Partial Differential Equations
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Lecture – 2.5
First Order Differential Equations
Relation between Characteristic Curves and Integral Surfaces for Quasilinear Equations

So, today, we will take the first steps towards solving Cauchy problems for Quasilinear equations. So, in this lecture, we are going to look at the relation between characteristic curves and integral surfaces.

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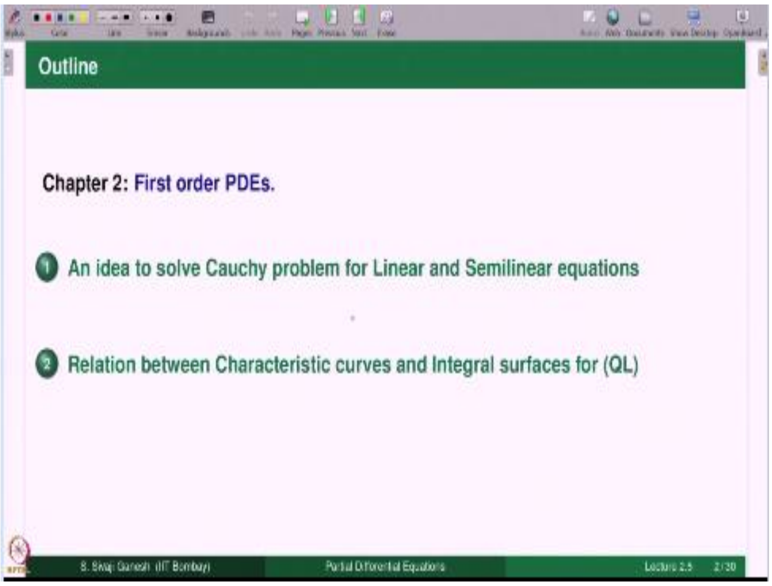

The outline of the today's lecture is, first we discuss an idea to solve Cauchy problems for linear and semilinear equations, which are clearly the simplest form of the Quasilinear equations, special forms but simplest. So, we try to solve the Cauchy problem for linear and semilinear equations and we see whether we encounter any difficulties. If we encounter, we see how to overcome them.

And when we generalise to Quasilinear what should be done will also be discussed. And then, we look at the relation between characteristic curves and integral surfaces for Quasilinear equations. This, we are back to look at this as a consequence of our analysis for the point 1. When we look at linear and semilinear equations and try to solve the Cauchy problem, naturally, we are let to asking certain questions that will lead us to our understanding of this.

It will lead us to inquire into the relation between characteristic curves and integral surfaces, which we will do more generally for the Quasilinear equations.

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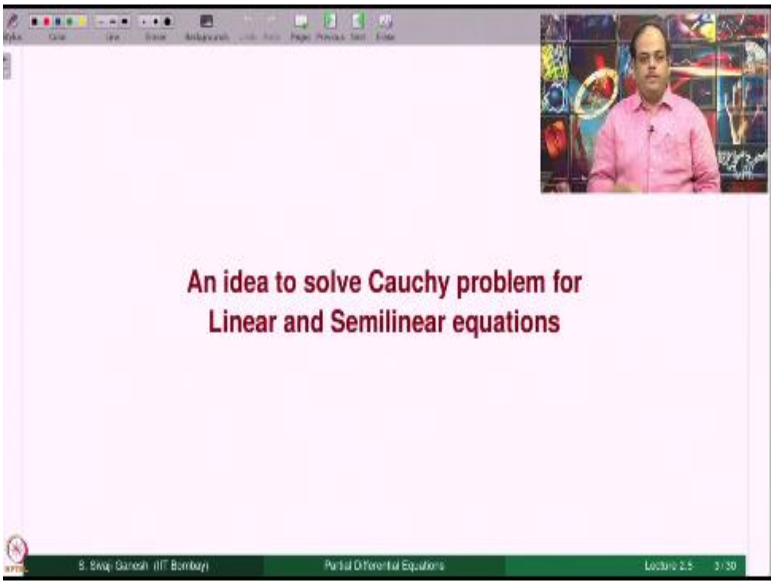

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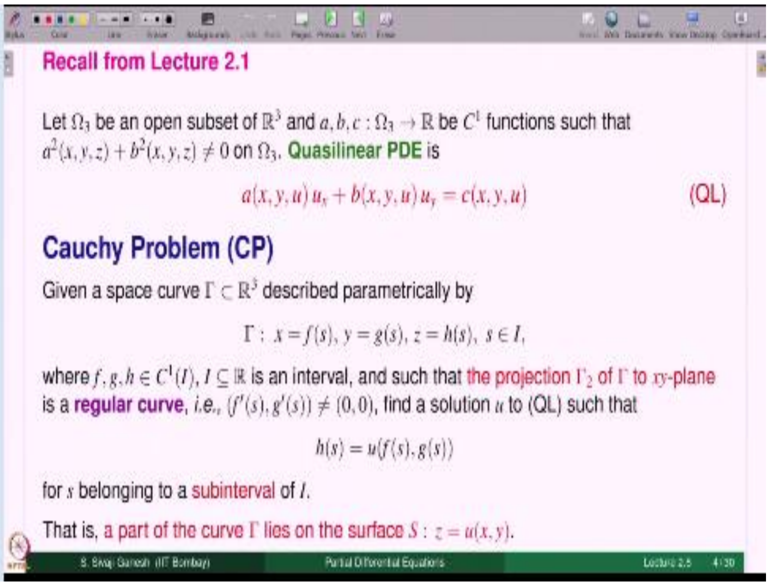

So, first we discuss an idea to solve Cauchy problem for linear and semolina equations. Let us recall from lecture 2.1, where we have introduced the notion of Cauchy problems and Quasilinear equations. So, this is a definition of the Quasilinear equation is an equation of the form a x y u, u x + b x y u u x = c x y u. Any reasonable analysis of this can be done at an elementary level.

For that, we need to assume that a b c are nice functions that is C 1 functions; their functions of 3 variables. So, they are defined in a domain omega 3 in R 3. And we want that a and b do not vanish simultaneously that is a, b is not equal to 0, 0 at every point of omega 3. So, that we have some form of the PDE. if both a and b vanish, then there is no PDE right, LHS is 0. Therefore, we require that at least one of these a and b is nonzero at every point of omega 3.

So, what is Cauchy problem? Here, we are given a space curve gamma, which is typically given a parametric version, where x component is given by F s; y g s; z h s as s varies in an interval in R and the functions f g h are C 1 functions on the interval. This is given. So, what is the Cauchy problem? It is to find the solution of the Quasilinear equation, but before that we need to assume something more that is this projection gamma 2 of gamma of this plane curve to R 2 to the xy plane is a what is called regular curve.

What it means is f prime, g prime is not 0, 0 at each of the points of gamma 2 and we require that this z is equal to h s right that h s must be given as u of x y that is what we expect. So, u of f s, g s that will be equal to h s. This is what we want to do. And as discussed before, we do not want s belong to the entire interval I, but we are happy if we can find such a function with these properties where s belongs to a subinterval of I that is a part of the curve gamma lies on the surface z = u x y.

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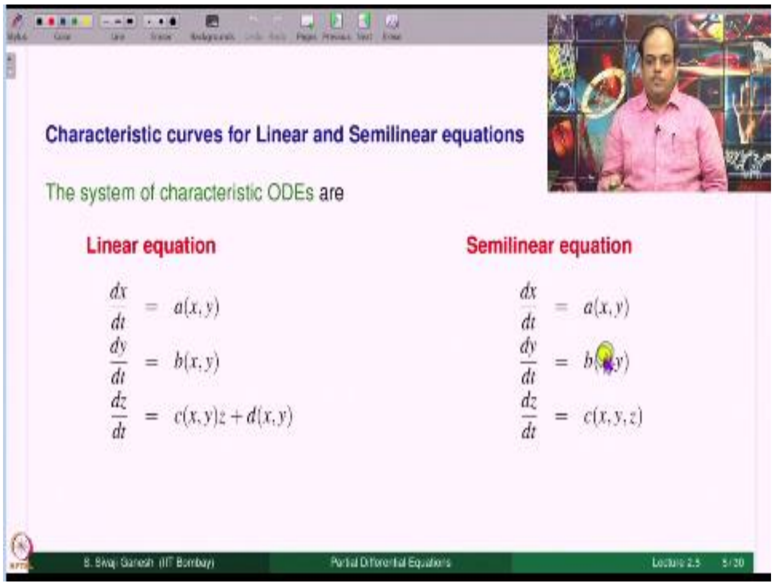

Now, what are characteristic curves for linear and semilinear equations? They are the trajectories of the solutions of a system of characteristic ODEs. In the linear case, dx by dt is a; dy by dt is b; dz by dt is of this form c z plus d. In the case of semilinear equation, it is c x y z basically a b c right, a u x + b u y equal to right hand side. Here also a u x + b u y equal to the right hand side. So, these are the characteristic system ODEs, their solutions and the images of the solution. They are the characteristic curves.

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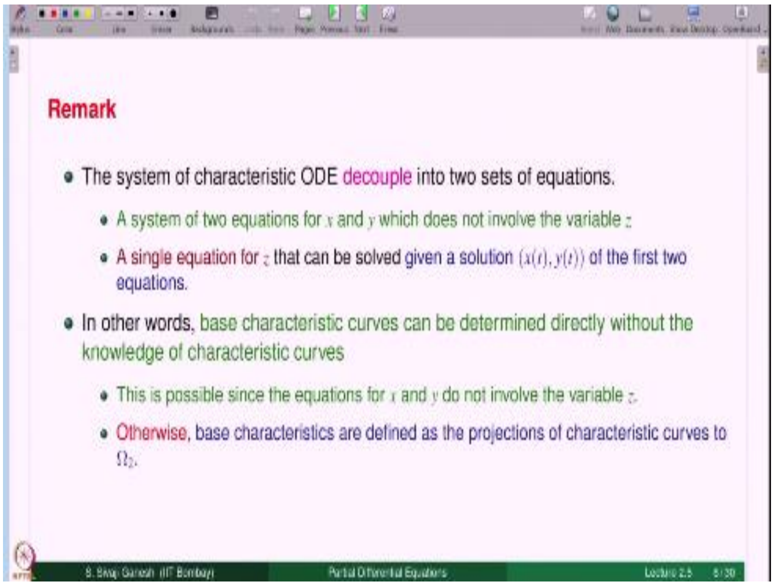

System of characteristic ODE decouple into 2 sets of equations that is what we need to observe. If it is a Quasilinear equation, we will have a of x y u; b of x y u and z of x y u. And because a and b do not depend on the third variable, the first 2 equations if you look, they feature only x and y; equation for x and y can be, is given in terms of right hand sides which are functions of x y only.

So, we do not require to solve this equation. We can solve these 2 independently. Similarly, in the semilinear equation, x and y can be solved for without bothering about the equation for z. So, it decouples into 2 sets of equation. A system of 2 equations for x and y, which do not involve the variable z, we saw that and a single equation for z which can be solved provided you give me what is x and y.

Given a solution x t, y t of the first 2 equations, we can solve for z. In other words, base characteristic curves can be determined directly without the knowledge of characteristic curves. Remember characteristic curves are the tracer of the triple x t, y t, z t as t varies the trace of this is a characteristic curve and when you project it to xy plane, you get x t, y t as t varies, but here x t, y t can be determined independent of z t.

Therefore, we say base characteristic curves are determined directly without the knowledge of characteristic curves. These possible since the equations for x and y do not involve the variable z. Otherwise, base characteristics are defined as a projection of the characteristic curves to omega 2.

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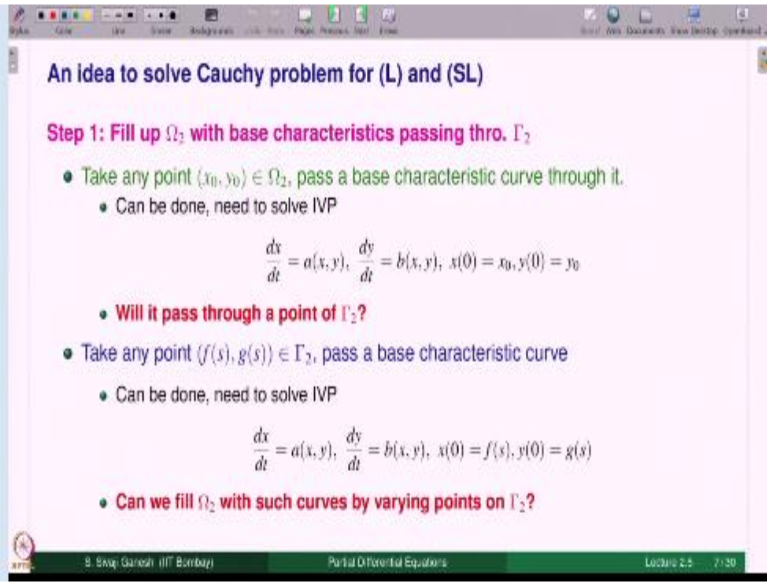

So, an idea to solve Cauchy problem for L and SL, we are dealing both L and SL together mainly because the part where the first order derivatives appear is the same a x + b y. What is the step 1? Fill up omega 2 with base characteristics passing through gamma 2. So, what we will do for this? Take a point x 0, y 0 in omega 2, pass a base characteristic curve through that point. How do you do that? It can be done.

You have to solve this initial value problem. This is the equation for base characteristic curves, the first 2 equations dx by dt, dv by dt equal a and b respectively. And you solve with this initial condition x 0 is x 0, y 0 is y 0. So, such solutions exists because we are assuming a b are C 1 functions. So, they are locally Lipschitz. This is an initial value problem and it has a unique solution which defined on some interval containing 0. All that is there.

Therefore, there is a base characteristic through any given point x 0, y 0. Now, the question is we are worried about gamma 2 that is where the data is given. So, question is: will it pass through a point of gamma 2? That is the question we ask. And because that is where the data is given. We need to solve Cauchy problem. So, that is why we asked this question. It will be clear in the next slides why we are asking this question.

On the other hand, we can do one more thing, take any point in gamma 2. Why you are taking omega 2? Take a gamma 2. And then pass a base characteristic through that. It can be done exactly same procedure. You will need to solve this initial value problem where x 0, y 0 there is replaced with f s, g s. Now, the question is, can we fill omega 2 with such curves by varying points and gamma 2? So, these are the 2 different points of view.

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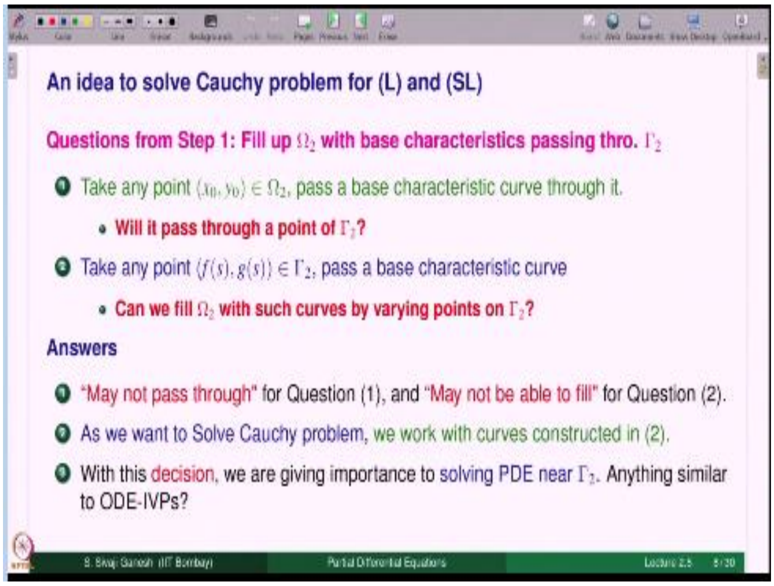

So, questions from step 1 are the following. Take any point in omega 2, pass a base characteristic curve through it. Will it pass through a point of gamma 1? One question. Second, take any point on gamma 2, pass a base characteristic curve through that point. Question is: can we feel omega 2 using such curve? Answers to the first question is, may not pass through; second question is, may not be able to fail.

For both questions, we do not seem to have outright positive answers. Both answers are having caution terms, may not, may not. But what we want to do? We want to solve Cauchy problem. Therefore, what is more relevant to us is gamma 2. Therefore, we work with curves which are constructed in these 2. So, therefore that is the reason why we are going to look at curves which are defined using this point to here.

So, we take points on gamma to pass base characteristic curves and then try to find a solution. That is the approach we will take because the initial data is given on gamma 2. x = f s; y = g s that is gamma 2. z has to be h s is the datum curve. So, with this decision, we are giving importance to solving PDE near gamma 2 because near gamma 2, why we say because we may not be able to fill up an entire omega 2 by using the curves that we get in as an answer to step 2.

We may not be able to fill that is the answer. So, we may not be able to fill but definitely we will cover gamma 2. So, we should be happy with that to start with. So, anything similar to ODE initial value problems here Yes, there is, there are similarities, there also when we solve dy by dx = f of x y and y x 0 = y 0 that is initial value problem. The existence theorem says there is a solution near x = x 0 that is why it is called local existence theorem. So, it is similar to that.

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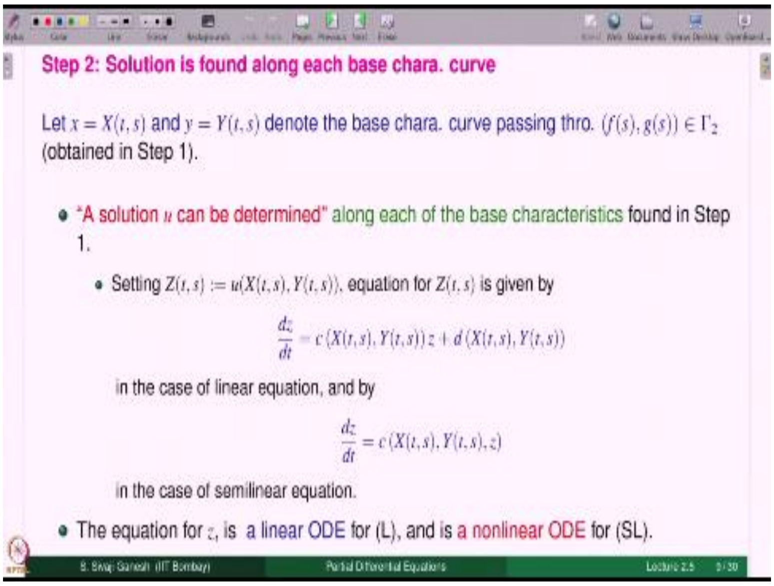

Now, what is the step 2? Solution is found along each base characteristic curve. What is that? We will explain. So, let x = x t s and y = y t s denote the base characteristic curves passing through a point f s, g s in gamma 2. So, this is what we have obtained in step 1. Now, I am writing this in quotes, it means that it may not be very rigorous, but these are feeling. When we prove theorems, everything is rigorous.

So, a solution u can be determined along each of the base characteristics found in step 1. So, setting Z t s is equal to u of X t s, Y t s. I look at what is the equation that Z t s satisfies because what is this u of X t s, Y t s means, I take a base characteristic curve X t s, Y t s for a fixed X, X is fixed here and then u of X t s, Y t s is there. I can consider. Now, that I call Z t s, this is actually u along the curve X t s, Y t s that is what I am calling Z t s.

So, equation for Z t s is given by of course, you are differentiate dz by dt = u x into dx by dt + u y into dy by dt; by dx by dt and dy by dt, they are solutions of base characteristic curves they will be A and B respectively. So, what you have here is a u x + b u y and that is equal to c if you use a solution to the PDE. Therefore, this is equal to C of, this in the linear case, c of X t s, Y t s into z + d X t s, Y t s.

And if it is semilinear equation, it is simply c of X t s, Y t s z. So, we got an equation a ODE for the solution along each base characteristic curve. So, the equation z is a linear ODE for the linear PDE and it is a nonlinear ODE for the semilinear PDE nonlinear because you know the dependence of c on z is not linear that is all it says.

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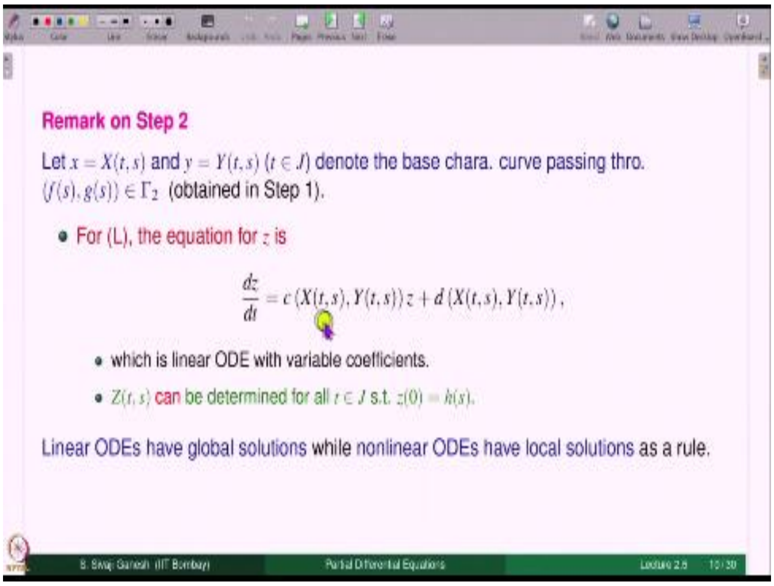

So, for L the equation for z was this, I am recalling the equations. It is a linear ODE with variable coefficients, because the coefficients depend on t here also these are non-homogeneous; you can take this to the other side, what you have is dz by dt = – c of X t s, Y t s into z + d X t s, Y t s. So, you have a right hand side term and you have variable coefficients here. Now, Z t s can be determined for all t in J. What is J? J is the interval on which you saw X t s and Y t s.

You have to remember that these equations for bass characteristics are solved. So, that the initial condition was f s g s. Therefore, this interval J actually depends on s. I do not want to confuse you at this moment. Therefore, I am not writing that dependents. Otherwise, we are tried t belongs to J s. We will see this in a forthcoming lecture. I will highlight that point. So, Z t s can be determined for all t in J such that z 0 is h s.

Why is that? Because linear ODEs have global solutions while nonlinear ODEs have local solutions as a rule. Nonlinear ODEs having global solutions is an exception. Linear ODEs will always have global solutions. Global solutions means, wherever your coefficients are defined in this case, the coefficients are all defined whenever t belongs to J. Therefore, the solution for t belongs to J.

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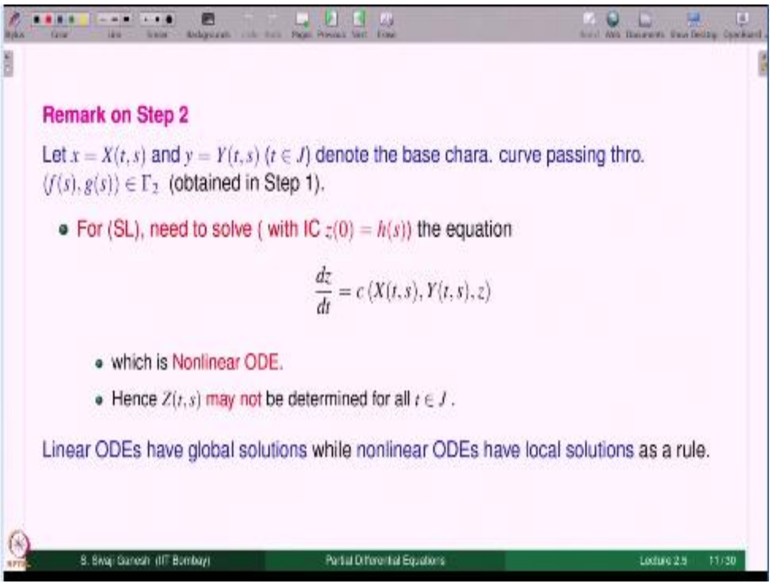

Now, let us comment same about the semilinear equation. This is the equation. This is nonlinear equation. Therefore, Z t s may not be determined for all t in J. It means that you may have to look at a smaller interval sub J sub interval sub J, where Z will be defined. Again linear ODEs have global solutions; nonlinear ODEs have local solutions as a rule.

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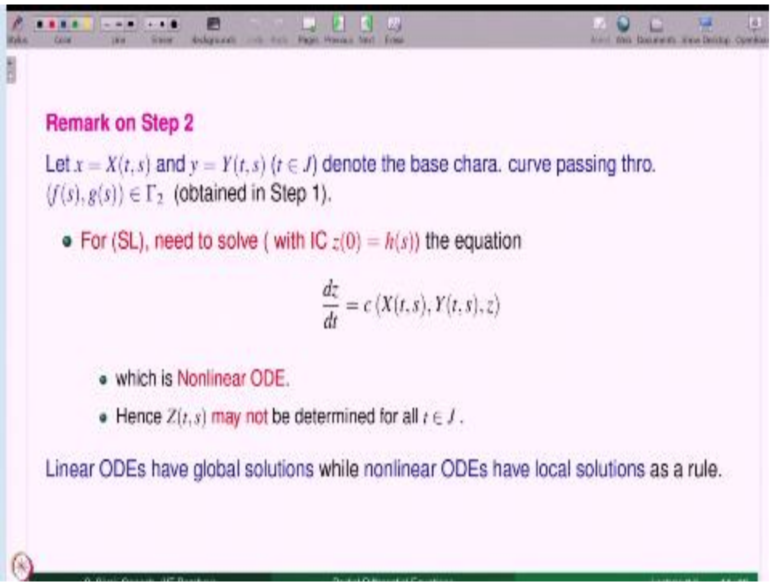

So, now, the third step is; what we have done so far, step 1, we have determined X t s, Y t s. Second step, we have determined Z t s. Now, we need to relate Z t s and X t s, Y t s. We hope that Z t s is u of X t s, Y t s.

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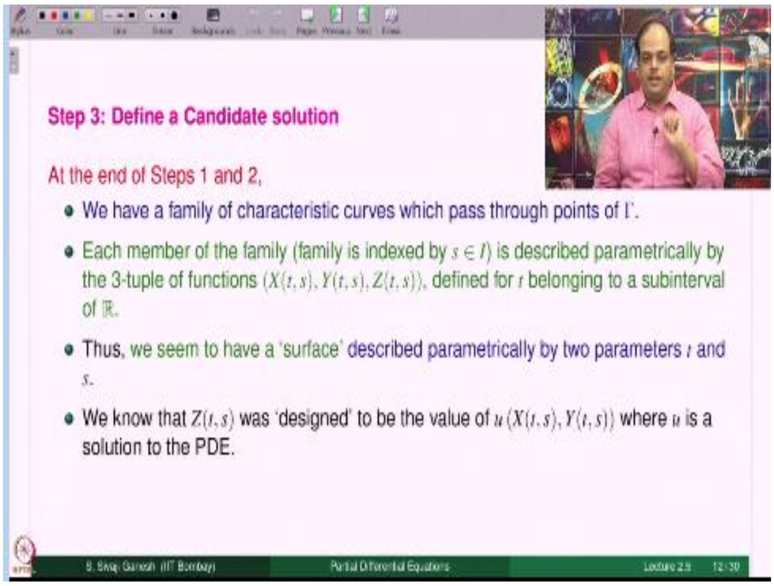

So, that is what is to be done here in Step 3, define a candidate solution. At the end of steps 1 and 2, we have a family of characteristic curves which pass through points of gamma, the families indexed by the point on gamma. For example, yes, so, first of all, we saw like this, we had a gamma 2, we took a point and passed one characteristic after that another point, maybe another characteristic, another character like that.

And after solving X t s, Y t s, now we also have Z t s. And what is the initial condition at t = 0? It is h s. Therefore, we have the family of character curves are now passing through points of gamma. Base characteristic curves are passing through the points of gamma 2 and they pass through the points of gamma 2. Each member of the family is described parametrically by 3-tuple functions, X t s, Y t s, Z t s defined for t belonging to a subinterval of R.

Thus, we seem to have a surface. It depends what your definition of surface is, but we seem to have a surface, let us be very vague here described parametrically by 2 parameters t and s, right. X t s, Y t s, Z t s, t and s are vary, so what you hope to get your surface. We know that Z t s was designed to be the value of this that was a definition of Z t s is u of X t s, Y t s where u is a solution to the PDE u was not known.

We thought there was a u and then Z t s, we set as u of X t s, Y t s, then we got an equation for z thinking that u is a solution to the PDE. We did all that.

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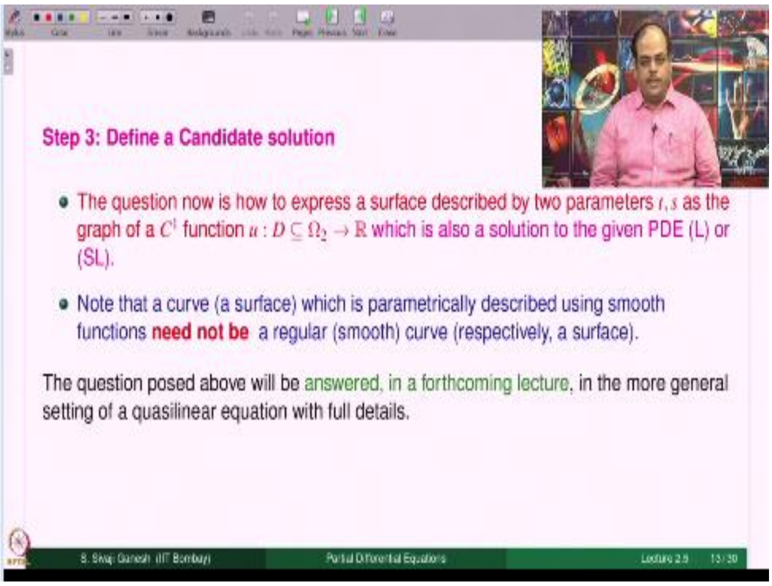

So, u is still has to be retrieved that is what we are saying here define a candidate solution. The question now is how to express a surface described by 2 parameters t and s as a graph of a function u defined on some domain D. Of course, we will be a subset of omega 2 for obvious reasons, which is also a solution to the given PDE. So, how to catch one of such a function whose graph will be equal to the parametric surface that we have got.

So, note that a curve or surface, this is very general thing; note that a curve or a surface which is parametrically described using smooth functions need not be a regular or smooth curve or surface. So, it can deceive us, it may have a very good description in terms of s and t, but actually it is a bad surface. It may have singularities. We will see that in many examples that we are going to consider in this course.

The question posed above will be answered, not now, but in a forthcoming lecture. In the more general setting of a Quasilinear equation, we will do that with fully details.

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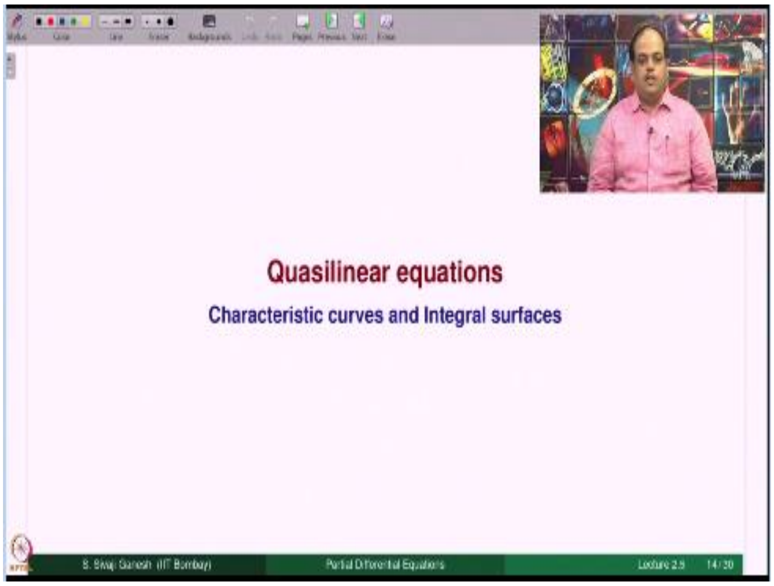

Now, we have done for linear and semilinear equations. We use base characteristics and then caught characteristics and then from there, we had a 2 parameter surface, a surface that what we think is a surface describe the 2 parameters. So, why not do the same thing here for Quasilinear?

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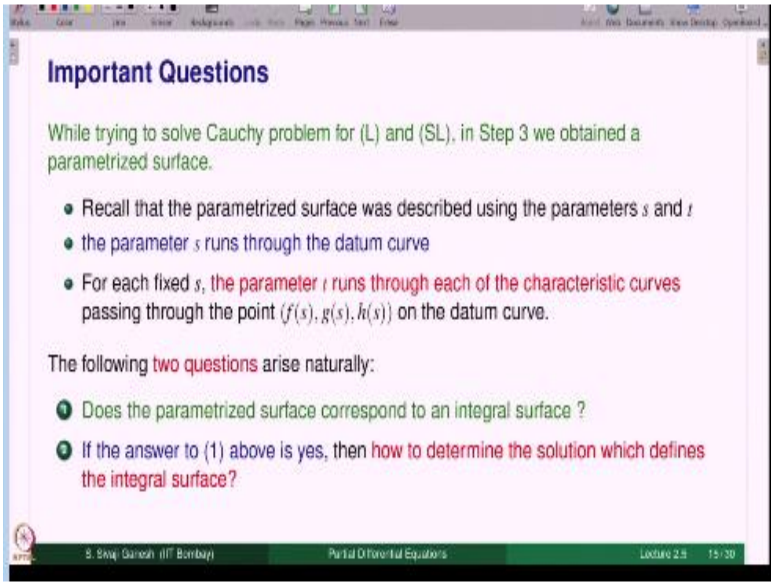

Therefore, we asked the same questions while trying to solve Cauchy problem for linear and semilinear equations. In step 3, we obtained a parameter surface, at the end of the step 3. Recall that the parameters surface was described using the parameters s and t. The parameter s runs to the datum curve. s is indexed by the datum curve and for each fixed s as t varies, where is the t varying?

The point will vary on the characteristic curve that is passing through the point f s, g s, h s. Therefore, the following 2 questions arise naturally. What are they? Does a parameter a surface correspond to an integral surface? In other words, the question is: is z is equal to x y where the solution to the PDE defined on some domain? If the answer is yes, sometimes you may say that yes, it is possible.

Now, the question is, if the answer is yes, then how to determine such function? How to determine that solution which defines the integral surface?

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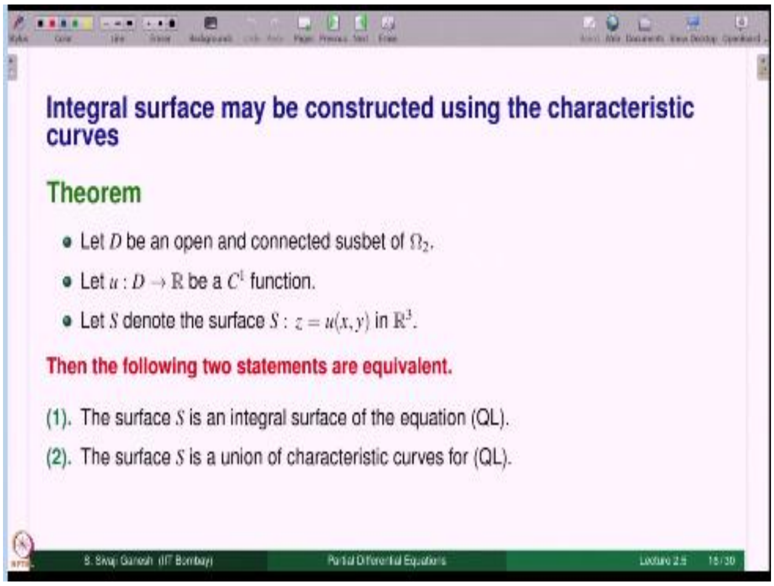

In fact, this question can be more general. I need not right integral surface. Does the parameter surface correspond to surface z = x y for some function u? Then I asked whether that function u is a solution to the PDE. Whenever we see z = x y and u is also PDE such as surface is called integral surface. Now, we have an assertion here. Integral surface may be constructed using the characteristic curves. That is a theorem.

Let D be an open and connected subset of omega 2. Let u be a function defined on D which is a C 1 function. I am not saying is a solution to the PDE or anything, it just a solution. It is just a function and consider the surface z = u x y. Then the following 2 statements are equivalent. What is that? First one, the surface is an integral surface of the equation QL. In other words, what this statement says is that u is a solution to Quasilinear equation QL. Second one, the surface s is a union of characteristic curves for QL.

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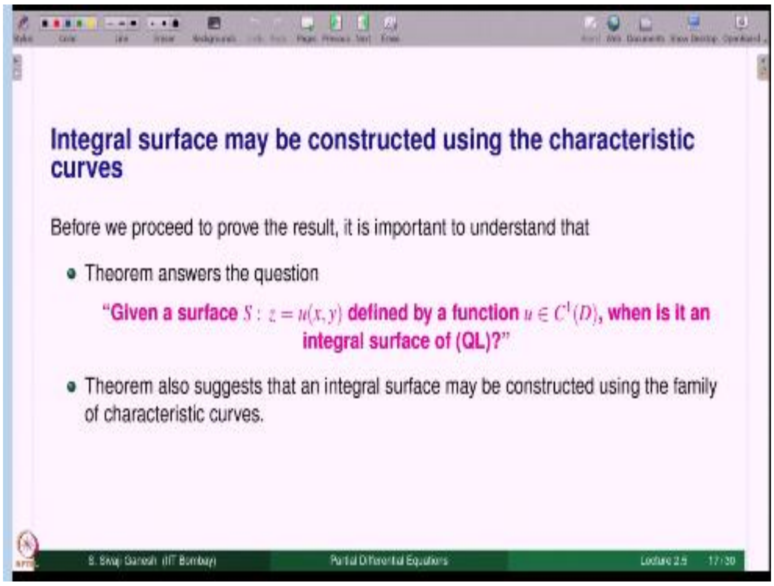

So, before we proceed to prove the result, we need to understand what this previous theorem is about. And what it is not about? Theorem answers the following question. Given a surface, I give you a surface z = x y which is defined by a function which is C 1 function defined on some domain D. When is it an integral surface of Q L? That is what the theorem answers.

Theorem also suggests that an integral surface may be constructed using the family of characteristic curves. The small missing point is that that u how do you catch hold of a function u that is where the twist lies.

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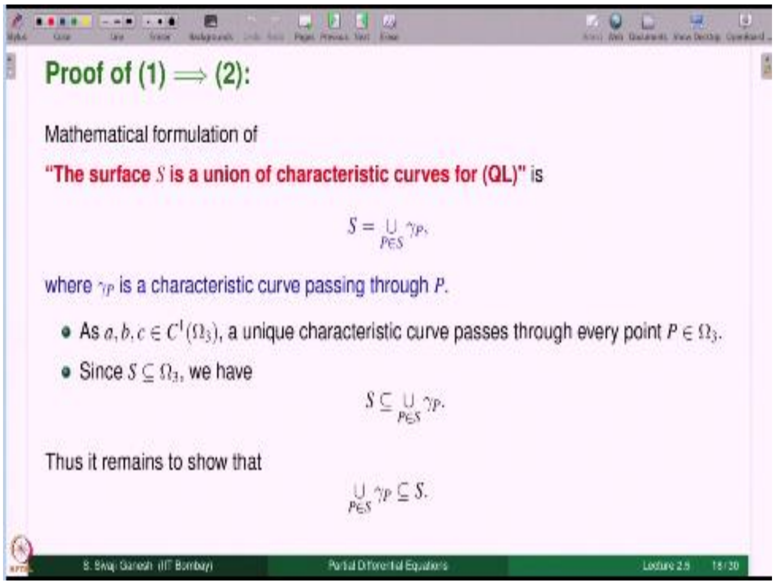

Let us prove 1 implies 2. What is 1? One is the surface is given to be an integral surface, I want to show that is union of characteristic curves. Therefore, what do you mean by this? Surface s is a union of character stickers for QL. What is the mathematical formulation of this? It means that s is a union of gamma P as P varies, yes of course, gamma P is a characteristic curve passing through P.

a b c is C 1 of omega 3. They are smooth functions even functions. Therefore, a unique characteristic passes through every point in omega 3. Since s is a subset of omega 3. We have s is a subset of this union. Take any point in s, that guy is going to belong to gamma P for some P. In fact, the same P, let P belongs to s, then gamma P passes through P. Therefore, this is a very simple containment, this we have shown. So, what remains to prove is s contains this unit?

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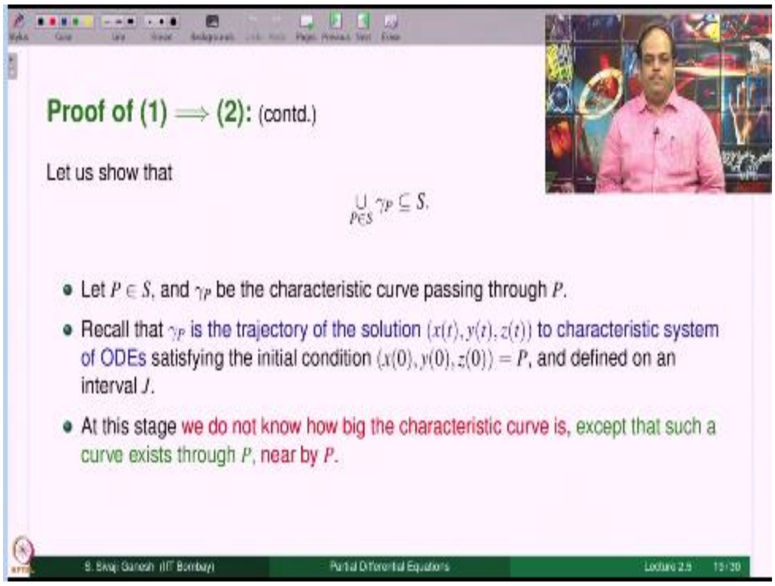

So, we want to show this. How do we show this? Take somebody on the LHS. How somebody analysis looks like? He looks like a gamma P for some P in s. So, let P belongs to s, take the gamma P. What is gamma P? It is a characteristic curve passing through P. Now, we want to show that gamma P is contained in s. It is on s. What is s? S is given by the third coordinate z = u of the first 2 coordinates x and y, u of x y.

So, we are to show that, but recall that gamma P is a trajectory of the solution x t s, y t s at t to the characteristic system of ODEs satisfying the initial conditions at t = 0. It passes through the point P. So, x 0 y 0 z 0 is P and defined on an interval J. At this stage, we do not know how big the characteristic curve is, except that such a curve exists through P and nearby P.

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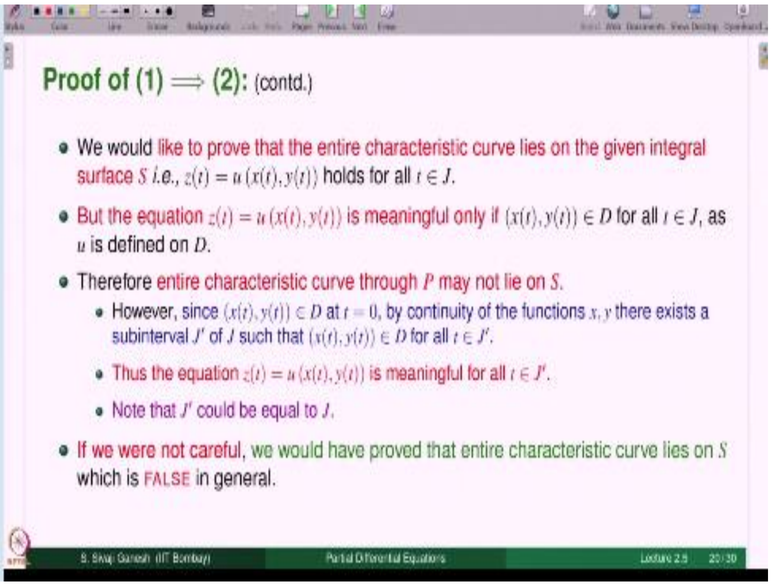

So, we would like to prove that the entire characteristic curve lies on the given integral surface s that is the third component is a function of the first 2 components. So, z t is equal to u of x t, y t. This holds for every t in J. We would like to show this. But this equation is meaningful if and only if this x t, y t belongs to the domain of u. Domain of u is already fixed. The u is given to us. It is defined on a domain D.

Therefore, it meaningful only if x t, y t belongs to D for every t in J. Whether it is true or false is a different issue. First, whether equation makes sense what question we are asking is it legitimate question or not? So, u is defined only on D that is the reason. Therefore, entire characteristic curve through P may not lie on s. See, characteristic curves are coming by solving characteristic equations, whereas u is given to you.

So, they do not talk to each other. Therefore, we cannot assert that the entire characteristic curve lies on s because this may not be make sense. What we can ask is for whichever t for which x t, y t belongs to the domain D, then does it hold? That is a good question. However, x 0 y 0 is in D. Therefore, by continuity x t, y t will be in D and you can find a subinterval of J such that x t, y t belongs to D.

Fine. x t, y t itself is defined on J but at 0, you are in D. D is an open set. Therefore, for some time t in some interval J dash which is a subinterval of J, u will be in D. Thus, this equation is meaningful for all t's that subinterval J dash. Note that J dash could be equal to J. we are not denying that but what we are saying is that we cannot assert that. We cannot assert that J dash is equal to J. It can be equal.

If you are not careful in observing this point 1 and 2, we would have proved that the entire characteristic curve lies on s. This is a place where one makes mistake, because we are not careful because what we are going to do later is simply differentiate this. Equation itself may not make sense. If you ignore that, you simply differentiate, go through the procedure and show that yes, the entire curve lies on s which is not correct. It is false in general. You have to be very careful.

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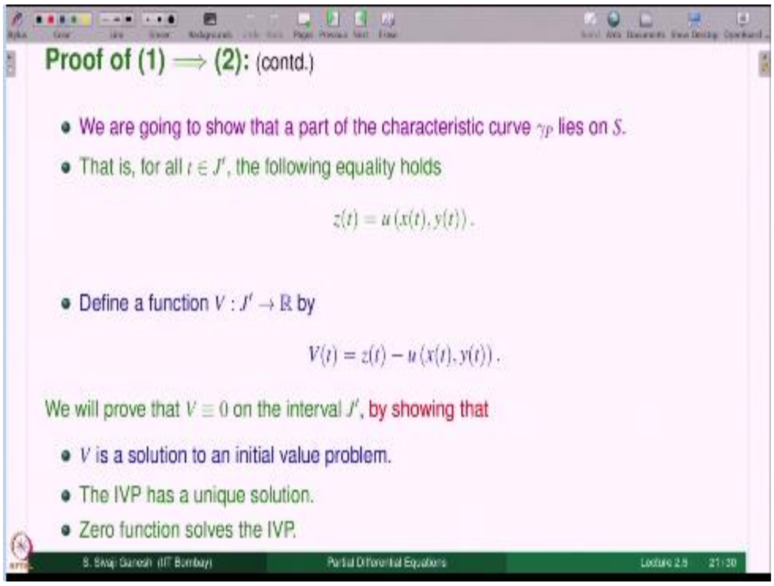

So, we are going to show that a part of the characteristic curve gamma P lies on s that is z t = u x t y t holds for t belongs to a J dash, fine. So, define this function, this very much makes sense for t in J dash because J dash was chosen such that x t, y t is guaranteed to be in D which is the domain of u. So, these are good function, it makes sense. What you want to show is, this function is 0.

What is the usual strategy? Show that the derivative is 0 and at some point, it is 0. Therefore, it is identical equal to 0. If a function f dash is 0 and f at some point is 0, f will be 0 because if f dash is 0 means f is constant. But we are not that much lucky. But we can still show that V is identical equal to 0. How do we show this? We show that V is going to satisfy a certain initial value problem for ODE.

And it is known, meaning that initial value problem which V is a solution to, has only one solution. It is also known that 0 is a solution. Therefore, V t must be 0 solution. That is a strategy we are going to look. We will show that V is a solution to initial value problem. We will construct or we will design that initial value problem and the initial value problem has a unique solution.

And 0 is a solution to that. And the way we have constructed initial value problem shows that V is already a solution. We are also showing that 0 is solution. Therefore, V must be identical equal to 0. And we have shown that z t equal to u x t, y t holds for t in J dash that means 1 implies 2 is done.

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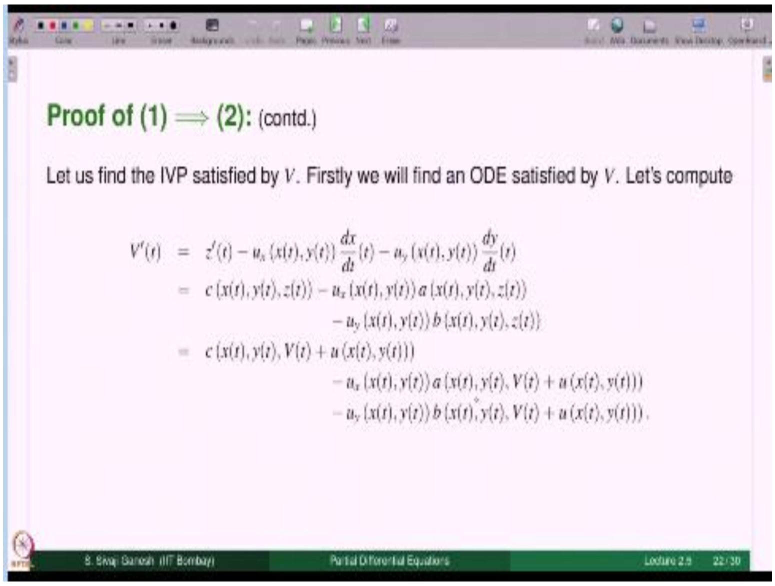

So, how do you find that initial value problem. We want this to be a solution to that. Therefore, we start differentiating V and get that first order ODE. So, we will find a ODE satisfied by V, simply by computing V dash. I am not going to this details. You have to just chain rule it is z t – u x t, y t. So, z t derivative is z dash t, this is u x and x dash, u y and y dash, but you know z dashes c.

You know the x dash is A and y dash is B and I want a ODE satisfied by V t. So, right hand side also, I would like to see V t and anything else known functions, which are x t, y t, z t. These are known functions. a b c are known functions. u is an of course, unknown function. So, I write this like this. So, there is a V here, V here and V here.

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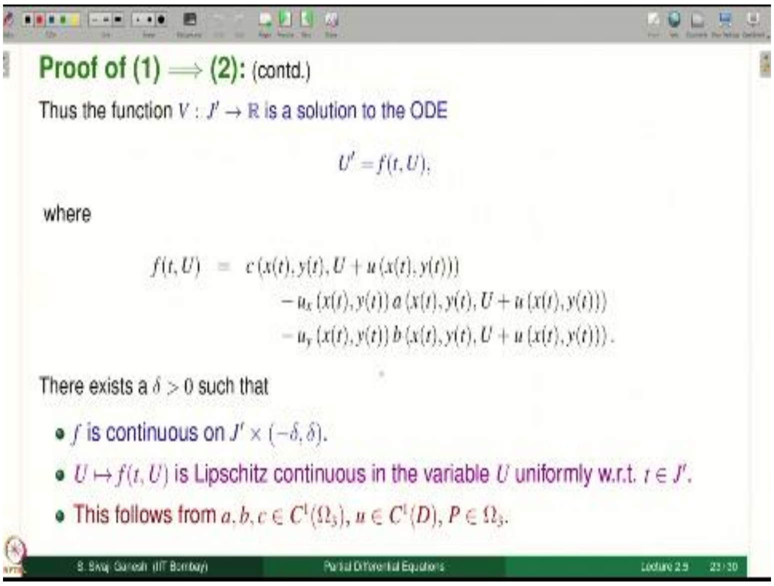

So, it satisfies the equation U dash equal to F t U. I am going to write what is F t U. So, this is what I want to think as f of t V, what is that. It is going to be c x t, y t U plus this u of x t, y t minus this is a known function I will keep it as it is, wherever V t is there, I put a V or U because the way I am going to write is in terms of U. So, this is the function. Recall that we are interested in applying Cauchy Lipschitz Picard’s theorem and conclude that the V which we have defined to be identical equal to 0.

We require the right hand side function f of t U to be ellipses with respect to u to apply Cauchy Lipschitz Picard’s theorem. So, is F, the function u going into f of t U, is it locally ellipses or not that is very important. Indeed, there exists a delta positive such that f is continuous on J prime cross minus delta, delta. So, the right hand side is continuous function. And U going to f of t U is locally ellipses continuous in the variable U that is what we need for applying Cauchy Lipschitz Picard’s theorem uniformly with respect to t in J dash.

And this follows from the smoothness of the a b c; a b c are C 1 of omega 3 and U itself is a C 1 function and P is a point in omega 3 because of that we get such an existence of such a delta and I leave it for you to fill in the details.

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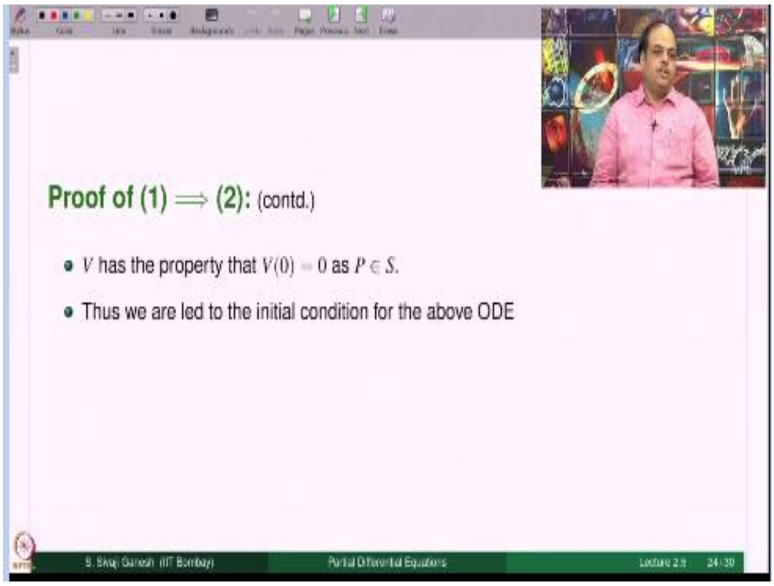

V has the property that V 0 is 0, we know that. What is V 0? z 0 – U x 0 y 0, z0 is h s, x 0 y 0 is f s g s. So, P is in S therefore **(()) (32:56)**. Thus, we are let to the initial condition for the above ODE that is U of 0 equals 0.

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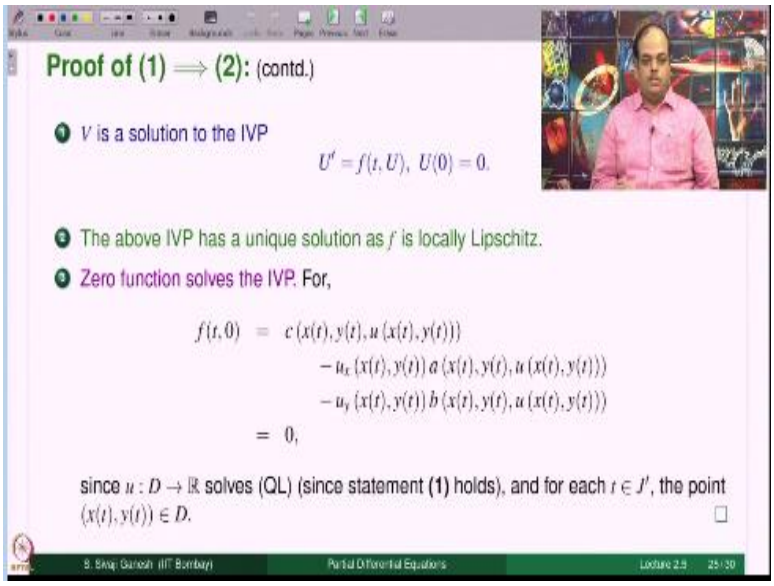

So, this is the initial value problem. U dash is f t U where f is described there and U 0 is 0. This has a unique solution, because f, we have shown you this is a locally ellipsis therefore, this initial value problem, we have unique solution. Zero function solves very obvious, you have to check that. Zero dash, derivative of 0 is 0 therefore, it is enough to check f of t, 0 is 0 which is verified here.

f of t, 0, you have to substitute for u; zero you get this and that is 0. Since you solve Quasilinear equation. We are using that because statement 1 is given that u is a solution that is it.

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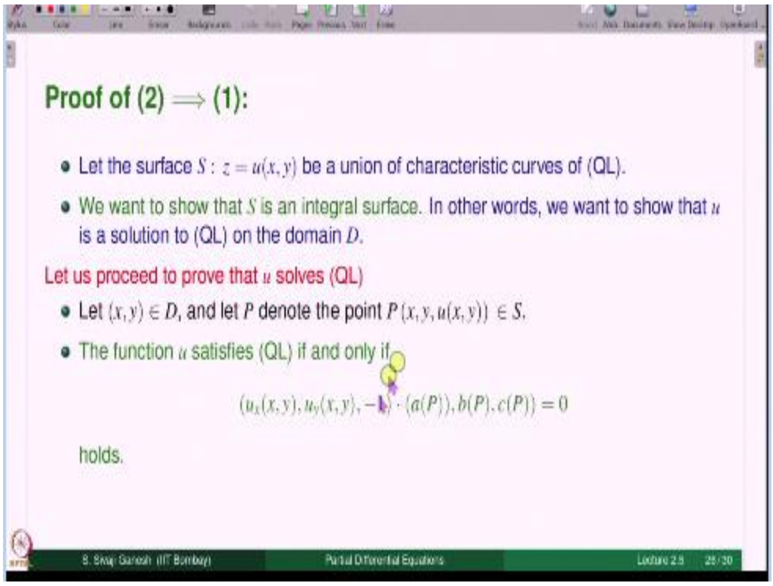

So, this shows 1 implies 2. Now, we go to 2 implies 1. So, we assumed that the surface is union of characteristic curve and we want to show that this u is a solution to Quasilinear equation. So, let us proceed to prove that u solve QL. So, take a point x y in D and let P denote this point x y, u of x y that will be on S, function u satisfies QL if and only if this equation is satisfied u x u y – 1 dot a P b P c P is 0. This holds.

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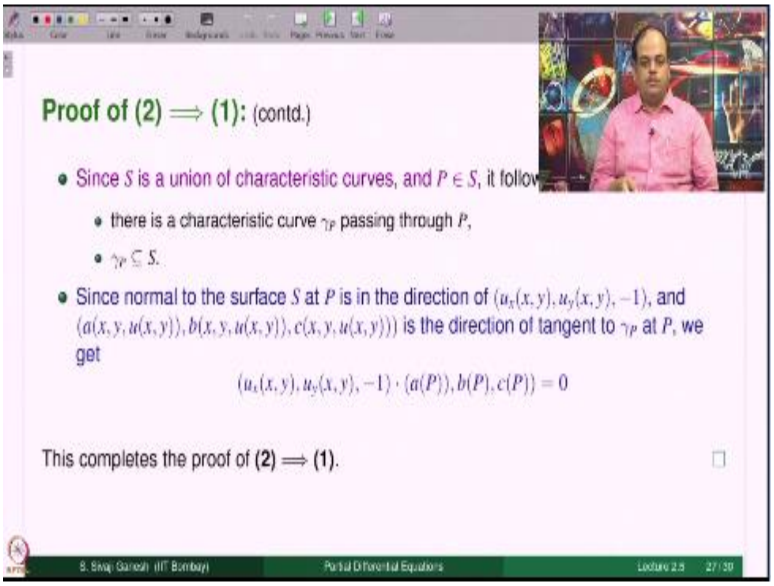

Since S is a union of characteristic curve and P in S, it follows that there is a characteristic curve gamma P passing through P, gamma P is contained in S. Since, normal to the surface S at the point P is in this direction u x u y – 1, z = u of x y, u x u y – 1 will be a normal direction and a b c at the point x y u x y is the direction of tangent to gamma P at P. Therefore, the dot product must be 0. So, this shows 2 implies 1.

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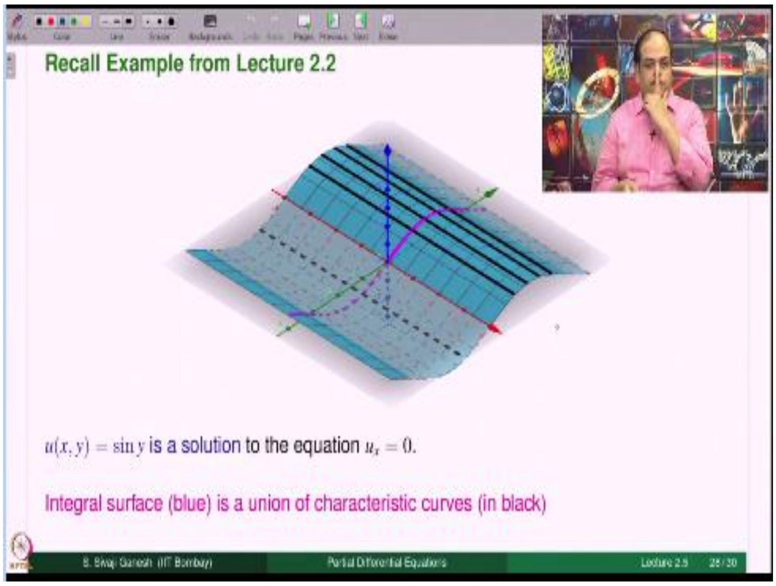

So, this is picture. As usual, this red x axis, green y axis, blue z axis, we saw this earlier. u x y = sin y is a solution to the equation u x equal to 0. Integral surface is blue. It is a union of characteristic curve. Characteristic curves are straight lines. We saw that already they were obtained as intersection of 2 surfaces. So, datum curve is in magenta. So, this is the datum curve. So, that is 0. Yes, sin S varies that is a datum curve.

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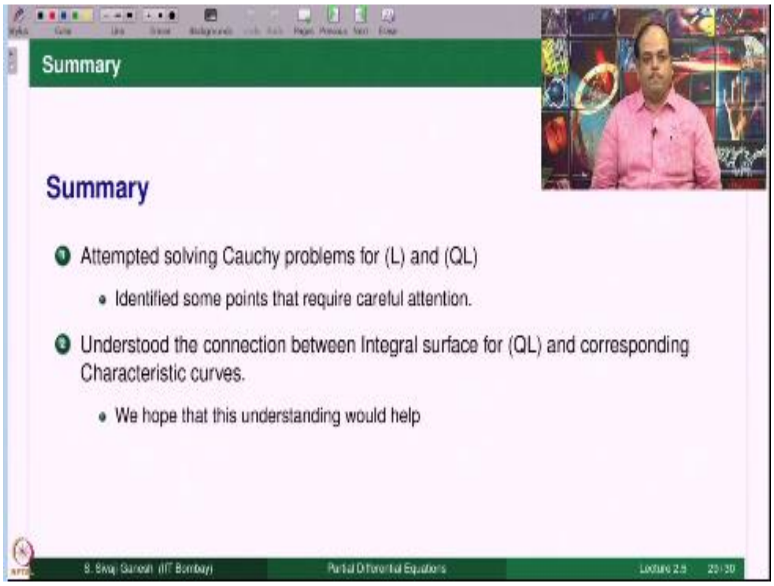

Let us summarise what we did. We attempted solving Cauchy problems L and QL. We were stuck where we had to find that function, we stopped there. So, we identified some point that require careful attention there and then we understood the connection between integral surface for a Quasilinear equation and corresponding characteristic curves. We hope that this understanding will help us in solving Cauchy problem for Quasilinear equation. Thank you.

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