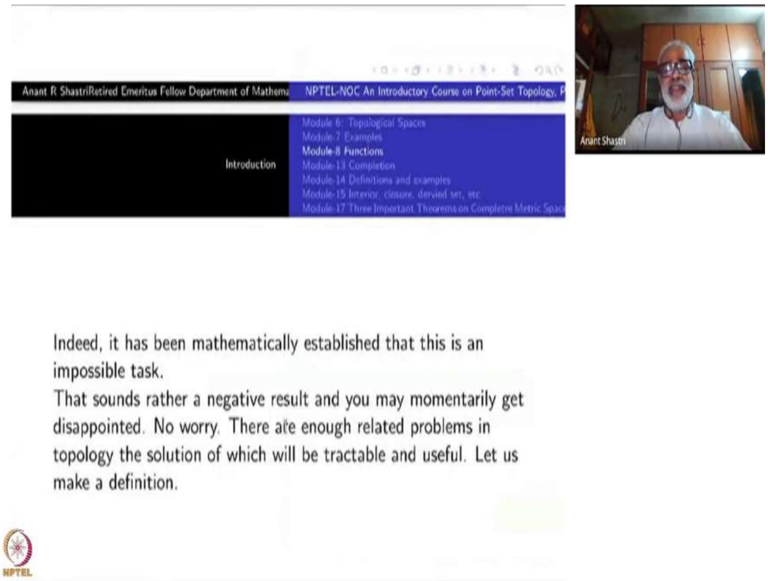

But here is something that is a mathematically established fact, that this is an impossible task. I cannot elaborate that right now because it involves deep logic, deep group theory and sufficiently deep, not so deep, but beyond this course, algebraic topology also ok? So, beyond this, I cannot say anything more than that it is impossible to classify all topological spaces. So, that sounds rather a negative result and you may momentarily get disappointed. Oh then why should we study topological spaces and so on. That is not the case. There is no worry on that score.

Because there are enough problems other than the central problem or the broad aim whatever you may call it. Many related problems are there in topology the solutions of which will be tractable ok? Tractable means what you can say, it just looks like you can try to solve it and sometimes solve it, or sometimes get partial answers and so on, but the finial answers would be useful ok. So, therefore, we should not get disappointed that is the whole idea ok.

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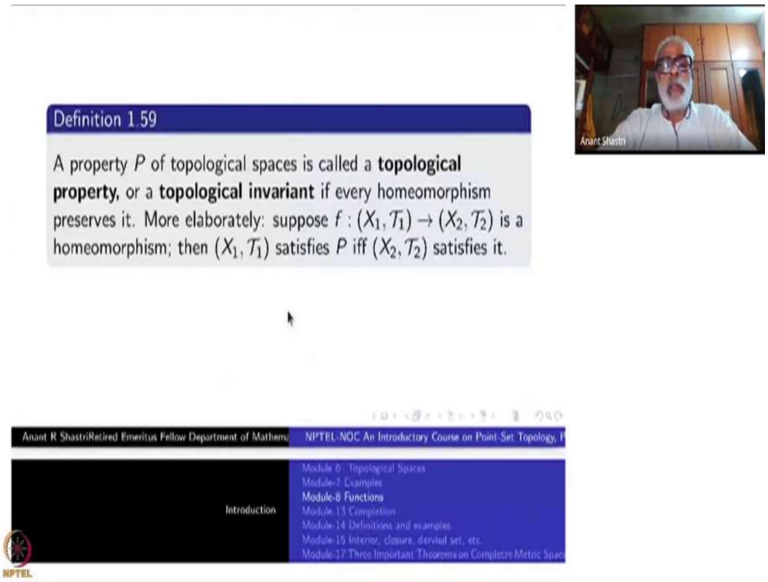

Let us begin with a definition now, a formal definition which looks like not a mathematical definition. A property P... what you mean by of property P that itself is a difficult thing to explain, so, take it as a you know just with a pinch of salt just take it as it is, a property P of topological spaces is called a topological property or a topological invariant if every homeomorphism preserves it. A little more elaborately, suppose you have a function f from $(X_1, \mathcal{T}_1)$ to $(X_2, \mathcal{T}_2)$ which is a homeomorphism then $(X_1, \mathcal{T}_1)$ has property P should imply $(X_2, \mathcal{T}_2)$ has it and conversely. $(X_1, \mathcal{T}_1)$ satisfies P if and only if $(X_2, \mathcal{T}_2)$ satisfies it. What are X_1 and X_2? They are homeomorphic to each other. So, that is the meaning of saying that P is a topological property.

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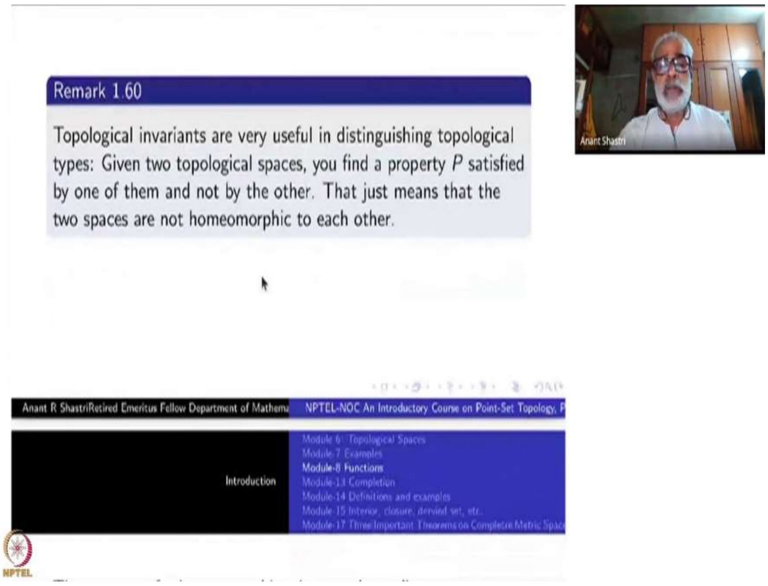

So, such a thing is called a topological invariant also, ok let me give you some examples. Then only you may will understand. Topological invariants are very useful in distinguishing topological types but what you mean by type? You know you have seen that homeomorphism defines an equivalence relation. So, these are equivalence classes ok.

Given two topological spaces you find a property P which is satisfied by one of them, but not the other then obviously, they will be in different classes they cannot be homeomorphic. So, this is the way you can distinguish the topological types somewhat easily of course. On the other hand, you may find that this property is satisfied by both of them. That does not prove that the two spaces are homeomorphic, because there may be some other property which is not satisfied. You may go on listing all your known properties satisfied by both the sides, still it will not prove that the two are homeomorphic.

To prove that something is homeomorphic something else, somehow you have to produce a homeomorphism. That is the only way. Like we did first, we showed all open balls are homeomorphic to each other then we showed that one single ball namely of radius one is homeomorphic to the whole of $\mathbb{R}^n$, now you can combine the two. So, that this is the kind of proof you have to produce in proving that given two spaces are homeomorphic to each other. Ok? just a collection of topological properties will not help in that case.

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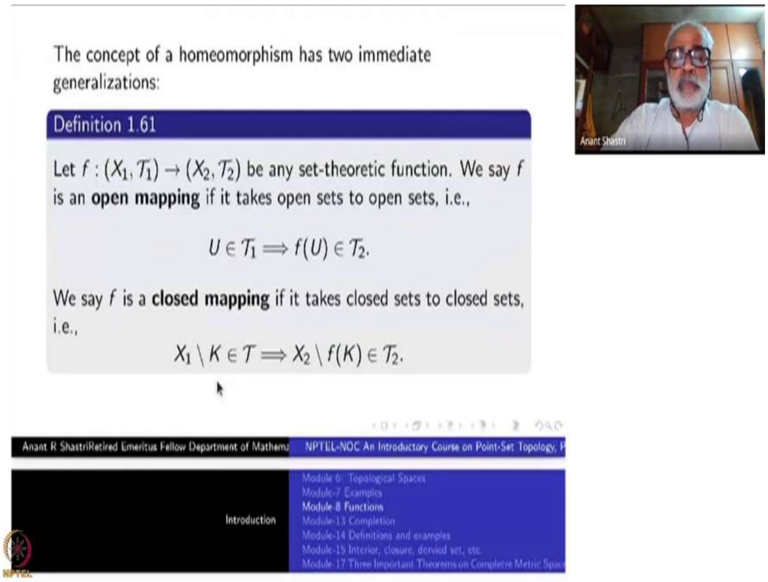

Nevertheless, half the way they come, namely in distinguishing topological types they are helpful ok. So, before going further, I will give you, I will take this opportunity to define two concepts, but we will actually meet them much later ok. Take a function from $(X_1, \mathcal{T}_1)$ to $(X_2, \mathcal{T}_2)$ we say this is an open mapping I told you this while defining continuous mapping such a thing also, now I am defining that, it is called an open mapping if it takes open sets into open sets.

That is all. f need not be continuous ok? If f inverse takes open sets to open sets, then it is defined as continuous function. Remember that U inside $\mathcal{T}_1$ to the implies $f(U)$ inside $\mathcal{T}_2$, that means the function is open. Similarly I can define a closed mapping also namely say K is a subset of X_1 and $X_1 \setminus K$ is inside $\mathcal{T}_1$ should imply $X_2 \setminus f(K)$ inside $\mathcal{T}_2$, which just means that K is closed in $\mathcal{T}_1$ should implies $f(K)$ is closed in $\mathcal{T}_2$. You remember closed sets are the compliments of open sets ok?

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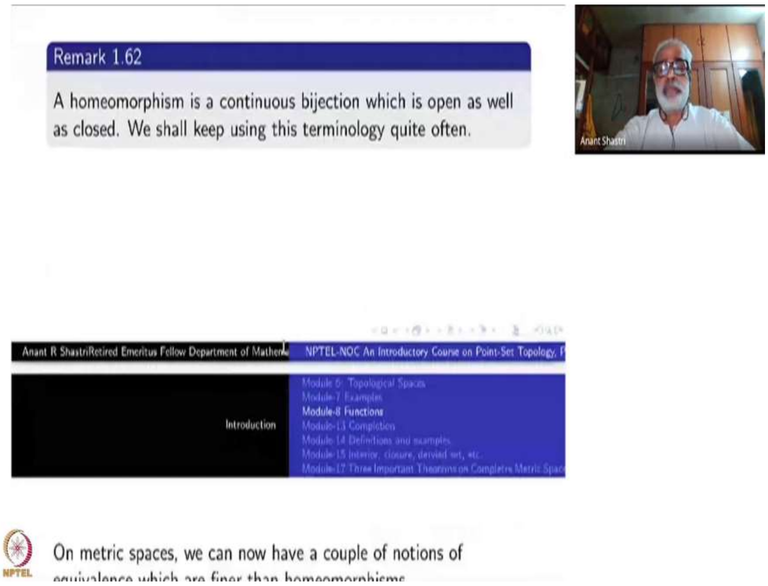

So, a homeomorphism, it is a continuous bijection which is open as well as closed. So, this is one way why open mappings and closed mappings are important, that is why I have introduced it here. you look at this one. Take a continuous function, what is it open subsets in the codomain, inverse of that will be open. In addition, suppose f is a bijection. What does that mean that means, f inverse function is an open mapping right?

Similarly, if suppose f is both open and continuous then it will imply that f inverse is continuous therefore, it will be a homeomorphism. Similarly if f is open as well as a closed mapping then also it will happen because you can see that the continuity can be defined in terms of closed sets also just by demorgan law. So, I have given you that lots of these things you can change from open sets to closed set by just by demorgan law. A function is continuous if for every closed set in the codomain, the inverse image is closed in the domain ok?

So, if f is a bijection then f inverse make sense as a function, (otherwise f inverse of a set always makes sense) as a function, f inverse make sense first of all and it will be continuous because f is a closed map. Therefore, you take a continuous function if it is open its already homeomorphism, I am sorry a continuous bijection, similarly a continuous bijection which is also a closed mapping is a homeomorphism ok.

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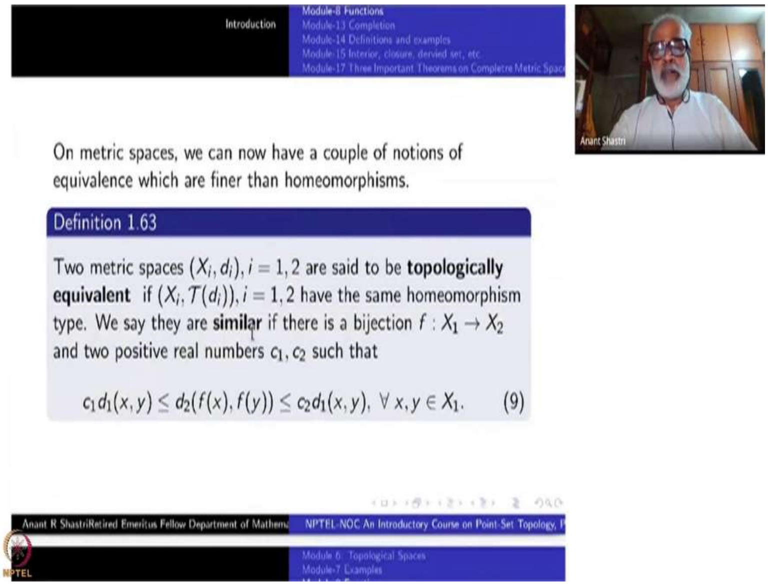

So, why sometimes it is easy to handle closed sets, sometimes it easy to handle open set that depends upon the situations. That is why we should have all these concepts clear to us. Now, we will come to metric spaces wherein there are other equivalences other notions of equivalences just like you had in your school geometry. You have congruent triangles as well as similar triangles and so on right? they are both equivalences. If one triangle is congruent to another triangle, this another one is congruent to a third one then the first one is congruent to third one and so on.

We know that congruence, as well as similarity are equivalence relations right? similar to that we will have different notions of equivalences here, but they are all metric related. So, let us start. Take two metric spaces (X_i, d_i). We say they are topologically equivalent if the underlying topological spaces $(X_i, \mathcal{T}(d_i))$ have the same homeomorphism type. That means, there is a homeomorphism between them ok? That is one thing namely topological type topological equivalence.

The second one is: We say (X_1, d_1) is similar to (X_2, d_2), I if you have a bijection from X_1 to X_2 and 2 positive real number c_1 and c_2 such that this $d_2(f(x), f(y))$ is trapped between $d_1(x, y)$ by these constants: $c_1 d_1(x, y)$ is less than or equal to $d_2(f(x), f(y))$ less than $c_2 d_1(x, y)$, again ok, c_1 and c_2 must be positive real numbers. If c_1 and c_2 are equal to 1, what is the meaning of this? they are all equal ok. That is a very strong thing of course, that also good, then we will say they are isometric. So, we will come to that also.

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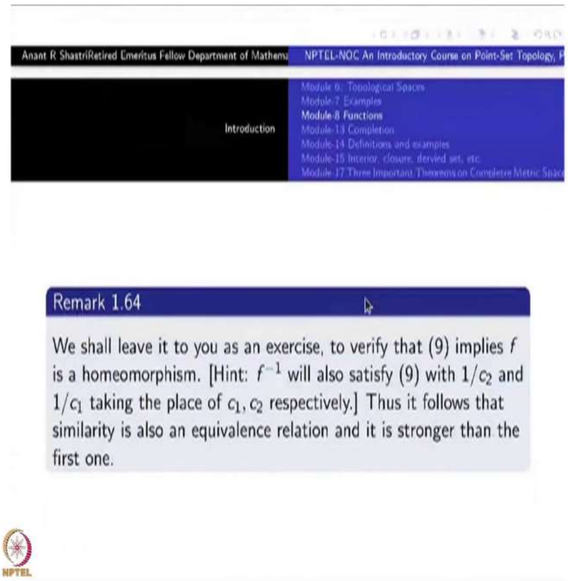

First of all why this similarity is an equivalence relation? This looks like one way I have defined f from X_1 to X_2, but it is a bijection ok. So, there is a map f^{-1} from X_2 to X_1, then I can try f^{-1} here and I take now a, b inside X_2. Then what should I take in place of c_1? I have take d_2 here $d_2(a, b)$ right and here also $d_2(a, b)$ what should I take in place of c_2?

You just look at this one. What should I take to bring c_1 to this side? here it is $1/c_1$. then d_1 of; I can write this as $d_1(f^{-1}(a), f^{-1}(b))$,and this will become $d_2(a, b)$ right? So, work it out I have already indicated how to do that ok. So, you have to use $1/c_1$ and $1/c_2$ ok in correct places and write down a similar inequality for f^{-1} that will show you that the similarity here is symmetric ok?

Then you have to show if f from X_1 to X_2 and another g from X_2 to X_3 are similarities there will be c_1' and c_2' and so on we have to look at $g \circ f$ from X_1 to X_3 as a similarity. Check it. So, that will show you transitivity of course.

I can take identity map here and c_1 equal to c_2 equal to 1, that will give you that identity map is a similarity. Therefore, the similarity, i.e., having a similarity map, similarity is an equivalence relation.

You have to do a little bit of verification. So, that I am leaving it as an exercise to you, but I have already indicated it and given you enough hints ok. So, this equivalence relation is stronger than the first one, why?

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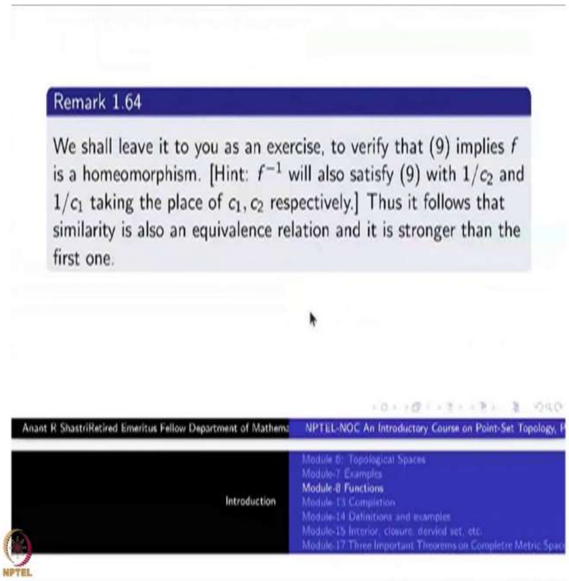

Because as soon as we have this one you will see that f is continuous ok. So, this part will give you f is continuous. So, that part will give you that f inverse is continuous therefore, it is homeomorphism similarity already implies hope topological equivalence. Thus similarity is a stronger equivalence here. So, that implies weaker equivalence that is all ok, but they are different is what I have to ensure. You have not seen it yet. ok? Let us go ahead.

So, I have already told if you put c_1 equal to c_2, what you get you will get that distance between $f(x)$ and $f(y)$, the d_2 distance, is equal to the d_1 distance between x and y, the points with which we started ok?

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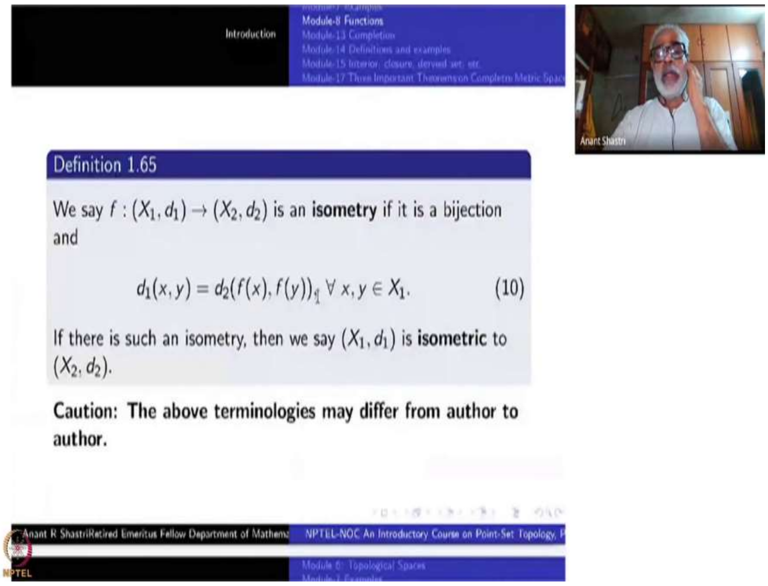

So, such a function is called an isometry if it is a bijection also. f is a bijection that condition is already there. The distance between x and y is preserved under f, $d_2(f(x), f(y))$ is equal to that $d_1(x, y)$. If there is such an isometry then you will call (X_1, d_1) isometric to (X_2, d_2). It is very easy to see that this is an equivalence relation ok whereas, in the similarity you have to work a little hard ok. So, I want to caution you that some authors may have different names for these concepts ok. So, be careful to read their definition before answering their questions or whatever.

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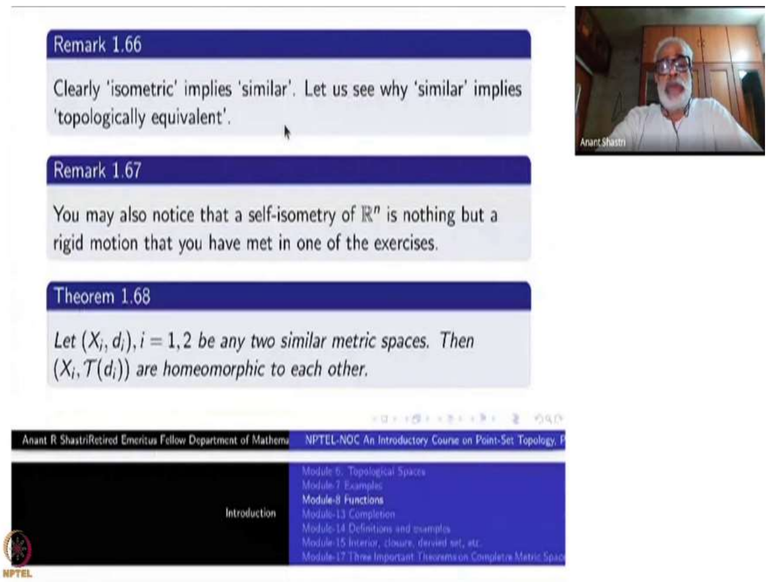

Clearly isometry implies similarity and similarity implies topological equivalence. So, there are three concepts which we are now studying with metric spaces one stronger than the other. this implies this this implies that. But are they really stronger? Maybe this is this will also imply that then they will be equivalent. So, let us see that. You may notice that a self-isometry of $\mathbb{R}^n$ is nothing, but a rigid motion. Hope by now, you must have at least read and tried the exercise on rigid motion. Then only we will understand this remark ok?

Rigid motion was defined as what? Any function which preserves the distance, like this. There was no condition of bijection. The only thing is that domain and co domain were both assumed to be $\mathbb{R}^n$. There is no need to do that. You can assume this one to be anything any topological space, but the same topological space on this side also. Sorry any metric space and same metric space on this side also ok. Rigid motions are usually within a metric space take (X_1, d_1) to (X_2, d_2), a map such that distance between x and y equals distance between $f(x), f(y)$ for every x, y. Automatically, you can check that it is injective.

But ontoness is an exercise in case X is $\mathbb{R}^n$. First of all you do not know whether it is onto, so that was the exercise. So, this is isometry is then stronger than a rigid motion. when X_1 and X_2 are the same (X_1, d_1) and (X_2, d_2) must be also equal. So, these two notions are not much different isometry and rigid motion. So, rigid motion is a weaker notion than this one ok? Also the notion of isometry is valid for between any two metric spaces ok?

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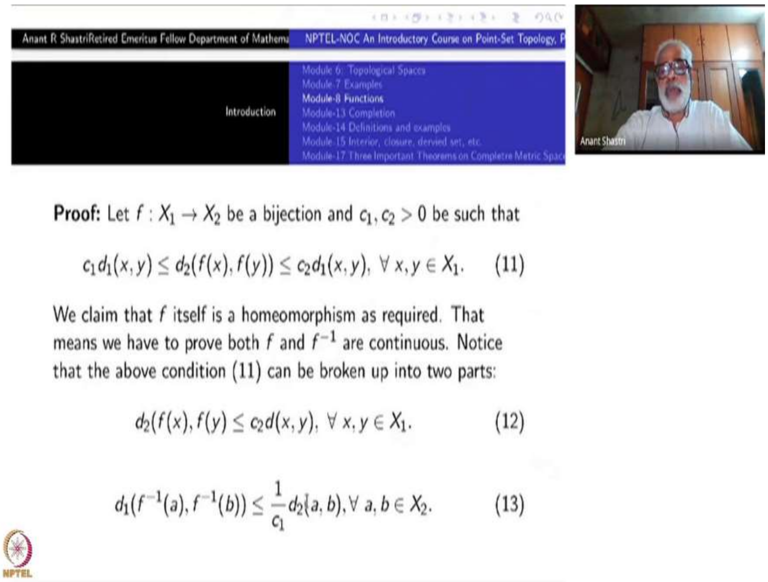

So, here is a theorem. Take two metric spaces which are similar to each other. Then they are homeomorphic to each other. I have already told you that I will elaborate on that one. Namely start with a bijection and c_1, c_2 positive constants such that this is true for every x, y right. So, that is definition of similarity I have to show from this one that f is continuous and f inverse is also continuous we claim that f itself is a homeomorphism, then it will follow that X_1 and X_2 are homeomorphic, that is all ok.

So, we have to prove both f and f inverse are continuous because bijection is already given. Notice that condition (11) can be broken up into two parts namely this later part $d_2(f(x), f(y))$ is less than $c_2 d_1(x, y)$ for every x, y inside X_1 ok?

Second part you take inverse here, $d_1(f^{-1}(a), f^{-1}(b))$ in place of $d_1(x, y)$ and bring c_1 on this side, we get $1/c_1$ times $d_2(f(f^{-1}(a)), f(f^{-1}(b)))$ that would $d_2(a, b)$ for every a, b inside X_2 ok? I am cleverly writing this using the definition using the fact that f is a bijection, so f inverse make sense. So, suppose we have these two conditions you can give back this condition. So, I am writing this condition one condition is equivalent to these two conditions here ok.

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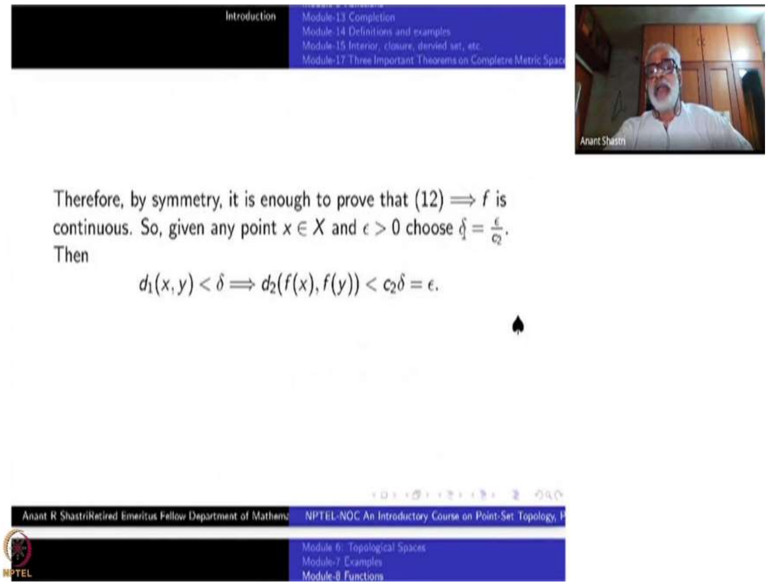

It is broken up into two parts right? once you have that by symmetry if I prove this implies f is continuous automatically this will imply f inverse is continuous ok. So, this (12) and (13) these two conditions are there. Suppose this (12) implies f is continuous then the other one is also continuous. So, what is given here? How do you prove the continuity of f? given ϵ there exist a δ blah blah blah right? that is what you have to do. So, whenever distance between this is less than δ this must be less than ϵ.

If this is less than δ, this entire thing will be less than $c_2\delta$. So, I have to only say that $c_2\delta$ is less than ϵ, then this will be less than ϵ. So, that is what I have to do here. Given $\epsilon > 0$ choose δ such that $c_2\delta$ is ϵ or δ equal to ϵ/c_2. Then $d_1(x, y)$ less than δ will implies $d_2(f(x), f(y))$, $f(x)$ and $f(y)$ here if this is less than $c_2\delta$ which is ϵ this is the seen ϵ, ok? Over. alright?

So, one equivalence relation other than homeomorphism you have studied, in fact two of them, but one we have shown that similarity implies homeomorphism the other one is easy right? because it is a special case. isometry is got by taking c 1 equal to c 2 equal to 1 here. So, it is similarity also. So, isometry implies similarity ok. So, next time we will see that homeomorphism does not imply similarity and similarity does not imply isometry ok so we will do it next time. Thank you.