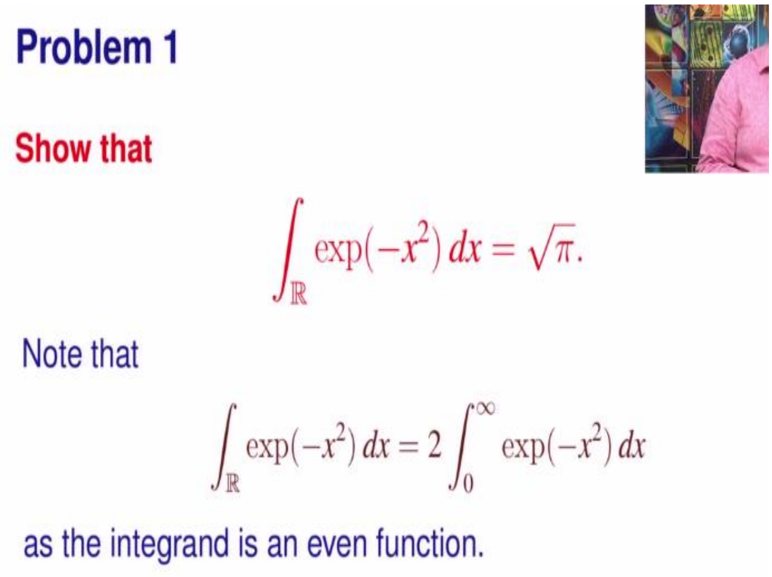
**Partial Differential Equations**
**Prof. Sivaji Ganesh**
**Department of Mathematics**
**Indian Institute of Technology – Bombay**

**Lecture – 59**
**Maximum Principle for Heat Equation**

**(Refer Slide Time: 00:28)**




Welcome to Tutorial on heat equation. In this tutorial, we are going to solve a few problems associated to heat equation. The first problem is a problem in calculus which is integral of e power –x square is root pi. We have used this information in our analysis of heat equation that is why I decided to do this proof. This involves a small trick, I will explain you that. So integral of e power –x square R is equal to two times the integral on 0 to infinity of the same integral e power –x square because it is an even function

**(Refer Slide Time: 00:57)**

## Solution to Problem 1

**Denote** $I = \int_0^\infty \exp(-x^2)\,dx$. **Then**

$$\begin{aligned} I^2 &= \left(\int_0^\infty \exp(-x^2)\,dx\right)\left(\int_0^\infty \exp(-y^2)\,dy\right) \\ &= \int_0^\infty \left(\int_0^\infty \exp(-x^2)\,dx\right)\exp(-y^2)\,dy \\ &= \int_0^\infty \int_0^\infty \exp(-x^2-y^2)\,dx\,dy \\ &= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=\infty} \exp(-r^2)\,r\,dr\,d\theta \end{aligned}$$

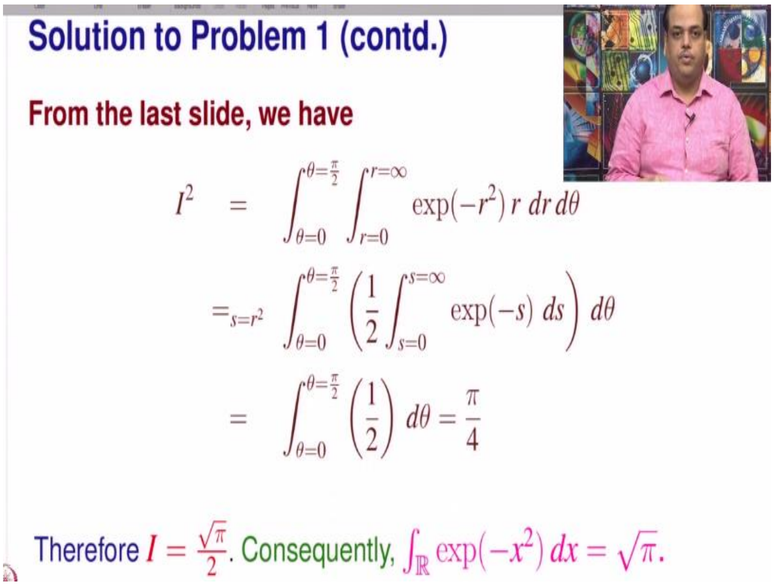
using Polar coordinates in the plane.

Therefore, it is enough to compute 0 to infinity e power –x square dx. Then the answer we are interested in will be two times I. So to compute I, we look at I square, I squared is I into I that is this quantity into the same quantity. Because I am writing two times I have used the x in one integral and y in the other integral. Anyway, this is so called a dummy variable, so I can use any variable I want. Now, what I do is that this quantity I take it inside.

Actually, I am assuming that this is finite quantity and that is why everything is fine. So I take this inside and I get this. Now, this is nothing but 0 to infinity 0 to infinity e power –x square into e power –y square is e power –x square –y square dx dy. And that is equal to in the polar coordinates r = square root of x square + y square. So therefore, –x square –y square becomes –r square and the area element dx dy becomes r dr d theta.

Now we have to find out the limits for the integration variables. Look at this integral, this integral is on 0 infinity cross 0 infinity that is the first quadrant. So the first quadrant, radius goes from 0 to infinity. But the part of the circle which in the first quadrant corresponds to theta = 0 to theta = pi by 2. For example, these are the circles I am integrating on. The angle is 0 to pi by 2 here, pi by 2 is the angle here.

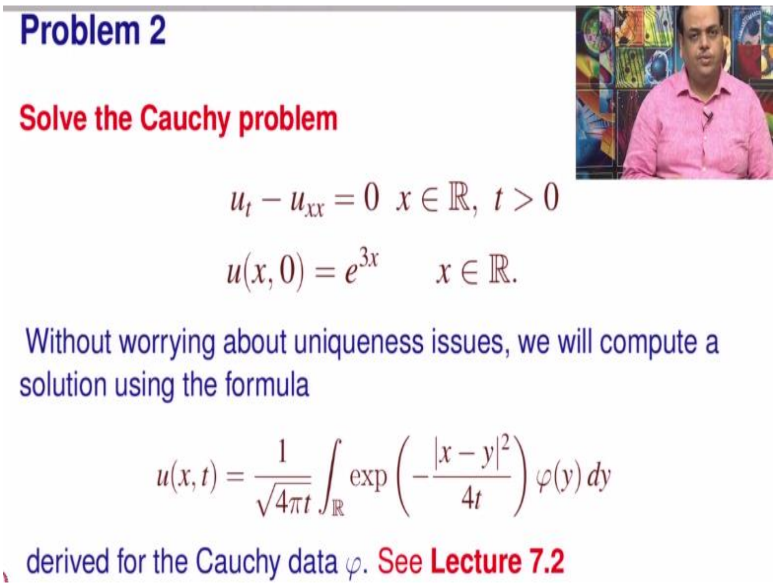
**(Refer Slide Time: 02:52)**




So I square equal to this, this is what we have seen on the last slide. Now it is very easy to integrate. Put s = r square, then the inside integral with respect to r becomes with respect to s in this form and this is easily integrable e power minus –s, integral is e power –s by –1 limits 0 to infinity. So you get 1 from there, so total is half. So integral with respect to theta from 0 to pi by 2 of 1 by 2 that is pi by 4. So I square is pi by 4, therefore I is root pi by 2.

Therefore, this integral which is two times I is root pi. This is a very simple trick that we use in calculus. To compute an integral of one variable, we go through computation of integral into variables. Luckily, because of the form of the integrand, we can convert that into polar coordinates and we could easily compute.

**(Refer Slide Time: 03:58)**



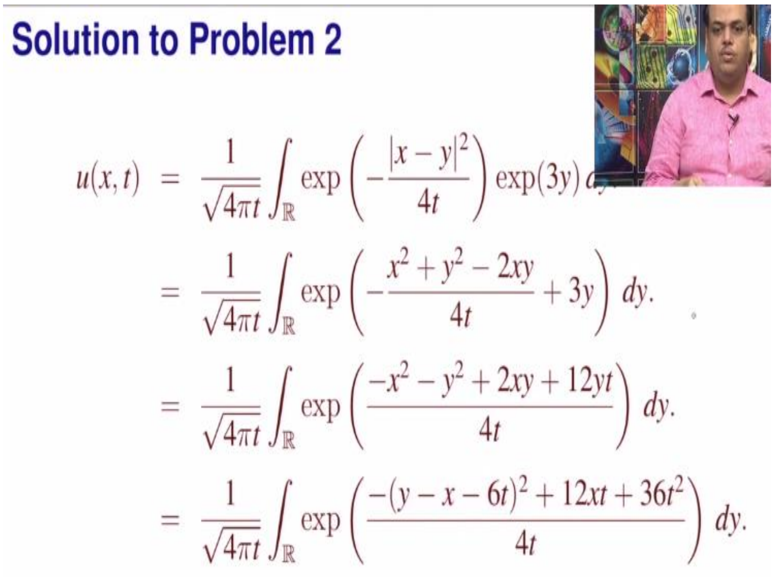
Now let us solve the Cauchy problem for homogeneous heat equation. Of course, we know a certain formula for the solution of the homogeneous heat equation and the corresponding Cauchy problem with a phi of x we have an integral, but that integral is rarely integrable, only in very special cases where we are very lucky. Namely C phi of x we have taken e power 3 x because the heat kernel is also exponential type, right.

So therefore, this exponential should help us that is how we are able to integrate, otherwise if it is some arbitrary function of x which is continuous and bounded, we do know that it gives rise to a solution but then that is it, we cannot really compute. We have to be really lucky to be able to compute. So without worrying about uniqueness issues because we know that Cauchy problem solution is not unique, it is unique only when you are looking at special classes.

Let us not worry about that and simply compute this integral and see what we get. So you will compute a solution, maybe that is the solution, that is the only solution in some class or maybe there are more solutions, we do not know it because this may not belong to the class for which uniqueness theorem holds. In fact, we are not discussed much about uniqueness theorems for which I have just given you the reference of book by DiBenedetto on PEDs.

You can look at at your leisure time. But as far as this course is concerned, we simply want to compute this integral with phi of y equal to e power 3 y, can we do it or not? So, this formula we have derived in lectures 7.2.

**(Refer Slide Time: 05:45)**



So, now this is exponential into exponential, therefore it is exponential of this plus this, so I have written down exactly that. So, this is same as this. So, I am only working in the power of this e, exponential e to the power something, so that power only I am manipulating. Now, I express that like this, so called completion of squares. The advantage now is that I am going to set y – x – 6 t by something equal to some variable.

**(Refer Slide Time: 06:24)**

**Solution to Problem 2 (contd.)**

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \exp\left(\frac{-(y-x-6t)^2 + 12xt + 36t^2}{4t}\right) dy.$$

Do a change of variable $p = \frac{y-x-6t}{\sqrt{4t}}$ and obtain

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \exp\left(-p^2\right) \exp(3x+9t)\, dp.$$

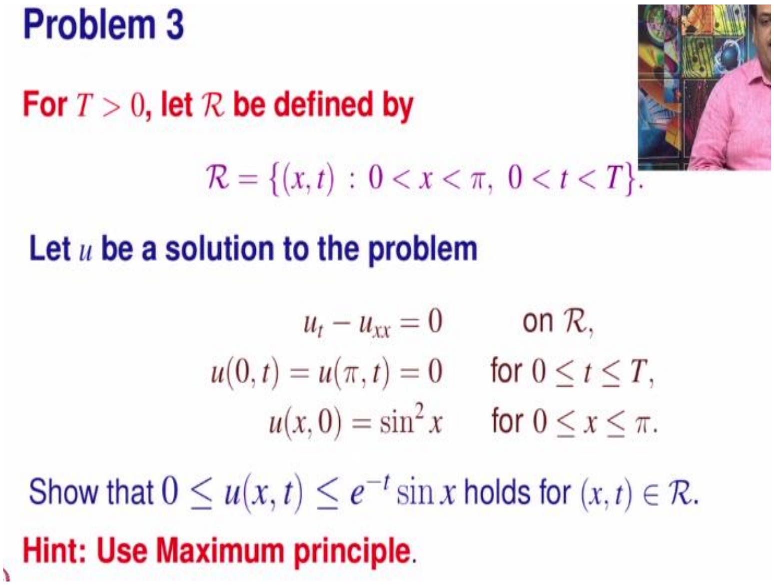
$$= \exp(3x+9t) \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \exp\left(-p^2\right) dp.$$

$$= \exp(3x+9t) \quad \square$$

So, I am going to put y – x – 6 t by root 4 t as P. So, then this term the first term by 4 t that simply becomes P square e power – P squared and what remains is 12 xt by 4 t, 36 t square by 4 t. So, this is e power – P square and what remains here is 12 xt by 4 t is 3 x; 36 t square 4 t is 9 t and dy 1 by root 4 t is your dp from here, dp is 1 by root 4 t dy. So, this is exactly same as this. Now, if you see exponential of 3 x + 9 t it does not depend on P, so it just comes outside.

And what we have is 1 by root pi into integral over R of e power – P square dp, we just found out this is a root pi. So root pi by root pi that is 1. So, what you have is exponential of 3 x + 9 t. So, we are very lucky that we could actually compute the integral.

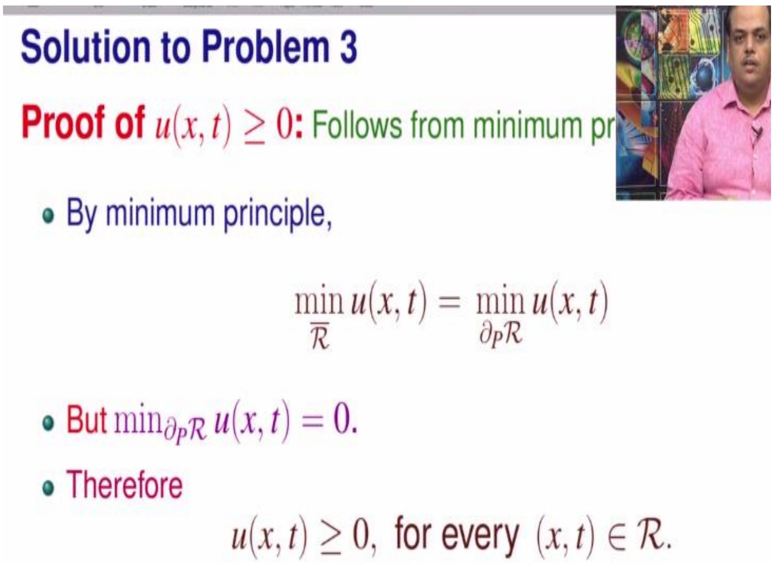
**(Refer Slide Time: 07:40)**




Now, let us move on to the problem 3. For T positive let R be defined by this rectangle 0, pi cross 0, T. Let u be a solution to this problem, which is initial boundary value problem, homogeneous heat equation, boundary conditions are 0, initial condition sine square x. Show that 0 is less than or equal to u of x, t is less than or equal to e power minus –t sine x holds for x, t in the rectangle. Hint is to use maximum principle.

The maximum principle allows us to compare certain things, right. So, therefore the maximum principle using that is the hint here. So, what we have to show is actually two inequalities. We have to show 0 is less than or equal to u x, t and u x, t is less than or equal to e power –t into sine x.

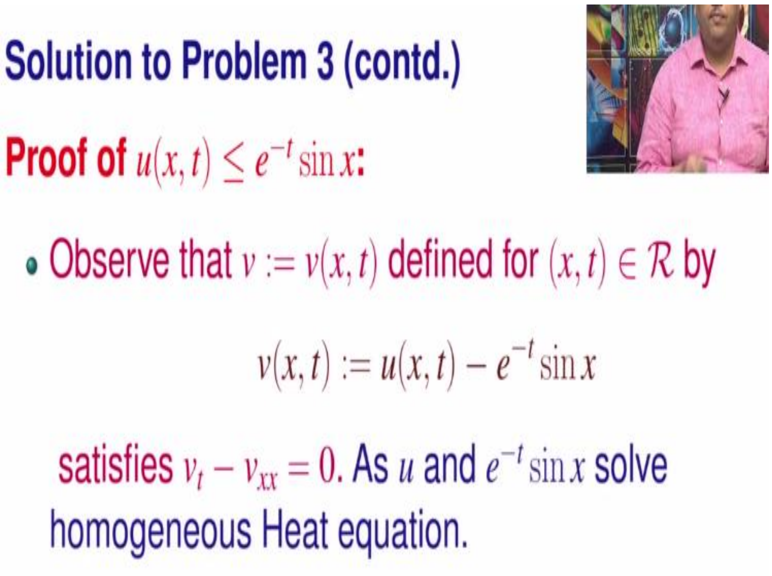
**(Refer Slide Time: 08:37)**



So let us prove the first one that is u x, t is greater than or equal to 0. It follows from minimum principle. By minimum principle, we know that minimum of u on R closure is actually minimum of u on the parabolic boundary of R. What is the parabolic boundary of R? This is R, so parabolic boundary is this, union of these three lines and u is 0 here, u is 0 here, and u is sine square x here that is what is given to us.

So, minimum of u on the parabolic boundary is 0 because of this because minimum of sine square x on this 0 to pi is 0. So minimum is 0. Therefore, minimum of u on R closure is 0, which means u is greater than or equal to 0 on R closure, in particular on R. So, this completes the proof of the first inequality.

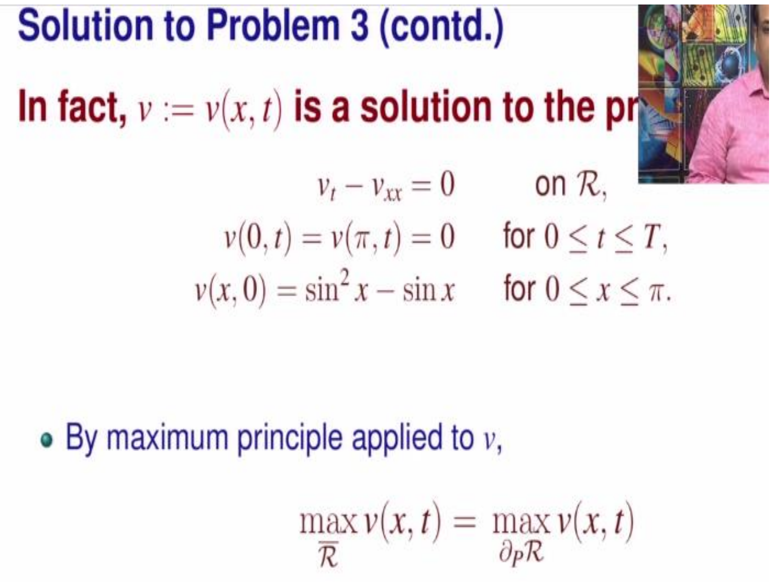
**(Refer Slide Time: 09:51)**




Now let us look at the second inequality, so we want to prove this. So if you want to apply maximum principle you can only apply that to solution solve heat equation. So, therefore, we naturally ask this question whether this is a solution of heat equation. Yes, we have already seen when we were solving heat equation using separation of variables method that these are the things which are coming as the terms in the series. So, is this a solution? Yes, it is a solution.

So, what we do now is we define a new function v which is u – e power –t sine x. This satisfies v t – v xx = 0 because u is already a solution to the homogeneous heat equation, this also solves homogeneous heat equation. Therefore, this equation being linear and homogeneous, sum of two solutions or the difference of two solutions is a solution still. So, v is a solution to this. And what is that we want to show v of x, t is less than or equal to 0. So, showing this is same as showing v is less than or equal to 0.

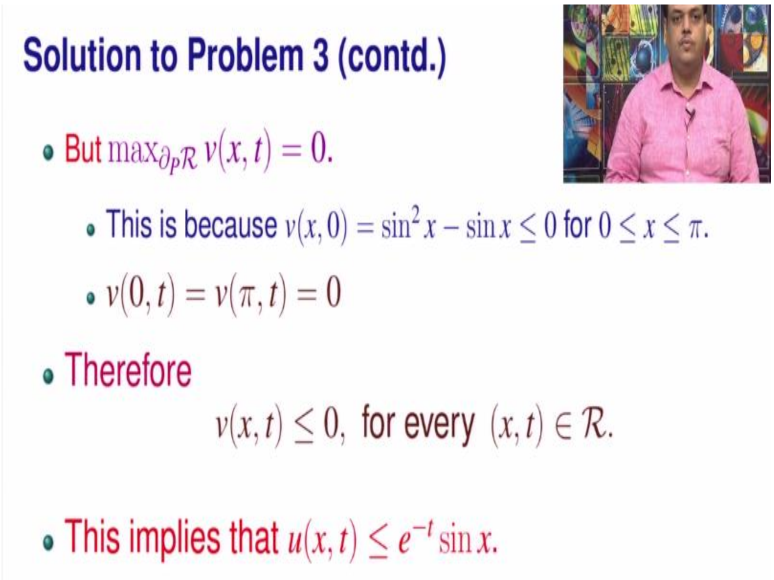
**(Refer Slide Time: 10:59)**




So, in fact, we will look at the problem that v satisfies. So, homogeneous heat equation v is a solution that we have already checked on R and v of 0, t and v of pi, t are also 0, v of x, 0 = sine square x – sine x, please check this, it is very simple. So, I am not going to check here. So, this is a problem that v satisfies. Now, by maximum principle applied to this v, maximum of v on R closure is nothing but maximum of v on the parabolic boundary.

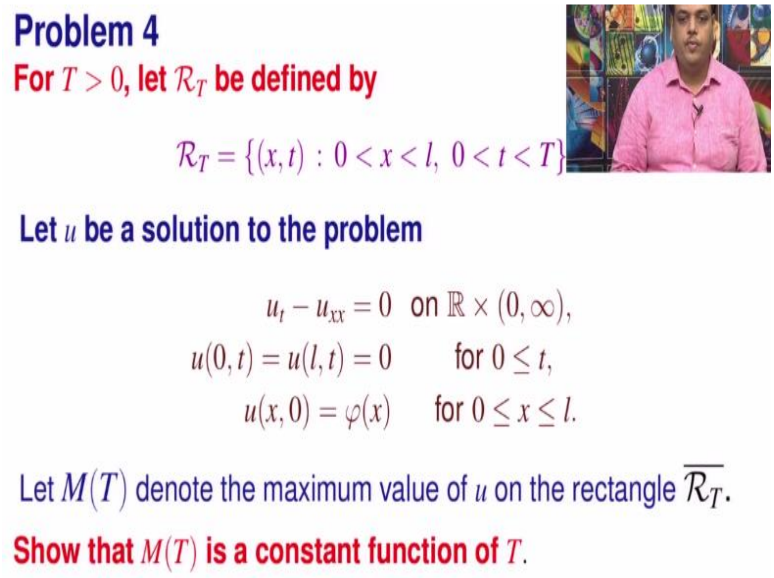
So, therefore we will look at what is the maximum of v on the parabolic boundary. Parabolic boundary consists of three lines; one is the line x = 0, pi on which v 0 okay and the other line is the x axis part between 0 to pi. So, we need to look at what happens to this sine square x – sine x. Sine square x – sine x is always less than or equal to 0 on this interval because sine x lies between 0 to 1 that is the reason, we will see the details.

**(Refer Slide Time: 12:12)**



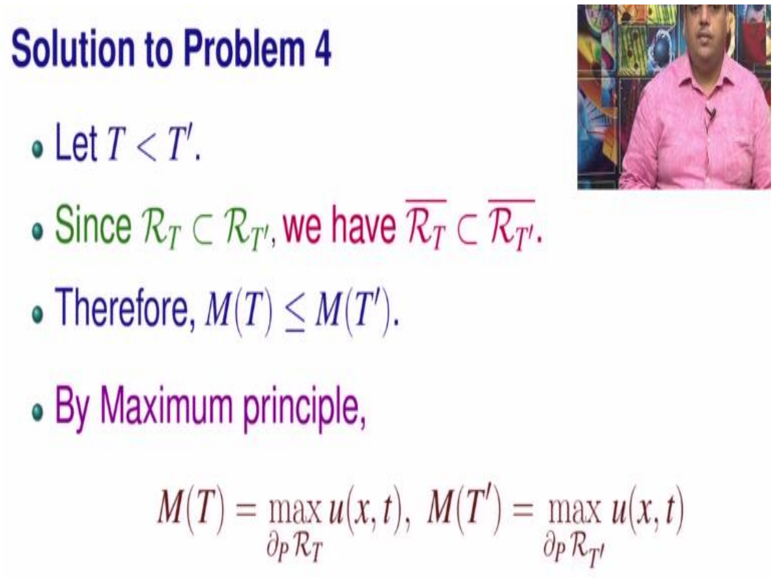

So, maximum of v on this is zero. This is because v of x, 0 is sine square x – sine x that is always less than or equal to 0 and v of 0, t and v of pi, t both of them are 0. Therefore, v of x, t is less than or equal to 0 for every x, t in R that implies u of x, t is less than or equal to e power –t sine x. This is what we wanted to show.

**(Refer Slide Time: 12:42)**



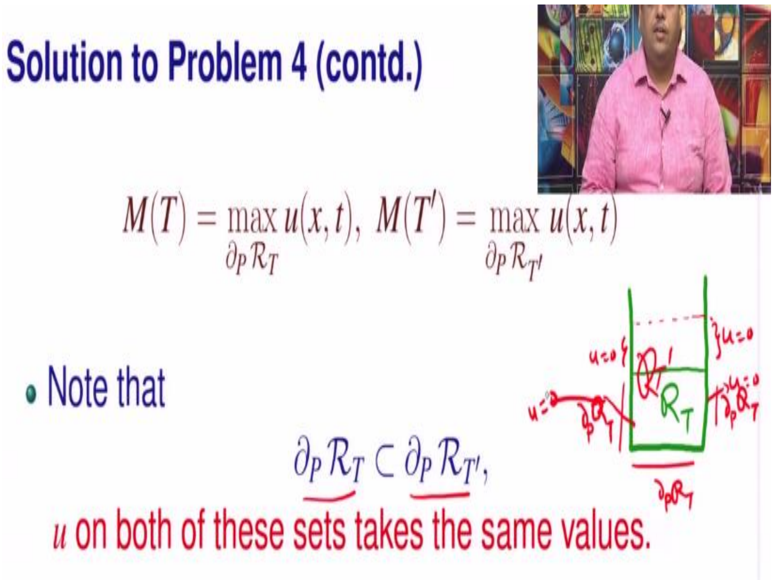

So, let us look at problem number 4. For T positive let R T be defined as this rectangle x between 0 to l and t between 0 to T, t = T. So, this is what is called R T. So, for example if you take T dash another line, t = T dash, this entire region will be R T dash. So, let u be a solution to the homogeneous heat equation in R cross 0, infinity and the boundary condition at x = 0 and x = l are 0 and u of x, 0 = phi x. Let M of T denote the maximum value of u on the rectangle R T and closure, R T closure. Show that M of T is a constant function of T.

**(Refer Slide Time: 13:50)**



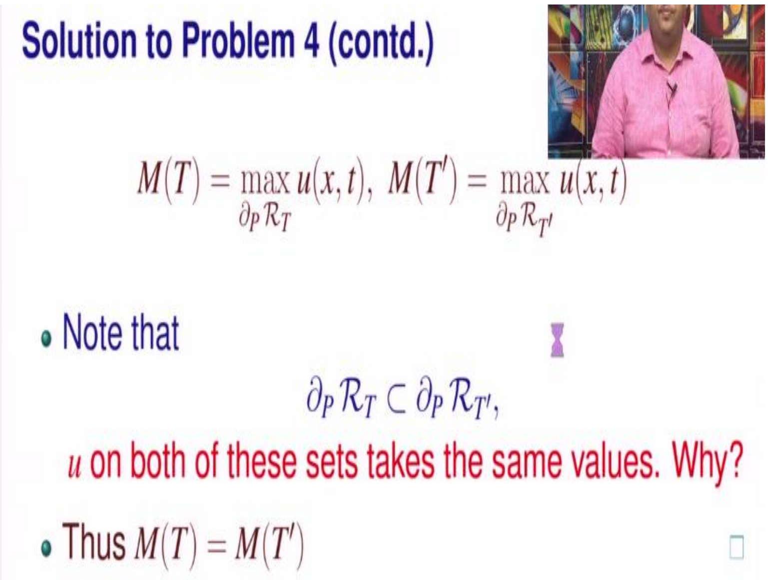

So let T be less than T dash and R T is a subset of R T dash, R T dash is a bigger rectangle. Therefore, R T closure also is contained in R T dash closure. Therefore, M of T is less than or equal to M of T dash. M of T after all is the maximum of u on a smaller set, M of T dash is a maximum on a bigger set, therefore this holds. So, by maximum principle what is M of T is a maximum of u on the parabolic boundary of R T. Similarly, M of T dash is maximum value of u on the parabolic boundary of R T dash.

**(Refer Slide Time: 14:27)**




Parabolic boundary of R T is a subset of parabolic boundary of R T dash, u on both these sets takes the same values. Let us just have a look at this picture. This is R T, this is R T dash. So this part, this part, this part is a parabolic boundary of R T. What is the parabolic boundary of R T dash? It is bigger than this. What is the new portion? It is only this much and what happens to you the u there? The u is 0 that is the boundary condition and u is already 0 here, so u does not take any new value so that maximum increases, therefore maximum is the same.

**(Refer Slide Time: 15:30)**




So, M of T = M of T dash. A similar statement can be made for the minimum also. Thank you.